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Research Paper

A study of method of reverberation-ray matrix for buckling of orthotropic multi-span plates

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ABSTRACT *The method of reverberation ray matrix (MRRM) is presented for buckling analysis of thin multi-span rectangular anisotropic plates. The reverberation ray matrix is formulated based on the modal solution for an orthotropic plate together with the continuity conditions among adjacent span plates and boundary conditions for the entire plate structure. The buckling loading can be determined from zero value of the determinant of difference between the unit matrix and the resulting reverberation ray matrix. The algebraic equation is analytically given for buckling loading under the action of a uniaxial pressure along a side of the multi-span plate. For the case of simply-supported boundary conditions, computational results for the buckling loading are presented, and compared with the data from the finite element simulation by using ANASYS code. The results show the validity and exactness in computation for MRRM. The resulting solution can be effectively and readily used to evaluation on the buckling for multi-span rectangular anisotropic plates.*
KEY WORDS *multi-span, thin rectangular plates, orthotropic, buckling, method of reverberation ray-matrix*

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I. INTRODUCTION

Plate structures have found a wide use in civil, mechanical, aeronautic and aerospace engineering. Analysis of buckling problem of such surface structures under in-plane compressive loadings is one of most important topics in engineering design and evaluation of safety. There exist extensive analyses of buckling of various plate-type structures (e.g., sandwich plates, multi-span plates and stiffened plates, and so on) by using state-space method, etc. [1-5].

Howard and Pao proposed a method of reverberation ray-matrix (MRRM) for analysis of dynamics response of two dimensional framed structures [6,7]. This method is based on analysis of a reverberation-ray matrix reflecting the properties of wave propagation in beam members. The matrix is resulted from the relation among the arriving and departing wave amplitudes for joints, and these amplitudes are unknown coefficients in steady-state solutions for extensional, bending and twisting wave motions in beam members. The MRRM has been extended to analysis of transient response and wave propagation in three dimensional frame structures and layered media [8-12], and also developed for static analysis of internal forces and deformations of framed structures [13]. More recently, a procedure of MRRM is proposed to determine analytically buckling loading of thin multi-span rectangular isotropic plates having internal line supports or stiffeners [14].

So far there is no related study of MRRM for buckling analysis of anisotropic plate structures under uniaxial or biaxial in-plane compressive load. This paper is aimed at seeking the solution for buckling of multi-span orthotropic plates in the frame of MRRM. For the structure having two opposite simply-supported edges with spans, a formulation is established by using the modal solution for an orthotropic plate together with the continuity conditions among adjacent span plates and boundary conditions for the entire plate structure. As an application, a three-span plate is analyzed, and an equation for buckling loading is presented analytically. The computational results from MRRM are given and compared with the numerical solutions from the finite element simulation. It is shown that the present formulation for MRRM has enough exactness in computation.

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II. SOLUTION FOR BUCKLING OF A THIN PLATE WITH TWO SIMPLY-SUPPORTED OPPOSITE EDGES

Let us consider a rectangular plate with *n* spans in the *Y*- direction, as shown in Fig. 1. The plate is of thickness h, width *a* and length b. The two opposite edges with spans (OB and AC) are simply-supported, and two remaining edges (OA and BC) may be in free, simply-supported or clamped edges. It is assumed that the plate is subjected to in-plane uniform pressures p_x, p_y in *X*- and *Y*-directions, respectively. The individual spans are indicated by lowercase letters *i*, *j*…. , and the *i*th span is of length l^{ij}. The origin of the coordinate system is set at the junction of edges OA and OB, and *Z* -direction follows the right-hand rule.

