Quest Journals
Journal of Research in Applied Mathematics
Volume 10 ~ Issue 1 (2024) pp: 24-28
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Analytical Solutions of the D-dimensional Klein-Gordon equation with Schiöberg potential by Greene-Aldrich approximation

Bijon Biswas
Department of Mathematics, Vivekananda College, Kolkata -700063, India

Abstract:
In this article, the D-dimensional Klein-Gordon equation within the framework of Greene-Aldrich approximations scheme for Schiöberg potential is solved for s-wave and arbitrary angular momenta. The energy eigenvalues and corresponding wave functions are obtained in an exact analytical manner via the Nikiforov-Uvarov (N-U) method.
Keywords: *Schiöberg potential, Greene-Aldrich approximation, Nikiforov-Uvarov(N-U) method.*
PACS codes: 03.65.Ge, 03.65.Pm, 03.65-w

***R**eceived 17 Jan., 2024; **R**evised 28 Jan., 2024; **A**ccepted 31 Jan., 2024*

I. Introduction :

The Klein-Gordon equation plays an important role in describing the behavior of relativistic spinless particles [1, 2]. The problem of finding analytical solutions of D-dimensional Klein-Gordon equation for a number of special potentials has been studied by numerous Scientists [3, 4, 5, 6, 7, 8, 9, 10]. The solutions are also crucial in quantum soluble systems. Methods involve in literature are Nikiforov-Uvarov method [11, 12, 13], asymptotic iteration method [14], Point-Cannonical transformation [15], Lie algebraic method [16], Laplace transform approach [17, 18], Factorization method [19] etc.

In this article, the approximate solutions of Klein-Gordon equation in D-dimensions is obtained for Schiöberg potential. The Schiöberg potential [20, 21], is an intermolecular potential and widely applied to molecular physics and quantum chemistry. The potential is given by:

$$V(r) = D[1 - \sigma coth(\alpha r)]^2 \qquad (1)$$

Where, D, α and σ are the adjustable positive parameters. Bearing in mind the deeper physical insight that analytical methodologies provide into the physics of problem, the most economic and powerful Nikiforov-Uvarov (N-U) method is applied in my calculations on the D-dimensions.

To investigate the behaviour of Schiöberg potential within the frame work of Klein-Gordon equation I use Greene-Aldrich approximation [22] and applying some simple constraints such that the equation can be solved by N-U method.

My work is organized as follows: - To make it self-contained a brief review of N-U method is given in section II. In section III, the D-dimensional Klein-Gordon equation is presented considering the Schiöberg potential as well as Greene-Aldrich approximation. In section IV, the energy eigenvalues and corresponding wave functions are obtained for the D-dimensional Klein-Gordon equation by using N-U method. Section V contains the concluding remark.

II. Nikiforov-Uvarov Method:

The N-U method is based on solving a second order linear differential equation by reducing it to a generalized hypergeometric type. In both relativistic and nonrelativistic quantum mechanics, the wave equation with a given potential can be solved by this method by reducing the one-dimensional K-G equation to an equation of the form:

$$\boldsymbol{\Psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\Psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\Psi(s) = 0} \qquad (2)$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of degree at most 2 and $\tilde{\tau}(s)$ is a polynomial of degree atmost 1. In order to find a particular solution to equation (2), we set the following wave function as a multiple of two independent parts

$$\Psi(s) = \Phi(s)y(s) \qquad (3)$$

Thus equation (2) reduces to a hyper-geometric type equation of the form:

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0$$

Where $\tau(s) = \tilde{\tau}(s) + 2\pi(s)$ satisfies the condition $\tau'(s) < 0$ and $\pi(s)$ is defined as

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + K\sigma(s)} \qquad (4)$$

in which K is a parameter. Determining K is the essential point in calculation of π(s). Since π(s) has to be a polynomial of degree at most one, the expression under the square root sign in Eq. (4) can be put into order to be the square of a polynomial of first degree [18], which is possible only if its discriminant is zero. So, we obtain K by setting the discriminant of the square root equal to zero. Therefore, one gets a general quadratic equation for K. By using

$$\lambda = K + \pi'(s) = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \qquad (5)$$

