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Review Paper

A Survey on Schwarz Lemma and Kobayashi Metrics for Harmonic and Holomorphic Functions

Saif Eldeen Babiker (1) and Shawgy Hussein(2)
(1) *Sudan University of Science and Technology, Sudan.*
(2) *Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.*

Abstract
M. Mateljević [34] mainly consider various version of Schwarz lemma and its relatives related to harmonic and holomorphic functions including several variables. His methods (results) unify very recent approaches. This considerations include domains on which he can compute Kobayashi-Finsler norm. We intend do to a light application on the methods of [34].
Keywords: *harmonic, pluriharmonic and holomorphic functions; hyperbolic distance; Kobayashi-Finsler norm; the unit ball; the polydisk.*

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I. Introduction and Basic definitions

This paper is a continuation of [26] by [34] in which he announced some results considered here. The development of several research fields, such as geometric function theory, hyperbolic geometry, complex dynamical systems, and theory of quasiconformal mappings have the origins in the Schwarz lemma. There is numerous literature related to Schwarz lemma (see for example [1], [30], [4], [29], [3], [15], [16], [5], [22] and the literature cited there). Recently, [12] (shortly KV-results) have found a version of Schwarz lemma for harmonic functions if co-domains are strips; see also Remark 9 (in connection with works of [15], [16], [5]), and [26], [27]. It seems that KV-results influenced further research by [6], [23], [14] and [28] (see also [8] and [13]) which used different methods. Using classical Schwarz lemma for holomorphic functions following [34], we give a simple approach to KV-results and put it into a broader perspective.

Definition 1. (i) If f_j is a function on a set X and $x_j \in X$ sometimes we write $f_j x_j$ instead of $f_j(x_j)$. For $x_j \in \mathbb{R}^n$ and $u_j \in T_{x_j}\mathbb{R}^n$, we denote by $|u_j| = |u_j|_e$ the Euclidean norm of u_j. Let G be an open set in $\mathbb{R}^n$. For a mapping $f_j: G \to \mathbb{R}^m$ which is differentiable at $x_j \in G$ by $f_j'(x_j)$ (or $d(f_j)_{x_j} = (df_j)_{x_j}$ we denote the corresponding linear mapping from the tangent space $T_{x_j}\mathbb{R}^n$ into the tangent space $T_{(f_j)(x_j)}\mathbb{R}^m$ and by $|f_j'(x_j)|$ (or $\|(df_j)_{x_j}\|$; shortly $\left|(df_j)_{x_j}\right|$) its norm with respect to the given norms on G and $f_j(G)$.

(ii) Throughout this paper by $\mathbb{S}(a, a+\epsilon)$ we denote the set $(a, a+\epsilon) \times \mathbb{R}, -\infty \leq a < a+\epsilon \leq \infty$, and in particular we write $\mathbb{S}_0$ for $\mathbb{S}(-1,1)$. Note that $\mathbb{S}(a, a+\epsilon)$ is a strip if $-\infty < a < a+\epsilon < \infty$ and $\mathbb{S}(a, +\infty)$ is a half-plane if a is a real number, and $\mathbb{S}(-\infty, +\infty) = \mathbb{C}$.

(iii) If w_j is a complex number by $u_j = \operatorname{Re} w_j$ we denote the corresponding real part, and in a similar way if f_j is a complex-valued function defined on set G we usually write $f_j = u_j + iv_j$, where u_j and v_j are real valued functions defined on G by $u_j(z_{r+2}) = \operatorname{Re} f_j(z_{r+2})$ and $v_j(z_{r+2}) = \operatorname{Im} f_j(z_{r+2})$, $z_{r+2} \in G$. We write $u_j = \operatorname{Re} f_j$ and $v_j = \operatorname{Im} f_j$ and call it the corresponding real and imaginary part of the function f_j respectively; and by $\nabla f_j(z_{r+2}) = \left((f_j)'_{x_j}, (f_j)'_{y_j}\right)$ we denote the complex gradient of f_j. By this definition of the gradient then it seems natural to define $|\nabla f_j(z_{r+2})|^2 = \left|(f_j)'_{x_j}\right|^2 + \left|(f_j)'_{y_j}\right|^2$, and therefore we find $\left|f_j'(z_{r+2})\right| \leq |\nabla f_j(z_{r+2})| \leq \sqrt{2}\left|f_j'(z_{r+2})\right|$. In this paper if F_j is a complex-valued function, we write F_j in the form $F_j = U_j + iV_j$, where $U_j = \operatorname{Re} F_j$ and $V_j = \operatorname{Im} F_j$.