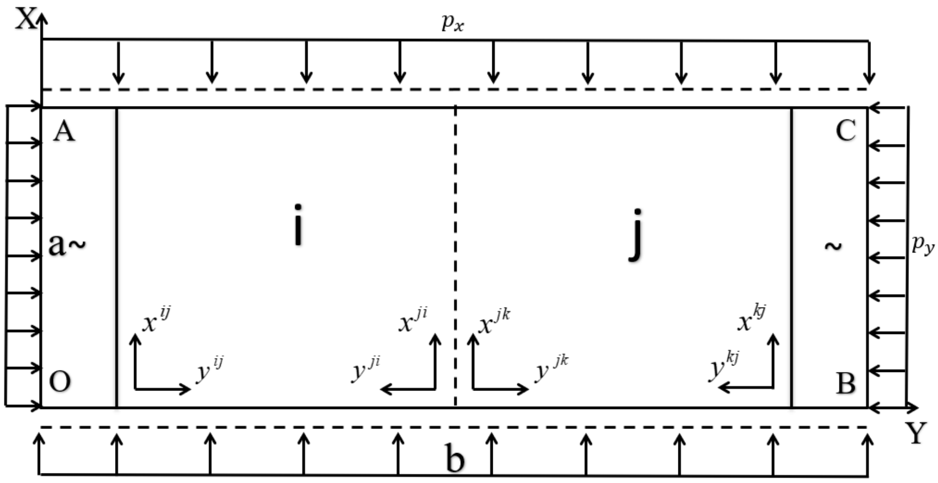

Fig. 1 Geometry and coordinate system of a thin multi-span plate

The governing equation for buckling of a thin rectangular orthotropic plate under the uniform pressures is written by

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}+p_x\frac{\partial^2 w}{\partial x^2}+p_y\frac{\partial^2 w}{\partial y^2}=0 \tag{1}$$

where w is the deflection, and D_1, D_2, D_3 are flexural rigidities with respect to axes Y and X and twisting rigidity respectively expressed by

$$D_1=\frac{E_1h^3}{12(1-\mu_1\mu_2)},\ D_2=\frac{E_2h^3}{12(1-\mu_1\mu_2)},\ D_3=\mu_2D_1+2D_k=\mu_1D_2+2D_k,\ D_k=\frac{Gh^3}{12} \tag{2}$$

in which E_1, E_2 and μ_1, μ_2 are Young's modulus in the directions of axes X and Y and corresponding Poisson's ratios, respectively; *G* and is shear modulus in the XY plane, and D_k is twisting rigidity around Z-axis.

For the plate with two opposite simply supported edges with spans (i.e., OB and AC), the solution for the deflection is expressed by

$$w=\sum_{n=1}^{\infty}w_n(x,y)=\sum_{n=1}^{\infty}Y_n(y)\sin\frac{n\pi x}{a} \tag{3}$$

where n is the half-wave number in *x* direction. For each half-wave number n, substitution of Eq. (3) into (1) results in

$$D_2Y_n^{(4)}-2D_3(\frac{n\pi}{a})^2Y_n^{(2)}+\left[D_1(\frac{n\pi}{a})^2-p_x\right](\frac{n\pi}{a})^2Y_n=0 \tag{4}$$

The solution for the above equation has the form of

$$Y_n(y)=a_1e^{\alpha y}+a_2e^{\beta y}+d_1e^{-\alpha y}+d_2e^{-\beta y} \tag{5}$$

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where

$$\alpha=\sqrt{\frac{2D_3\left(\frac{n\pi}{a}\right)^2-p_y}{2D_2}+\sqrt{\left\{\left[\frac{2D_3\left(\frac{n\pi}{a}\right)^2-p_y}{2D_2}\right]^2-\left[\frac{D_1\left(\frac{n\pi}{a}\right)^2-p_x}{D_2}\right]\left(\frac{n\pi}{a}\right)^2\right\}}},$$

$$\beta=\sqrt{\frac{2D_3\left(\frac{n\pi}{a}\right)^2-p_y}{2D_2}-\sqrt{\left\{\left[\frac{2D_3\left(\frac{n\pi}{a}\right)^2-p_y}{2D_2}\right]^2-\left[\frac{D_1\left(\frac{n\pi}{a}\right)^2-p_x}{D_2}\right]\left(\frac{n\pi}{a}\right)^2\right\}}}. \tag{6}$$

Then we rewrite the deflection for half-wave number *n* as

$$w_n\left(x,y\right)=(a_1e^{\alpha y}+a_2e^{\beta y}+d_1e^{-\alpha y}+d_2e^{-\beta y})w_n\left(x\right), \tag{7}$$

where $w_n\left(x\right)=\sin\frac{n\pi x}{a}$, and a_i and d_i $\left(i=1,2\right)$ are undetermined coefficients.