The values of K can used for the calculation of energy eigenvalues. Polynomial solutions $y_n(s)$ are given by the Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\left(\frac{d}{ds}\right)^n [\sigma^n(s)\rho(s)] \qquad (6)$$

in which Bn is a normalization constant and ρ(s) is the weight function satisfying

$$\rho(s) = \frac{1}{\sigma(s)} exp \int \frac{\tau(s)}{\sigma(s)} ds \qquad (7)$$

on the other hand, second part of the wave function $\Phi(s)$ in relation (3) is given by

$$\Phi(s) = exp \int \frac{\pi(s)}{\sigma(s)} ds \qquad (8)$$

III. The Klein-Gordon equation in D-dimensions:

The time independent D-dimensional Klein-Gordon equation in the atomic units $\hbar$ = c = μ = 1, may be written as [23],

$$\nabla_D^2\Psi(r,\Omega_D) + \left[\left(E - V(r)\right) - \left(M + S(r)\right)\right]\Psi(r,\Omega_D) = 0 \qquad (9)$$

where M denotes the particle mass, E is the energy, V(r) and S(r) are vector and scalar potentials respectively. The D-dimensional Laplacian operator ∇_D^2 is given by [24],

$$\nabla_D^2 = r^{1-D}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) + \frac{L_D^2(\Omega_D)}{r^2} \qquad (10)$$

Where, $L_2^D(\Omega_D)$ is the ground angular momentum [25]. In addition, we know that $\frac{L_D^2(\Omega_D)}{r^2}$ is a generalization of the centrifugal barrier for the D-dimensional space and involves angular coordinates Ω_D and the eigenvalues of $L_2^D(\Omega_D)$ [24]. $L_2^D(\Omega_D)$ is a partial differential operator on the unit space S^{D-1} define analogously to a three-dimensional angular momentum [25] as $L_D^2(\Omega_D) = -\sum_{i\geq j}^{D}\left(L_{ij}^2\right)$, where $L_{ij}^2 = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}$ for all Cartesian component xi of the D-dimensional vector $(x_1, x_2, \ldots\ldots, x_D)$.

To eliminate the first order derivative, the total wave function may be defined as

$$\Psi(r,\Omega_D) = r^{\frac{(D+1)}{2}} R_{nl}(r)Y_{lm}(\Omega_D) \qquad (11)$$

Where, $Y_{lm}(\Omega_D)$is the generalized spherical harmonic function. The eigenvalues equation for the generalized angular momentum operator is given by $L_2^D(\Omega_D) = l(l + D - 2)Y_{lm}(\Omega_D)$. With this, we can write the radial part of the D-dimensional Klein-Gordon equation as follows:

$$\frac{d^2R_{nl}(r)}{dr^2} + \left[(E^2 - M^2) - 2\left(MS(r) + EV(r)\right) + V^2(r) - S^2(r) - \frac{(2l + D - 1)(2l + D - 3)}{4r^2}\right]R_{nl}(r) = 0 \qquad (12)$$

Assuming V (r) = S(r), equation (12) becomes

$$\frac{d^2R_{nl}(r)}{dr^2} + \left[(E^2 - M^2) - 2V(r)(M + E) - \frac{(2l + D - 1)(2l + D - 3)}{4r^2}\right]R_{nl}(r) = 0 \qquad (13)$$

The solution for the above equation with $l \neq 0$ is mainly depending on replacing the orbital centrifugal term of singularity with the help of a suitable approximation scheme. The approximation scheme used in this article to deal with the centrifugal term is Greene-Aldrich approximation scheme given by:

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \qquad (14)$$

Inserting the potential function in exponential form (i.e. $V(r) = D[1-\sigma coth(\alpha r)]^2 = D\left[1-\sigma\frac{1+e^{-2\alpha r}}{1-e^{-2\alpha r}}\right]^2$ **and the modified centrifugal term as given Eqn. (1) and Eqn. (14) respectively in Eqn. (13), we have**

$$\frac{d^2R_{nl}(r)}{dr^2} + \left[(E^2-M^2) - 2D(M+E)\left[1-\sigma\frac{1+e^{-2\alpha r}}{1-e^{-2\alpha r}}\right]^2 - \frac{(2l+D-1)(2l+D-3)\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}\right]R_{nl}(r) = 0 \qquad (15)$$