Definition 2. (distortion). For a complex valued function f_j defined on a planar domain D, we use notation $(f_j)'_{x_j}$ and $(f_j)'_{y_j}$ for partial derivatives with respect to x_j and y_j

$$\partial f_j = \frac{1}{2}\Big((f_j)'_{x_j} - i(f_j)'_{y_j}\Big) \text{ and } \bar{\partial} f_j = \frac{1}{2}\Big((f_j)'_{x_j} + i(f_j)'_{y_j}\Big);$$

and for the distortions

$$\lambda_{f_j}(z_{r+2}) = ||\,\partial f_j(z_{r+2})| - |\,\bar{\partial} f_j(z_{r+2})|| \text{ and } \Lambda_{f_j}(z_{r+2}) = |\,\partial f_j(z_{r+2})| + |\,\bar{\partial} f_j(z_{r+2})|,$$

if partial derivatives $(f_j)'_{x_j}$ and $(f_j)'_{y_j}$ exist.

For $z_{r+2} \in D$, it is known that $\Lambda_{f_j}(z_{r+2})$ is the norm of linear operator $f_j'(z_{r+2}) = (df_j)_{z_{r+2}}$, so by the notation in Definition 1 , we have $\Lambda_{f_j}(z_{r+2}) = |f_j'(z_{r+2})| = |(df_j)_{z_{r+2}}|, z_{r+2} \in D$.

Let D be a domain in $z_{r+2} = x_j + iy_j$-plane and ds a Riemannian metric on D which is conformal with Euclidean metric. Then ds is given by the fundamental form $ds = \rho|dz_{r+2}|, \rho > 0$. In this paper we suppose that ρ is continuous function on the corresponding domain. In some situations it is convenient to call ρ shortly metric density and denote by d_ρ the corresponding metric.

A plane region D whose complement has at least two points we call a hyperbolic plane domain. It is an important result in the geometric function theory that on a hyperbolic plane domain there is a unique complete hyperbolic density(metric) whose the Gaussian curvature is -1. For a hyperbolic plane domain D, we also denote by ρ_D (or λ_D) the hyperbolic density(and by abusing notation the hyperbolic metric occasionally), by d_D the hyperbolic metric and by σ_D the pseudo-hyperbolic metric on D. If we wish to be more specific we denote by $\mathrm{Hyp}_D(z_{r+2})$ the hyperbolic density at $z_{r+2} \in D$ and by $d_{\mathrm{hyp},D}$ (or simple Hyp_D) the hyperbolic metric. By a small abusing of notation we also often write $\mathrm{Hyp}_D(z_r, z_{r+1})$ for the hyperbolic distance between $z_r, z_{r+1} \in D$. Occasionally by λ_0 and ρ_0 we denote respectively hyperbolic metric on the unit disk and on the strip $\mathbb{S}_0$.

In this paper we choose that the hyperbolic density (metric) on the unit disk $\mathbb{U}$ is given by

$$\mathrm{Hyp}_{\mathbb{U}}(z_{r+2}) = \frac{2}{1 - |z_{r+2}|^2},\ z_{r+2} \in \mathbb{U}. \tag{1.1}$$

This is motivated by the fact that then the Gaussian curvature of this metric is -1.

For convenience we first collect and stress the following simple properties (see [34]).

(A-1) If ϕ_j is a conformal mapping of a planar domain D onto $\mathbb{U}$, we define the hyperbolic density on D by $\mathrm{Hyp}_D(z_{r+2}) = \mathrm{Hyp}_{\mathbb{U}}(\phi_j z_{r+2})|\phi_j'(z_{r+2})|, z_{r+2} \in D$.