III. REVERBERATION RAY MATRIX FOR A MULTI-SPAN ORTHOTROPIC PLATE

Let us consider adjacent spans, e.g., the *j*th, *k*th, *l*th, … spans, as shown in Fig. 2. The *x*-direction of coordinates is assumed to be the same, and *y*- direction is opposite to each other for a set of dual local coordinates. The *z*-direction still follows the right-hand rule. Every coordinate is labeled by two superscripts. The first one is the support where the origin of the coordinates locates, and the second one is the other support of the same span.

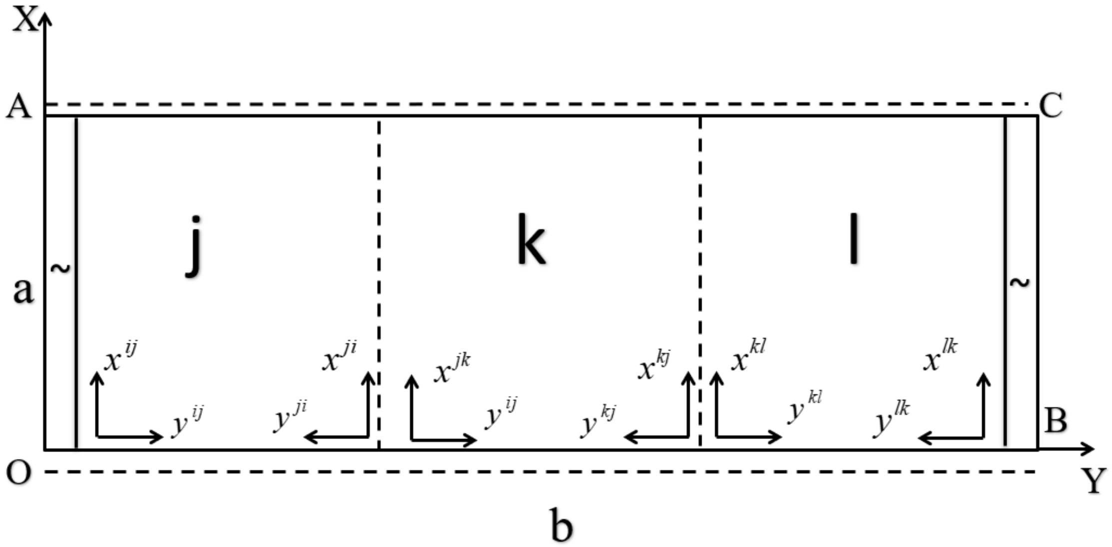

Fig. 2 Dual local coordinates in adjacent spans

For the *j*th span, in the dual local coordinates $\left(x^{ij},y^{ij},z^{ij}\right)$ and $\left(x^{ji},y^{ji},z^{ji}\right)$, the deflection can be expressed as

$$w_n^{ij}=(a_1^{ij}e^{\alpha y^{ij}}+a_2^{ij}e^{\beta y^{ij}}+d_1^{ij}e^{-\alpha y^{ij}}+d_2^{ij}e^{-\beta y^{ij}})w_n\left(x\right), \tag{8}$$

$$w_n^{ji}=(a_1^{ji}e^{\alpha y^{ji}}+a_2^{ji}e^{\beta y^{ji}}+d_1^{ji}e^{-\alpha y^{ji}}+d_2^{ji}e^{-\beta y^{ji}})w_n\left(x\right). \tag{9}$$

where $w_n\left(x\right)=\sin n\pi x/a.$ For the *k*th span, in the dual local coordinates $\left(x^{jk},y^{jk},z^{jk}\right)$ and

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$\left(x^{kj}, y^{kj}, z^{kj}\right)$, the corresponding transverse displacements, w_n^{jk}, w_n^{kj}, can be also written by replacing superscripts *ij* and *ji* by *jk* and *kj* in Eqs.(8) and (9) respectively.