IV. Solutions of the D-dimensional Klein-Gordon equation:

In order to solve Eqn. (15) by the N-U method, we need to recast it into a solvable form. To do so, I introduce a new variable $s = e^{-2\alpha r}$ and Eqn. (15) takes the form

$$\frac{d^2R(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dR(s)}{ds} - \frac{1}{s^2(1-s)^2}[((1+\sigma)^2\beta^2-\epsilon^2)s^2 + (2\epsilon^2+(1-\sigma^2)\beta^2-\gamma^2)s - (\epsilon^2-(1-\sigma)^2\beta^2)]R(s) = 0 \qquad (16)$$

Where I have used the notations

$$\epsilon^2 = \frac{E^2-M^2}{4\alpha^2}, \quad \beta^2 = \frac{2D(M+E)}{4\alpha^2}, \quad \gamma^2 = (2l+D-1)(2l+D-3)$$

Comparing Eqn.(16) with Eqn. (2),

$$\tilde{\tau}(s) = (1-s), \qquad \sigma(s) = s(1-s),$$

$$\tilde{\sigma}(s) = -[((1+\sigma)^2\beta^2-\epsilon^2)s^2 + (2\epsilon^2+(1-\sigma^2)\beta^2-\gamma^2)s - (\epsilon^2-(1-\sigma)^2\beta^2)] \qquad (17)$$

Substituting them into relation (4) leads to

$$\pi(s) = -\frac{s}{2} \pm \sqrt{\left(\frac{1}{4}-\epsilon^2+(1+\sigma)^2\beta^2-K\right)s^2 + (2\epsilon^2+2(1-\sigma^2)\beta^2-2\gamma^2+K)s + (-\epsilon^2+(1-\sigma)^2\beta^2)} \qquad (18)$$

Further, the discriminant of the upper expression under the square root has to be set equal to zero. So, one can easily obtain

$$\Delta = (2\epsilon^2+2(1-\sigma^2)\beta^2-2\gamma^2+K)^2 - 4(-\epsilon^2+(1-\sigma)^2\beta^2)\left(\frac{1}{4}-\epsilon^2+(1+\sigma)^2\beta^2-K\right) = 0 \qquad (19)$$

Solving Eqn.(19) for the constant K, the double roots are obtained as $K_{1,2} = -2(2(1-\sigma)\beta^2-\gamma^2) \pm 2ab$ **, where** $a = \sqrt{-\epsilon^2+(1-\sigma)^2\beta^2}$ **and** $b = \sqrt{\frac{1}{4}+2(1+\sigma^2)\beta^2+2\gamma^2}$ **.**

Thus, substituting these values for each K into equation (18), one can easily obtained:

$$\pi(s) = -\frac{s}{2} \pm \begin{cases} (b-a)s-a; & for\ K_1 = -2(2(1-\sigma)\beta^2-\gamma^2)+2ab \\ (b+a)s-a; & for\ K_2 = -2(2(1-\sigma)\beta^2-\gamma^2)-2ab \end{cases} \qquad (20)$$

By choosing an appropriate value for K in π(s) which satisfies the condition $\tau'(s) < 0$, one gets $\pi(s) = -\left(a+b+\frac{1}{2}\right)s + a \quad for \quad K_2 = -2(2(1-\sigma)\beta^2-\gamma^2)-2ab$ **; giving the function:**

$$\tau(s) = -2(a+b)s + 1 + 2a \qquad (21)$$

As per Eqn. (5), the constant λ is defined as

$$\lambda = -4(1-\sigma)\beta^2 - 2\gamma^2 - 2ab - (a+b) \qquad (22)$$

Also, by Eqn. (5):

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \qquad (23)$$

Here,

$$\tau'(s) = -2(a + b) \quad and \quad \sigma''(s) = -2 \tag{24}$$

Carrying out some simple algebraic calculation with the equations (22), (23) and (24), we have

$$a = \frac{1}{2}\left[\frac{(1-\sigma^2)\beta^2 + \left(n + \frac{1}{2} + b\right)^2}{n + \frac{1}{2} + b}\right] \tag{25}$$