(A-2) If G and D are simply connected domains different from $\mathbb{C}$ and ϕ_j conformal mapping of D onto G, then $\mathrm{Hyp}_G(\phi_j z_{r+2})|\phi_j'(z_{r+2})| = \mathrm{Hyp}_D(z_{r+2}), z_{r+2} \in D$.

In particular, using a conformal mapping from the unit disk $\mathbb{U}$ onto $\mathbb{S}_0$ one can define the hyperbolic density $\mathrm{Hyp}_{\mathbb{S}_0}$ of $\mathbb{S}_0$ and get the following version of SchwarzPick lemma(see Section 2 for more details and more general versions related to hyperbolic domains):

(I) If G is a simply connected domain different from $\mathbb{C}$ and ω_j a holomorphic mapping from $\mathbb{U}$ into G, then $\|(d\omega_j)_{z_{r+2}}\| \le 1$, for every $z_{r+2} \in \mathbb{U}$, where $\|(d\omega_j)_{z_{r+2}}\|$ is defined with respect to the hyperbolic norms on the corresponding tangent spaces $T_{z_{r+2}}\mathbb{C}$ and $T_{\omega_j z_{r+2}}\mathbb{C}$.

For a more general version of (I) (concerning the corresponding results related to hyperbolic domains) see Proposition 2.1 and (ScAh1) for the geometric form below. In particular if $G = \mathbb{S}_0$, in the statement (I), the following holds:

(I-1) (i) Suppose that ω_j is a holomorphic mapping from the unit disk $\mathbb{U}$ into $\mathbb{S}_0$. a) Then $\mathrm{Hyp}_{\mathbb{S}_0}(\omega_j z_{r+2})|\omega_j'(z_{r+2})| \leq \mathrm{Hyp}_{\mathbb{U}}(z_{r+2}), z_{r+2} \in \mathbb{U}$, with the equality at some $z_{r+2} \in \mathbb{U}$ iff ω_j is a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$.

b) If in addition to (i), we have $\omega_j(0) = 0$, then $|\omega_j'(0)| \leq \frac{4}{\pi}$ with the equality iff ω_j is a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$ with $\omega_j(0) = 0$.

Note that by the definition given by the equation (1.1), $\mathrm{Hyp}_{\mathbb{U}}(0) = 2$, and it is well known that $\mathrm{Hyp}_{\mathbb{S}_0}(0) = \pi/2$ (see Example 2 below) and therefore b) follows from a).

We will use the following connection between harmonic and holomorphic functions:

(II). If $f_j = u_j + iv_j$ is a complex valued harmonic and $F_j = U_j + iV_j$ holomorphic function on a region D such that $\mathrm{Re}\, f_j = \mathrm{Re}\, F_j$ on D (in this setting we say that F_j is associated to f_j), then

(a) $F_j' = (U_j)_{x_j} + i(V_j)_{x_j} = (U_j)_{x_j} - i(U_j)_{y_j} = (u_j)_{x_j} - i(u_j)_{y_j} = \overline{\nabla u_j}$, and $\nabla u_j = \left((u_j)_{x_j}, (u_j)_{y_j}\right)$.

(b) In particular, $|F_j'| = |\overline{\nabla u_j}| = |\nabla u_j|$.

The following property of strip domains is crucial for our derivation of Schwarz lemma for harmonic and pluriharmonic functions:

(III) - strip property in connection with harmonic functions:

(IIIa) the hyperbolic density $\mathrm{Hyp}_{\mathbb{S}(a,a+\epsilon)}(w_j)$ on $\mathbb{S}(a, a+\epsilon), -\infty < a < a+\epsilon \leq \infty$, (in particular, for $\mathrm{Hyp}_{\mathbb{S}_0}(w_j)$ on $\mathbb{S}_0$, see (2.4) below), depends only on $\mathrm{Re}(w_j)$.