The continuity conditions for the support connecting two adjacent *i*th and *j*th plates are

$$\begin{cases} \left(w_n^{ji}\right)_{y^{ji}=0} = \left(w_n^{jk}\right)_{y^{jk}=0} = 0 \\ \left(\varphi_y^{ji}\right)_{y^{ji}=0} = \left(\varphi_y^{jk}\right)_{y^{jk}=0} \\ \left(M_y^{ji}\right)_{y^{ji}=0} + \left(M_y^{jk}\right)_{y^{jk}=0} = 0 \end{cases} \tag{10}$$

in which $\varphi_{ny} = \dfrac{\partial w_n}{\partial y}$. Eq.(10) can be expressed in the form of matrix

$$\mathbf{D^J d^J} = \mathbf{A^J a^J}, \tag{11}$$

where $\mathbf{d^J} = \left[d_1^{ji}, d_2^{ji}, d_1^{jk}, d_2^{jk}\right]^{\mathrm{T}}$, $\mathbf{a^J} = \left[a_1^{ji}, a_2^{ji}, a_1^{jk}, a_2^{jk}\right]^{\mathrm{T}}$, and

$$\mathbf{D^J} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ \alpha & \beta & -\alpha & -\beta \\ -\alpha^2 & -\beta^2 & -\alpha^2 & -\beta^2 \end{bmatrix}, \mathbf{A^J} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \alpha & \beta & -\alpha & -\beta \\ \alpha^2 & \beta^2 & \alpha^2 & \beta^2 \end{bmatrix}.$$

Eq.(11) can be rewritten by

$$\mathbf{d^J} = \mathbf{S^J_{4\times4} a^J}, \tag{12}$$

where $\mathbf{S^J} = \left(\mathbf{D^J}\right)^{-1} \mathbf{A^J}$ is the local scattering matrix.

The boundary conditions for the left and right edges are

$$\mathbf{D^0 d^0} = \mathbf{A^0 a^0},\ \mathbf{D^n d^n} = \mathbf{A^n a^n} \tag{13}$$

where $\mathbf{D^0}$, $\mathbf{D^n}, \mathbf{A^0}, \mathbf{A^n}$ can be determined by boundary conditions for two opposite edges OA and BC. Eq.(13) can be written in a similar form of Eq.(12)

$$\mathbf{d^0} = \mathbf{S^0_{2\times2} a^0}, \tag{14}$$

$$\mathbf{d}^n = \mathbf{S}^n_{2\times2} \mathbf{a}^n. \tag{15}$$

Global scattering matrix can be assembled by all local scattering matrices

$$\mathbf{d} = \mathbf{S}\mathbf{a}, \tag{16}$$

where $\mathbf{d} = \left[d_1^{01}, d_2^{01}, \ldots, d_1^{n(n-1)}, d_2^{n(n-1)}\right]^{\mathrm{T}}$ and $\mathbf{a} = \left[a_1^{01}, a_2^{01}, \ldots, a_1^{n(n-1)}, a_2^{n(n-1)}\right]^{\mathrm{T}}$ are unknown global vectors, and $\mathbf{S}_{4n\times4n} = \begin{bmatrix} \mathbf{S^1} & & & & \\ & \ddots & & & \\ & & \mathbf{S^J} & & \\ & & & \ddots & \\ & & & & \mathbf{S^P} \end{bmatrix}$ is global scattering matrix.

For the *j*th span, the transverse displacements in two local coordinates satisfy

$$\begin{gathered} x^{ij} = x^{ji}, y^{ij} = l^{ij} - y^{ji}, z^{ij} = -z^{ji} \\ w_n^{ij}(y^{ij}) = -w_n^{ji}(l_j - y^{ji}) \end{gathered} \tag{17}$$

which is expressed in the form of matrices below

$$\begin{Bmatrix} a_1^{ij} \\ a_2^{ij} \end{Bmatrix} = \mathbf{P}^{ij} \begin{Bmatrix} d_1^{ji} \\ d_2^{ji} \end{Bmatrix}, \begin{Bmatrix} a_1^{ji} \\ a_2^{ji} \end{Bmatrix} = \mathbf{P}^{ij} \begin{Bmatrix} d_1^{ij} \\ d_2^{ij} \end{Bmatrix}, \tag{18}$$