Substituting the values of a and b in Eqn. (25) and simplifying, we have

$$\epsilon_n^2 = (1-\sigma)^2\beta^2 - \frac{1}{4}\left[\frac{(1-\sigma^2)\beta^2 + \left(n + \frac{1}{2} + b\right)^2}{n + \frac{1}{2} + b}\right]^2 \tag{26}$$

This constitutes the energy eigenvalue equation for Schiöberg potential and the approximate energy eigenvalue (by putting the values of notations ϵ, β, γ) is of the form:

$$E_{nl} \approx \frac{D(1-\sigma)^2}{2} - \frac{\alpha^2}{2M}\left[\frac{\frac{MD(1-\sigma^2)}{\alpha^2} + \chi^2}{\chi}\right]^2 \tag{27}$$

Where, $\chi = n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{MD(1+\sigma^2)}{\alpha^2} + (2l + D - 1)(2l + D - 3)}$

From (7) it can be shown that the weight function $\rho(s)$ is $\rho(s) = s^{2a}(1-s)^{2b}$ and by substituting $\rho(s)$ into the Rodrigues relation (6) one gets

$$y_n(s) = \frac{B_n}{s^{2a}(1-s)^{2b}}\left(\frac{d}{ds}\right)^n \left[s^n(1-s)^n s^{2a}(1-s)^{2b}\right] = \frac{B_n}{s^{2a}(1-s)^{2b}} P_n^{(2a,2b)}(1-2s) \tag{28}$$

Where $P_n^{(2a,2b)}(1-2s)$ stands for Jacobi polynomial [26, 27] and B_n is the normalizing constant. The other part of the wave function is simply found from (8) as,

$$\Phi(s) = s^a(1-s)^{\left(\frac{1}{2}+b\right)} \tag{29}$$

Finally, the wave function is obtained as follows

$$R(s) = B_n s^a (1-s)^{\left(\frac{1}{2}+b\right)} P_n^{(2a,2b)}(1-2s) \tag{30}$$

With the notations, $a = \sqrt{-\epsilon^2 + (1-\sigma)^2\beta^2}$ **and** $b = \sqrt{\frac{1}{4} + 2(1+\sigma^2)\beta^2 + 2\gamma^2}$ **.**

V. Conclusions :

In this article, the solutions of the D-dimensional Klein-Gordon equation with equal scalar and vector potentials for the **Schiöberg** potential using N-U method upon application of Greene-Aldrich approximation to the centrifugal term. The relativistic energy eigenvalues are obtained and the corresponding wave functions in terms of the Jacobi polynomials are presented.

References

[1]. W. Greiner, "Relativistic Quantum Mechanics", Springer, Berlin, 2000.

[2]. T. Ohlsson, "Relativistic Quantum Physics- From Advanced Quantum Mechanics to Introductory Quantum Field Theory" Cambridge University Press, Cambridge, 2011.

[3]. C.P. Onyenegecha, A.I. Opara, I.J. Njoku, S.C. Udensi, U.M. Ukewuihe, C.J. Okereke, A. Omame, "Analytical solutions of D-dimensional Klein-Gordon equation with modified Mobius squared potential", Results in Physics, vol.25, 2021, 104144.

[4]. A.N. Ikot, H. P. Obong, H. Hassanabadi, N. Salehi, O. S. Thomas, "Solutions of Ddimensional Klein-Gordon equation for multiparameter exponential-type potential using supersymmtric quantum mechanics", Ind J Phys., vol.89, 2015, p.649.

[5]. A.N. Ikot, H. Hassanabadi, E. Maghsoodi, S. Zarrinkamar, "D-Dimensional Dirac Equation for Energy-Dependent Pseudoharmonic and Mie-type Potentials via SUSYQM", Commun Theor Phys., vol.61, 2014, p.436.

[6]. M.E. Udoh, U.S. Okorie, M.I. Ngwueke, E.E. Ituen, A.N. Ikot, "Rotationvibrational energies for some diatomic molecules with improved Rosen-Morse potential in D-dimensions", J Mol Mod., vol.25, 2019, p.170.

[7]. C.A. Onate, A.N. Ikot AN, M.C. Onyeaju, M.E. Udoh, "Bound state solutions of D-dimensional Klein-Gordon equation with hyperbolic potential", Karbala Int J Mod Sci., vol.3, 2016, p.1.

