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Research Paper

Fractional Integrals of Power of Fractional Sine and Cosine Function

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ABSTRACT: *This article mainly studies the fractional integrals of power of fractional sine and cosine function based on the Jumarie type of modified Riemann-Liouville fractional derivatives. We make use of binomial theorem of fractional analytic functions, fractional DeMoivre's formula, and several properties to obtain the answers of these two types of fractional integrals.*

KEYWORDS: *Fractional Integrals, Fractional Sine Function, Fractional Cosine Function, Jumarie Type of Modified Riemann-Liouville Fractional Derivatives, Binomial Theorem, Fractional DeMoivre's Formula*

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I. INTRODUCTION

The mathematical thought of fractional calculus was developed by mathematicians Leibniz (1695), Liouville (1834), Riemann (1892) and others. Fractional calculus attracted the attention of many scientists and engineers, and has been widely used in many fields such as chemistry, physics, engineering, applied mathematics, biology, and economics [1-18].

In this paper, we mainly study the following two types of fractional integral problems:

$$\left({}_0I_x^\alpha\right)\left[[cos_\alpha(x^\alpha)]^{\otimes n}\right], \quad (1)$$

$$\left({}_0I_x^\alpha\right)\left[[sin_\alpha(x^\alpha)]^{\otimes n}\right], \quad (2)$$

where $0 < \alpha \le 1$, n is a positive integer, cos_α, sin_α are α-fractional cosine function and sine function respectively. Using binomial theorem of fractional analytic functions, fractional DeMoivre's formula, and some basic properties of fractional cosine function and sine function, we can easily evaluate these two types of fractional integrals.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1: Let α be a real number and p be a positive integer, then the modified Riemann-Liouville fractional derivatives of Jumarie type ([19]) is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}(x-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}(x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_{x_0}D_x^{\alpha-m}\right)[f(x)], & \text{if } p \le \alpha < p+1 \end{cases} \quad (3)$$

where $\Gamma(w) = \int_0^\infty t^{w-1}e^{-t}dt$ is the gamma function defined on $w > 0$. Moreover, we define the α-fractional integral of $f(x)$ by $\left({}_aI_x^\alpha\right)[f(x)] = \left({}_aD_x^{-\alpha}\right)[f(x)]$, where $\alpha > 0$. If $\left({}_aI_x^\alpha\right)[f(x)]$ exists, then $f(x)$ is called an α-fractional integrable function.

Definition 2.2 ([20])**:** Suppose that x, x_0 and a_n are real numbers, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as a α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval $(x_0 - r, x_0 + r)$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0, where r is the radius of convergence about x_0. If $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and is α-fractional analytic at every point in open interval (a,b), then we say that f_α is an α-fractional analytic function on $[a,b]$.

Definition 2.3 ([21])**:** Let $0 < \alpha \le 1$ and x be a real variable. Then $E_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}$ is called the α-fractional exponential function, and the period of $E_\alpha(ix^\alpha)$ is denoted as T_α. On the other hand, the

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α-fractional cosine and sine function are defined by

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{4}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \tag{5}$$

Proposition 2.4 (fractional Euler's formula) ([21])**:** *Let* $0 < \alpha \leq 1$, *then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha). \tag{6}$$

Proposition 2.5 (fractional DeMoivre's formula) ([21])**:** *If* $0 < \alpha \leq 1$, *and* n *is a positive integer, then*

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes n} = cos_\alpha(nx^\alpha) + isin_\alpha(nx^\alpha). \tag{7}$$

Proposition 2.6: *Suppose that* α, β, c *are real numbers,* $0 < \alpha \leq 1$, *and* $\beta \geq \alpha > 0$, *then*

$$\left({}_0D_x^\alpha\right)[x^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{8}$$

$${}_0D_x^\alpha[c] = 0, \tag{9}$$

$$\left({}_0D_x^\alpha\right)[sin_\alpha(x^\alpha)] = cos_\alpha(x^\alpha), \tag{10}$$

$$\left({}_0D_x^\alpha\right)[cos_\alpha(x^\alpha)] = -sin_\alpha(x^\alpha), \tag{11}$$

Proposition 2.7: *If* $0 < \alpha \leq 1$, *then the* α*-fractional integrals*

$$\left({}_aI_x^\alpha\right)[cos_\alpha(x^\alpha)] = sin_\alpha(x^\alpha), \tag{12}$$

$$\left({}_aI_x^\alpha\right)[sin_\alpha(x^\alpha)] = -cos_\alpha(x^\alpha). \tag{13}$$

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.8 ([22])**:** Let $0 < \alpha \leq 1$, $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha}, \tag{14}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha}. \tag{15}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) x^{k\alpha}. \end{aligned} \tag{16}$$

Definition 2.9: $[f_\alpha(x^\alpha)]^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the n-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$.