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where $\mathbf{P}^{ij} = \begin{bmatrix} -e^{-\alpha l^{ij}} & \\ & -e^{-\beta l^{ij}} \end{bmatrix}$ is the local phase matrix. Eq.(18) can be rewritten as

$$\begin{Bmatrix} a_1^{ij} \\ a_2^{ij} \\ a_1^{ji} \\ a_2^{ji} \end{Bmatrix} = \mathbf{P^j U^j} \begin{Bmatrix} d_1^{ij} \\ d_2^{ij} \\ d_1^{ji} \\ d_2^{ji} \end{Bmatrix}, \tag{19}$$

where $\mathbf{P^j} = \begin{bmatrix} \mathbf{P}^{ij} & \\ & \mathbf{P}^{ij} \end{bmatrix}$, $\mathbf{U^j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ is the local permutation matrix. Hence, the relation between **a** and **d** for a multi-span plate can be constructed as

$$\mathbf{a = PUd}, \tag{20}$$

in which $\mathbf{P}_{4n\times 4n} = \begin{bmatrix} \mathbf{P}^1 & & & & \\ & \text{O} & & & \\ & & \mathbf{P^j} & & \\ & & & \text{O} & \\ & & & & \mathbf{P}^n \end{bmatrix}$ and $\mathbf{U}_{4n\times 4n} = \begin{bmatrix} \mathbf{U^1} & & & & \\ & \text{O} & & & \\ & & \mathbf{U^j} & & \\ & & & \text{O} & \\ & & & & \mathbf{U}^n \end{bmatrix}$.

Using Eq.(16) and Eq.(20), one gets the equation for $\mathbf{d}$ in the following

$$\left(\mathbf{I}_{4n\times 4n} - \mathbf{R}\right)\mathbf{d} = \mathbf{0}, \tag{21}$$

where $\mathbf{R = SPU}$ is the reverberation ray matrix.

The buckling load p_x (p_y) can be determined by putting the determinate of coefficients for $\mathbf{d}$ in the above equation to be zero. In view of Eq.(21), the dimension of reverberation ray matrix, $\mathbf{R}$, increases with the number of spans. Based on Eq.(12), the direct relation between local coordinates $\left(x^{ij}, y^{ij}, z^{ij}\right)$ and $\left(x^{kj}, y^{kj}, z^{kj}\right)$ can be derived, which contains the phase relations in *j*th and *k*th spans. The following relation can be finally obtained [14]

$$\left(\mathbf{I}_{4\times 4} - \bar{\mathbf{R}}\right)\bar{\mathbf{d}} = \mathbf{0}, \tag{22}$$

where $\bar{\mathbf{R}} = \bar{\mathbf{S}}\bar{\mathbf{P}}$ is a reduced reverberation ray matrix, which includes the same information as the above conventional reverberation ray matrix, $\mathbf{R}$. However, the matrix, $\bar{\mathbf{R}}$, is always a matrix of 4×4 and independent of the number of the spans of a plate, and its use will enhance efficiency in computation for muti-span plate-type structures.

IV. NUMERICAL EXAMPLES

Let us consider a three-equal-span orthotropic plate with internal line support under the action of a uniaxial pressure p_x or p_y, as shown in Fig.3. Four edges are assumed to be all simply-supported. The geometrical parameters of the plate are $h = 0.5\text{cm}$, $a = 20\text{cm}$ and $b = 3l = 60\text{cm}$. The boron fiber/epoxy composite material is chosen as its material, and some used material constants are $E_1 = 211\text{GPa}$, $E_2 = 21.1\text{GPa}$ $G_{12} = 7.0\text{GPa}$ and $\mu_1 = 0.3$[15].

For the simply-supported plate subjected to a uniform uniaxial pressure, the determinant of the difference between the unit matrix and reverberation ray matrix in Eq.(31) can be written by