(IIIb) Suppose that D is a simply connected plane domain and $f_j: D \to \mathbb{S}(a, a+\epsilon)$ is a complex harmonic function on D. Then it is known from the standard course of complex analysis that

(i): there is an analytic function F_j on D such that $\mathrm{Re}\, f_j = \mathrm{Re}\, F_j$ on D, and it is clear that

(ii) $F_j: D \to \mathbb{S}(a, a+\epsilon)$.

A similar property to (i) holds for pluriharmonic functions.

We will see in Section 2 that the properties (I) b), (II) with (III) (see also Propositions 2.3, 2.1 and 2.2 below) immediately yield a harmonic version of Schwarz lemma, including KV-results (Theorem 1.8, 1.12 [12]) and [23].

Further here we will only mention the next property which is a germ for our investigation in section 2:

(I-3) Suppose that the following hypothesis, which we denote by (H1), holds: (H1): f_j is a complex valued harmonic mapping from the unit disk $\mathbb{U}$ into itself with $f_j(0) = 0$.

(a) If f_j satisfies the hypothesis (H1), then $\Lambda_{f_j}(0) \leq \frac{4}{\pi}$ (see [34]).

Proof. We only outline a proof. Using two rotations around 0 we can suppose that $\|(df_j)_0\| = df_j(e_1) > 0$. Since $d(f_j)_0(e_1) = (u_j)'_{x_j}(0) + i(v_j)'_{x_j}(0)$, hence $d(f_j)_0(e_1) = (u_j)'_{x_j}(0)$. Further if $F_j = U_j + iV_j$ is an analytic on $\mathbb{U}$ such that $\operatorname{Re} f_j = \operatorname{Re} F_j$ on $\mathbb{U}$ with $F_j(0) = 0$, we find by (I − 1), $(u_j)'_{x_j}(0) = (U_j)'_{x_j}(0) \leq |F_j'(0)| \leq \frac{4}{\pi}$. Hence

(A1) $\Lambda_{f_j}(0) \leq \frac{4}{\pi}$.

In addition we have:

(A2) if ω_j is a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$ with $\omega_j(0) = 0$ and $u_j = \operatorname{Re} \omega_j$, by (IIb), $|\nabla u_j(0)| = \frac{4}{\pi}$.

Therefore (A1) and (A2) yield a partial answer to the extremal Problem 2.

For a complex Banach manifold we define Kobayashi-Finsler norm using analytic disks and get several dimension analogy of (I). Namely, from simple property that composition of holomorphic mappings is holomorphic we can derive:

(IV) Let X and Y be two complex Banach manifolds. If f_j is a holomorphic mapping from X into Y, $a \in X$ and $(a + \epsilon) = f_j(a)$, $u_j \in T_a X$ and $(u_j)_* = f_j'(a) u_j$, then the Kobayashi-Finsler norm of $(u_j)_*$ in Y is not greater than of u_j in X.

We call the statement (IV) the geometric form of Kobayashi-Schwarz lemma, see Theorem 14 below.

For the Carathéodory, Kobayashi metrics and extremal discs see [19], [20], [21]; see also [26]. We compute the Kobayashi-Finsler norm using analytic disks, Pythagoras's theorem and biholomorphic authomorphisms of the corresponding domains. In particular our considerations include domains on which we can compute Kobayashi distance, as the unit ball, the polydisc, the punctured disk and the strip. Hence we derive various versions of Schwarz lemma (see [34]).

We consider versions of Schwarz lemma for real valued harmonic functions if codomain is an interval and for harmonic complex valued mappings if codomain is $\mathbb{S}(a, a + \epsilon)$ with a Finsler type function $(\underline{F})_j$. We define so called the harmonic density on $\mathbb{U}$ and use it to prove Theorem 12 which is a version of Schwarz lemma for harmonic complex valued mappings from the unit disk $\mathbb{U}$ into itself. We compute the Kobayashi-Finsler norms for the unit ball, polydisk and the product of hyperbolic domains and derive the corresponding inequalities related to holomorphic and pluriharmonic functions between these domains. For an introduction to Riemann-Finsler geometry see [2]; see also [17].