Theorem 2.10 (binomial theorem of fractional analytic functions)**:** *Assume that* $0 < \alpha \leq 1, n$ *is a positive integer, and* $f_\alpha(x^\alpha)$ *and* $g_\alpha(x^\alpha)$ *are two* α*-fractional analytic functions, then*

$$[f_\alpha(x^\alpha) + g_\alpha(x^\alpha)]^{\otimes n} = \sum_{m=0}^{n} \binom{n}{m} [f_\alpha(x^\alpha)]^{\otimes(n-m)} \otimes [g_\alpha(x^\alpha)]^{\otimes m}. \tag{17}$$

III. MAIN RESULTS AND EXAMPLES

In the following, we will obtain the fractional integrals of power of fractional sine and cosine function.

Theorem 3.1: *If* $0 < \alpha \leq 1$, *and* n *is a positive integer. Then*

$$\begin{aligned} & \left({}_0I_x^\alpha\right)[[cos_\alpha(x^\alpha)]^{\otimes n}] \\ &= \begin{cases} \frac{1}{2^{n-1}} \sum_{m=0}^{\frac{n-1}{2}} \frac{\binom{n}{m}}{n-2m} sin_\alpha((n-2m)x^\alpha) & if\ n\ is\ odd \\ \frac{1}{2^{n-1}} \sum_{m=0}^{\frac{n-2}{2}} \frac{\binom{n}{m}}{n-2m} sin_\alpha((n-2m)x^\alpha) + \frac{\binom{n}{n/2}}{2^n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha & if\ n\ is\ even \end{cases} \end{aligned} \tag{18}$$

Proof

$$\begin{aligned} & \left({}_0I_x^\alpha\right)[[cos_\alpha(x^\alpha)]^{\otimes n}] \\ &= \left({}_0I_x^\alpha\right)\left[\left[\frac{1}{2}\left(E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)\right)\right]^{\otimes n}\right] \\ &= \frac{1}{2^n}\left({}_0I_x^\alpha\right)\left[\left[\left(E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)\right)\right]^{\otimes n}\right] \\ &= \frac{1}{2^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n} \binom{n}{m} [E_\alpha(ix^\alpha)]^{\otimes(n-m)} \otimes [E_\alpha(-ix^\alpha)]^{\otimes m}\right] \\ &= \frac{1}{2^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n} \binom{n}{m} E_\alpha(i(n-m)x^\alpha) \otimes E_\alpha(-imx^\alpha)\right] \\ &= \frac{1}{2^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n} \binom{n}{m} E_\alpha(i(n-2m)x^\alpha)\right] \\ &= \frac{1}{2^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n} \binom{n}{m} cos_\alpha((n-2m)x^\alpha)\right] \\ &= \frac{1}{2^n} \sum_{m=0}^{n} \binom{n}{m} \left({}_0I_x^\alpha\right)[cos_\alpha((n-2m)x^\alpha)] \end{aligned}$$

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$$=\begin{cases}\frac{1}{2^{n-1}}\sum_{m=0}^{\frac{n-1}{2}}\frac{\binom{n}{m}}{n-2m}sin_\alpha((n-2m)x^\alpha) & if\ n\ is\ odd\\ \frac{1}{2^{n-1}}\sum_{m=0}^{\frac{n-2}{2}}\frac{\binom{n}{m}}{n-2m}sin_\alpha((n-2m)x^\alpha)+\frac{\binom{n}{n/2}}{2^n}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha & if\ n\ is\ even\end{cases}$$

Q.e.d.

Theorem 3.2: *Suppose that* $0<\alpha\leq 1$, *and* n *is a positive integer. Then*

$$\left({}_0I_x^\alpha\right)[[sin_\alpha(x^\alpha)]^{\otimes n}]$$
$$=\begin{cases}\frac{(-1)^{\frac{n+1}{2}}}{2^{n-1}}\sum_{m=0}^{\frac{n-1}{2}}\frac{(-1)^m\binom{n}{m}}{n-2m}cos_\alpha((n-2m)x^\alpha) & if\ n\ is\ odd\\ \frac{(-1)^{\frac{n}{2}}}{2^{n-1}}\sum_{m=0}^{\frac{n-2}{2}}\frac{(-1)^m\binom{n}{m}}{n-2m}sin_\alpha((n-2m)x^\alpha)+\frac{\binom{n}{n/2}}{2^n}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha & if\ n\ is\ even\end{cases}. \quad (19)$$