[8]. J. Gao, M.C. Zhang, "Analytical Solutions to the D-Dimensional Schrodinger Equation with the Eckart Potential", Chin. Phys Lett., vol.33, 2016, p.010308.

[9]. V.H. Badalov, B. Baris, K. Uzun, "Bound states of the D-dimensional Schrodinger equation for the generalized Woods-Saxon potential", Mod Phys Lett., vol.34, 2019, p.1950107.

[10]. C.A. Onate, O. Ebomwonyi, K.O. Dopamu, J.O. Okoro, M.O. Oluwayemi, "Eigen solutions of the D-dimensional Schrödinger equation with inverse trigonometry scarf potential and Coulomb potential", Chin J Phys., vol.56, 2018, p.2538.

[11]. A.F. Nikiforov, V.B. Uvarov, "Special Functions of Mathematical Physics", Birkh¨auser,Basel, 1988.
[12]. A.N. Ikot, A.B. Udoimuk, L.E. Akpabio, "Bound states solution of Klein-Gordon Equation with type-I equal vector and Scalar Poschl-Teller potential for Arbitray I-State", American J. of Scientific and Industrial Research, vol.2.2, 2011, p.179.
[13]. L.H. Zhang, X.P. Li, C.S. Jia, "Analytical Approximation to the Solution of the Dirac Equation with the Eckart Potential Including the Spin-Orbit Coupling Term", Phys. Lett. A, vol.372, 2008, p.2201.
[14]. A.D. Alhaidari, "Nonrelativistic Green's Function for Systems with Position- Dependent Mass", Int. J. Theor. Phys., vol.42, 2003, p.2999.
[15]. S.H. Dong, "Wave equations in Higher Dimensions", Springer-Verlag, New York, 2011.
[16]. A.R. Plastino,A. Rigo, M. Casas, F. Garcias, A. PLastine, "Super symmertric approach to quantum systems with position dependent effective mass", Phys. Rev. A, vol.60 no.6, 1999, p.4318.
[17]. B. Biswas, S. Debnath, "Bound states of the Dirac-Kratzer-Fues potential with spin and pseudo-spin symmetry via Laplace transform approach", Bulg. J. Phys., vol.43, 2016, pp.89-99.
[18]. B. Biswas, S. Debnath, "Bound states of the pseudoharmonic potential of the Dirac equation with spin and pseudo-spin symmetry via Laplace transform approach", Acta Physica Polonica A, vol.130 no.5, 2016, pp.692-696.
[19]. S.H. Dong, "Factorization Method in Quantum Mechanics", Springer Science and Business Media, 2007.
[20]. D. Schiöberg, "The energy eigenvalues of hyperbolical potential functions ", Molecular Physics, Vol59(5), 1986, pp.123-1137.
[21]. A. Diaf, "Unified treatment of the bound states of the Schio¨berg and the Eckart potentials using Feynman path integral approach", Chin. Phys. B, Vol. 24, No. 2, 2015, 020302.
[22]. R.L., Greene, C. Aldrich, "Variational wave functions for a screened Coulomb potential(Review)", Physical Review A, vol.14 no.6, 1976, pp.2363-2366.
[23]. T.T. Ibrahim, K.J. Oyewumi, M. Wyngaadt, "Analytical solution of Ndimensional Klein-Gordon and Dirac equations with Rosen-Morse potential", Eur. Phys. J. Plus, vol.127, 2012, p.100.
[24]. H. Hassanabadi, S. Zarrinkamar, H. Rahimov, "Approximate Solution of D-Dimensional Klein-Gordon Equation with Hulthen-Type Potential via SUSYQM", Comm. Theor. Phys., vol.56, 2011, p.423.
[25]. K. J. Oyewumi, E. A. Bangudu, "Isotropic Harmonic Oscillator plus Inverse Quadratic Potential in N-Dimensional Space", The Arabian Journal for Science and Engineering, vol.28 no.2A, 2003, p.173.
[26]. S. M. Ikhdair, "Approximate solutions of the Dirac equation for the Rosen-Morse potential including the spin orbit centrifugal term", Math. Phys., vol.51, 2010, pp.023525:1-16.
[27]. S.I.Gradshteyn, I.M.Ryzhik, "Table of Integrals, Series and Products", Elsevier Academic Press, USA, 2007.