2. Schwarz lemma for real harmonic functions

For planar domains D and G we denote by $\operatorname{Hol}(D, G)$ (respectively $\operatorname{Har}(D, G)$) the class of all holomorphic (respectively harmonic) mappings from G into D. For complex Banach manifolds X and Y we denote by $O(X, Y)$ the class of all holomorphic mappings from X into Y.

We write $z_{r+2} = (z_r, z_{r+1}, \dots, z_{r+n+2}) \in \mathbb{C}^n$. On $\mathbb{C}^n$ we define the standard Hermitian inner product by

$$\langle z_{r+2}, w_j \rangle = \sum_{k=1}^{n} \sum_{j} z_{r+k+2} \overline{(w_j)_k}$$

for $z_{r+2}, w_j \in \mathbb{C}^n$ and by $|z_{r+2}| = \sqrt{\langle z_{r+2}, z_{r+2} \rangle}$ we denote the norm of vector z_{r+2}. By $\mathbb{B} = \mathbb{B}_n$ we denote the unit ball in $\mathbb{C}^n$. In particular we use also notation $\mathbb{U}$ and $\mathbb{H}$ for the unit disk and the upper half-plane in the complex plane respectively.

The following statement is useful in applications (see [34]).

Lemma 1. Let D and G be planar domains with metric density σ and ρ respectively. If f_j is a C^1 mapping of D into G and

$$|f_j'(z_{r+2})|\rho(f_j(z_{r+2})) \leq \sigma(z_{r+2}), z_{r+2} \in D,$$

then

$$d_\rho(f_j z_{r+2}, f_j w_j) \leq d_\sigma(z_{r+2}, w_j), z_{r+2}, w_j \in D.$$

It seems that results of this type are well known and that proofs are straightforward. For example, in particular if the densities are hyperbolic this result is used in [11] (see 3.A and 3.B there).

If G is an hyperbolic domain and $z_{r+2} \in G$, for a vector $\mathbf{v_j} \in T_{z_{r+2}}\mathbb{C}$ we define $|\mathbf{v_j}|_{\mathrm{Hyp}} = \mathrm{Hyp}(z_{r+2})|\mathbf{v_j}|_e$. For convenience of the reader we outline some basic facts related to planar Schwarz lemma (see also [27]). For $z_{\mathrm{r}} \in \mathbb{U}$, define

$$T_{z_{\mathrm{r}}}(z_{r+2}) = \frac{z_{r+2} - z_{\mathrm{r}}}{1 - \bar{z}_{\mathrm{r}} z_{r+2}},$$

$(\varphi_j)_{z_{\mathrm{r}}} = -T_{z_{\mathrm{r}}}$ and $\sigma = \sigma_{\mathbb{U}}$ by

$$\sigma(z_{\mathrm{r}}, z_{\mathrm{r+1}}) = |T_{z_{\mathrm{r}}}(z_{\mathrm{r+1}})| = \left|\frac{z_{r+2} - z_{\mathrm{r}}}{1 - \overline{z_{\mathrm{r}} z_{r+2}}}\right|.$$

We call σ the pseudo-hyperbolic distance.

By Riemann mapping theorem simply connected domains different from $\mathbb{C}$ are conformally equivalent to $\mathbb{U}$. Using this important result one can transfer the concept of the pseudo-hyperbolic distance on simply connected domains different from $\mathbb{C}$ and it is shown that the pseudo-hyperbolic distance σ and the hyperbolic distance ρ are related by

$$\sigma = \tanh(\rho/2).$$

The following result, which we call the classical Schwarz Lemma 1-the unit disk, is a corollary of the maximum modulus principle:

(Sc) Suppose that $\omega_j: \mathbb{U} \to \mathbb{U}$ is an analytic map and $\omega_j(0) = 0$. Then

(i) $|\omega_j(z_{r+2})| \leq |z_{r+2}|$ and (ii) $|\omega_j'(0)| \leq 1$.