Proof $\left({}_0I_x^\alpha\right)[[cos_\alpha(x^\alpha)]^{\otimes n}]$

$$=\left({}_0I_x^\alpha\right)\left[\left[\frac{1}{2i}\left(E_\alpha(ix^\alpha)-E_\alpha(-ix^\alpha)\right)\right]^{\otimes n}\right]$$
$$=\frac{1}{(2i)^n}\left({}_0I_x^\alpha\right)\left[\left[\left(E_\alpha(ix^\alpha)-E_\alpha(-ix^\alpha)\right)\right]^{\otimes n}\right]$$
$$=\frac{i^n}{(-2)^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n}\binom{n}{m}[E_\alpha(ix^\alpha)]^{\otimes(n-m)}\otimes[-E_\alpha(-ix^\alpha)]^{\otimes m}\right]$$
$$=\frac{i^n}{(-2)^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n}(-1)^m\binom{n}{m}E_\alpha(i(n-m)x^\alpha)\otimes E_\alpha(-imx^\alpha)\right]$$
$$=\frac{i^n}{(-2)^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n}(-1)^m\binom{n}{m}E_\alpha(i(n-2m)x^\alpha)\right]$$
$$=\frac{i^n}{(-2)^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n}(-1)^m\binom{n}{m}cos_\alpha((n-2m)x^\alpha)+i\sum_{m=0}^{n}(-1)^m\binom{n}{m}sin_\alpha((n-2m)x^\alpha)\right]$$
$$=\begin{cases}\frac{i^{n+1}}{-2^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n}(-1)^m\binom{n}{m}sin_\alpha((n-2m)x^\alpha)\right] & if\ n\ is\ odd\\ \frac{i^n}{2^n}\left({}_0I_x^\alpha\right)\left[\sum_{m=0}^{n}(-1)^m\binom{n}{m}cos_\alpha((n-2m)x^\alpha)\right] & if\ n\ is\ even\end{cases}$$
$$=\begin{cases}\frac{(-1)^{\frac{n+1}{2}}}{-2^n}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\left({}_0I_x^\alpha\right)[sin_\alpha((n-2m)x^\alpha)] & if\ n\ is\ odd\\ \frac{(-1)^{\frac{n}{2}}}{2^n}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\left({}_0I_x^\alpha\right)[cos_\alpha((n-2m)x^\alpha)] & if\ n\ is\ even\end{cases}$$
$$=\begin{cases}\frac{(-1)^{\frac{n+1}{2}}}{2^{n-1}}\sum_{m=0}^{\frac{n-1}{2}}\frac{(-1)^m\binom{n}{m}}{n-2m}cos_\alpha((n-2m)x^\alpha) & if\ n\ is\ odd\\ \frac{(-1)^{\frac{n}{2}}}{2^{n-1}}\sum_{m=0}^{\frac{n-2}{2}}\frac{(-1)^m\binom{n}{m}}{n-2m}sin_\alpha((n-2m)x^\alpha)+\frac{\binom{n}{n/2}}{2^n}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha & if\ n\ is\ even\end{cases}.$$

Q.e.d.

Example 3.3: Suppose that $0<\alpha\leq 1$. By using Theorem 3.1, we obtain the α-fractional integrals

$$\left({}_0I_x^\alpha\right)[[cos_\alpha(x^\alpha)]^{\otimes 3}]=\frac{1}{12}sin_\alpha(3x^\alpha)+\frac{3}{4}sin_\alpha(x^\alpha). \quad (20)$$

And

$$\left({}_0I_x^\alpha\right)[[cos_\alpha(x^\alpha)]^{\otimes 4}]=\frac{1}{32}sin_\alpha(4x^\alpha)+\frac{1}{4}sin_\alpha(2x^\alpha)+\frac{3}{8}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (21)$$

Example 3.4: Let $0<\alpha\leq 1$. Using Theorem 3.2 yields the α-fractional integrals

$$\left({}_0I_x^\alpha\right)[[sin_\alpha(x^\alpha)]^{\otimes 3}]=\frac{1}{12}cos_\alpha(3x^\alpha)-\frac{3}{4}cos_\alpha(x^\alpha). \quad (22)$$

And

$$\left({}_0I_x^\alpha\right)[[sin_\alpha(x^\alpha)]^{\otimes 4}]=\frac{1}{32}sin_\alpha(4x^\alpha)-\frac{1}{4}sin_\alpha(2x^\alpha)+\frac{3}{8}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (23)$$

IV. CONCLUSION

The main purpose of this paper is to find two fractional integrals of power of fractional sine and cosine function based on Jumarie type of Riemann-Liouville fractional derivatives. We take advantage of binomial theorem of fractional analytic functions, fractional DeMoivre's formula, and some properties of fractional cosine function and sine function to obtain the answers of these two types of fractional integrals. In the future, we will use some techniques to study the problems of fractional calculus and fractional differential equations.

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