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$$\left|\mathbf{I}_{4\times4} - \bar{\mathbf{R}}\right| = \frac{\Omega_1}{\Omega_2}, \tag{23}$$

where

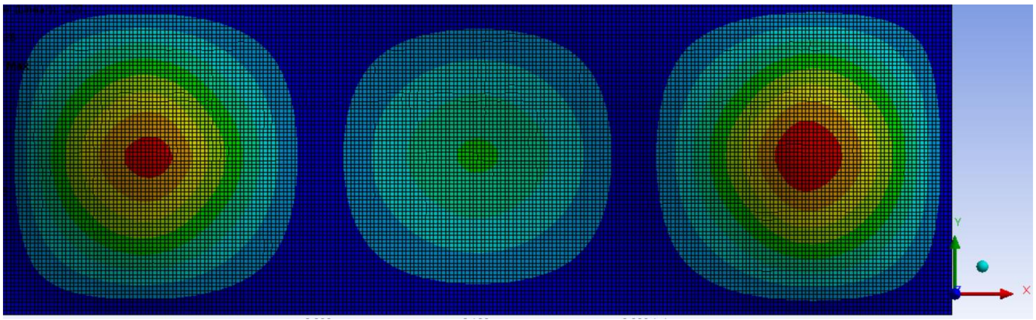
$$\begin{aligned}
\Omega_1 = {} & (\alpha^2 - 2\alpha\beta)e^{2l(\alpha+\beta)} + (-3\alpha^2 + 2\alpha\beta)(e^{2l(3\alpha+2\beta)} + e^{2\beta l}) \\
& + (-3\beta^2 + 2\alpha\beta)(e^{2l(2\alpha+3\beta)} + e^{2\alpha l}) + \beta^2 e^{6l(\alpha+\beta)} - (\alpha^2 + 2\alpha\beta + \beta^2)(e^{6\alpha l} + e^{6\beta l}) \\
& + (3\alpha^2 + 2\alpha\beta)(e^{4\beta l} + e^{2l(3\alpha+\beta)}) + (3\beta^2 + 2\alpha\beta)(e^{4\alpha l} + e^{2l(\alpha+3\beta)}) \\
& - 4\alpha\beta(e^{l(3\alpha+5\beta)} + e^{l(5\alpha+3\beta)} + e^{l(\alpha+3\beta)} + e^{l(3\alpha+\beta)}) \\
& + 2\alpha\beta(e^{l(\alpha+\beta)} - e^{4l(\alpha+\beta)} + e^{5l(\alpha+\beta)} - e^{6l(\alpha+\beta)} - e^{2l(\alpha+2\beta)} - e^{2l(2\alpha+\beta)} + e^{l(\alpha+5\beta)} \\
& + e^{l(5\alpha+\beta)}) + 8\alpha\beta e^{3l(\alpha+\beta)} + \alpha^2 - 2\alpha\beta + \beta^2, \\
\Omega_2 = {} & (\alpha^2 + \beta^2)e^{2l(\alpha+\beta)} - \alpha^2 e^{2\alpha l} - \beta^2 e^{2\beta l} + 2\alpha\beta e^{l(\alpha+\beta)} - 2\alpha\beta e^{2l(\alpha+\beta)}.
\end{aligned}$$

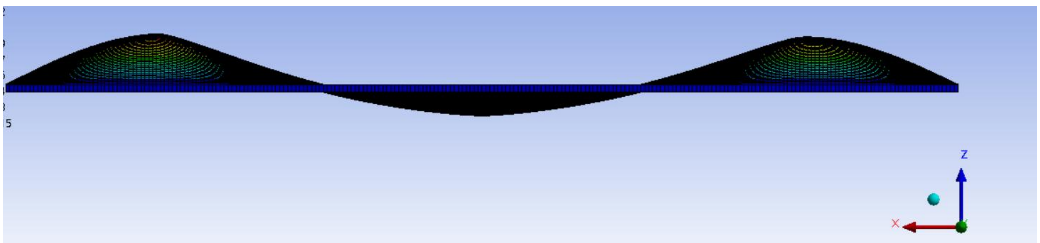
The results of the buckling loadings by MRRM and finite element method (FEM) by using ANSYS code are listed in the Table 1. In finite element simulation, a shell63 element is used. The total numbers of nodes and elements are 19521 and 19200, respectively. The results show that the present MRRM has enough accurateness in computation.

Table 1 Comparison of buckling loading

	$p_x(N/m)$	$p_y(N/m)$
MRRM	547240.0	1198177.0
FEM	544120.0	1227100.0

The first-order mode of buckling of the plate under the pressure along the long side for FEM is displayed in Fig. 3. A comparison for the mode of buckling between MRRM and FEM is shown in Fig. 4. In contrast to the mode in FEM model (Fig. 4b), when the plate is constrained along its long side, the deflection has a change in the form of the trigonometric function along the direction of the short side of the plate, and a half-cycle curve can be derived, and it agrees with the corresponding result for MRRM, as shown in Fig. 4a.