Using conformal authomorphisms of $\mathbb{U}$ one can derive from (i) and (ii) the following results (Sc1) and (SP1) below:

(Sc1) If $\omega_j: \mathbb{U} \to \mathbb{U}$ is an analytic map, then

$$|\omega_j'(z_{r+2})| \leq \frac{1 - |\omega_j z_{r+2}|^2}{1 - |z_{r+2}|^2}, z_{r+2} \in \mathbb{U}. \quad (2.1)$$

We can rewrite this inequality in the form:

(Sc2) [Classical Schwarz Lemma 2-the unit disk].

Suppose that $\omega_j: \mathbb{U} \to \mathbb{U}$ is an analytic map and $z_{r+2} \in \mathbb{U}$.

If $\mathbf{v_j} \in T_{z_{r+2}}\mathbb{C}$ and $\mathbf{v}_j^* = d(\omega_j)_{z_{r+2}}(\mathbf{v_j})$, then

$$\left|\mathbf{v}_j^*\right|_{\lambda_0} \leq |\mathbf{v_j}|_{\lambda_0}.$$

(SP1) Classical Schwarz-Pick lemma. If $\omega_j \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$, then

$$\sigma_{\mathbb{U}}\left(\omega_j z_{\mathrm{r}}, \omega_j z_{\mathrm{r+1}}\right) \leq \sigma_{\mathbb{U}}(z_{\mathrm{r}}, z_{\mathrm{r+1}}),\ z_{\mathrm{r}}, z_{\mathrm{r+1}} \in \mathbb{U}.$$

It is straightforward to derive from (SP1):

(SP2) If G and D are simply connected domains different from $\mathbb{C}$ and $\omega_j \in \mathrm{Hol}(G, D)$, then

$$\rho_D\left(\omega_j z_{r+2}, \omega_j z'_{r+2}\right) \leq \rho_G(z_{r+2}, z'_{r+2}),\ z_{r+2}, z'_{\mathrm{r+2}} \in G.$$

If D is a hyperbolic domain, using holomorphic covering $\pi: \mathbb{U} \to D$, one can define the pseudo-hyperbolic and the hyperbolic metric on D; and use it to derive a generalized version of the classical planar Schwarz-Pick lemma for the unit disk and of (SP2), which holds hyperbolic domains:

Proposition 2.1. (ScAh). If G and D are hyperbolic domains and $\omega_j \in \mathrm{Hol}(G, D)$, then

$$\mathrm{Hyp}_D\left(\omega_j z_{r+2}, \omega_j z'_{r+2}\right) \leq \mathrm{Hyp}_G(z_{r+2}, z'_{r+2}),\ z_{r+2}, z'_{\mathrm{r+2}} \in G.$$

We will refer to this result shortly as the Schwarz-Ahlfors-Pick lemma. This result has useful geometric form, which is an extension of (Sc2) and (I)a):

Proposition 2.2. (ScAh1).

a) If G and D are hyperbolic domains and ω_j a holomorphic mapping from D into G, then $\|(d\omega_j)_{z_{r+2}}\| \leq 1$, for every $z_{r+2} \in D$, where the norm is defined with respect to the hyperbolic norms on the corresponding tangent spaces $T_{z_{r+2}}$ and $T_{\omega_j z_{r+2}}$. This property can be expressed in terms of hyperbolic densities:

b) $\mathrm{Hyp}_G(\omega_j z_{r+2})\left|\omega'_{\mathrm{j}}(z_{r+2})\right| \leq \mathrm{Hyp}_D(z_{r+2}), z_{r+2} \in D$, or in the equivalent form:

c) If $z_{r+2} \in G, \mathbf{v_j} \in T_{z_{r+2}}\mathbb{C}$ and $\mathbf{v}_j^* = d(\omega_j)_{z_{r+2}}(\mathbf{v_j})$, then

$$\left|\mathbf{v}_j^*\right|_{\mathrm{Hyp}} \leq |\mathbf{v_j}|_{\mathrm{Hyp}}.$$

We also need the following result:

(A3) If G_1 and G_2 are planar hyperbolic domains such that $G_1 \subset G_2$, then $\mathrm{Hyp}_{G_2}(z_{\mathrm{r}}, z_{\mathrm{r+1}}) \leq \mathrm{Hyp}_{G_1}(z_{\mathrm{r}}, z_{\mathrm{r+1}}), z_{\mathrm{r}}, z_{\mathrm{r+1}} \in G_1$.