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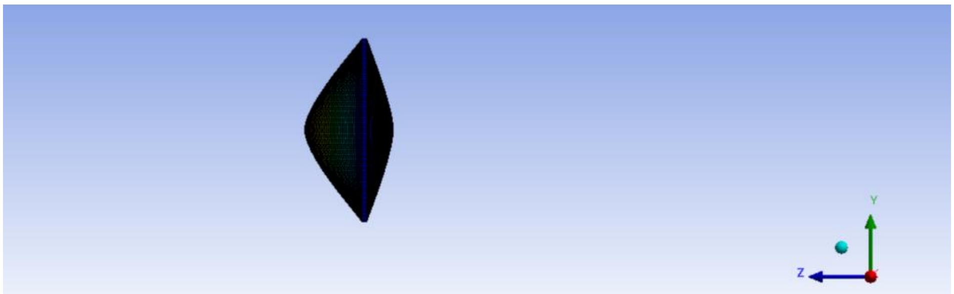
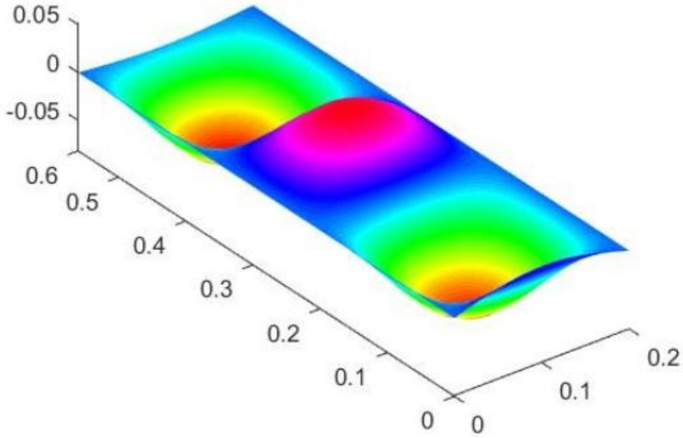
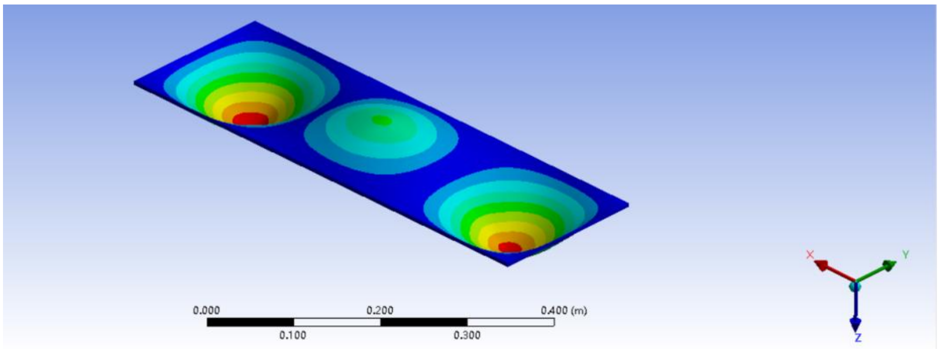


Fig. 3 The first-order mode of buckling for FEM



(a)



(b)

Fig. 4 Comparison of mode of buckling (a) MRRM; and (b) FEM

V. CONCLUSIONS

In this paper a method of reverberation ray matrix (MRRM) is formulated for buckling of thin multi-span rectangular orthotropic plates. A corresponding reverberation ray matrix is derived based on the modal solution for an orthotropic plate together with the continuity conditions among the adjacent span plates and boundary conditions for the entire plate structure. The equation for buckling loading is analytically given for the case of a uniaxial pressure along a side of the multi-span plate. Numerical examples are given to show the validity and exactness in computation for MRRM. The present formulation of solution is helpful to effective evaluation on the buckling behavior of multi-span rectangular anisotropic plates.

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