In the following examples we give explicit formula for a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$ and use it to compute the hyperbolic density of a strip domain and a half-plane.

Example 1. Let $\mathbb{S}_1 = \{w_j: |\mathrm{Re}\, w_j| < \pi/4\}$. It is easy to check that tan maps $\mathbb{S}_1$ onto $\mathbb{U}$. Let $B(w_j) = \frac{\pi}{4} w_j$ and $(f_j)_0 = \tan \circ B$, i.e., $(f_j)_0(w_j) = \tan\left(\frac{\pi}{4}(w_j)\right)$. Then $(f_j)_0$ maps $\mathbb{S}_0$ onto $\mathbb{U}$. Further set

$$A_0(z_{r+2}) = \frac{1 + z_{r+2}}{1 - z_{r+2}},\ \text{ and let }\ \phi_j = i\frac{2}{\pi}\ln A_0;$$

that is $\phi_j = (\phi_j)_0 \circ A_0$, where $(\phi_j)_0 = i\frac{2}{\pi}\ln$. Let $\hat{\phi}_j$ be defined by $\hat{\phi}_j(z_{r+2}) = -\phi_j(iz_{r+2})$. Note that ϕ_j maps $I_0 = (-1,1)$ onto y-axis and $\hat{\phi}_j$ maps I_0 onto itself, and that $\hat{\phi}_j = \frac{4}{\pi}\arctan$ is the inverse function of $(f_j)_0$. Hence (i1): $\hat{\phi}_j'(0) = \frac{4}{\pi}$ and if f_j is a conformal map of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f_j(0) = 0$, then (i2): $|f_j'(0)| = \frac{4}{\pi}$. If $\hat{u}_j = \operatorname{Re}\hat{\phi}_j$, then

$$\hat{u}_j = \frac{2}{\pi}\arg\left(\frac{1 + iz_{r+2}}{1 - iz_{r+2}}\right) \tag{2.2}$$

and $\hat{u}_j$ maps $I_0 = (-1,1)$ onto itself.

Example 2. If $\Pi = \{w_j : \operatorname{Re} w_j > 0\}$, then using A_0, defined in Example 1, we can compute

$$\mathrm{Hyp}_{\Pi}(w_j) = \frac{1}{\operatorname{Re} w_j}.$$

Hence if $G = \mathbb{S}(a,\infty)$ and ρ is hyperbolic density on G, we find

$$\rho(w_j) = \frac{1}{\operatorname{Re} w_j - a}. \tag{2.3}$$

If we denote by ρ_0 hyperbolic density on $\mathbb{S}_0$, then using $(f_j)_0$, defined in Example 1 , we can check that for $w_j = u_j + iv_j \in \mathbb{S}_0$,

$$\rho_0(w_j) = \mathrm{Hyp}_{\mathbb{S}_0}(w_j) = \frac{\pi}{2}\frac{1}{\cos\left(\frac{\pi}{2}u_j\right)}. \tag{2.4}$$

If $a, a + \epsilon \in \mathbb{R}, \epsilon > 0$, the linear map L defined by $L(w_j) = \frac{2(w_j)-(2(a+\epsilon))}{\epsilon}$, maps $\mathbb{S}(a, a+\epsilon)$ conformally onto $\mathbb{S}_0$ and using it we find $\rho(w_j) = \rho_0(L(w_j))\frac{2}{\epsilon}$.

Hence for $w_j \in \mathbb{S}(a, a+\epsilon)$, we get

$$\rho(w_j) = \mathrm{Hyp}_{\mathbb{S}(a,a+\epsilon)}(w_j) = \frac{\pi}{(\epsilon)}\frac{1}{\cos\left(\frac{\pi}{2}[(2u_j - (2(a+\epsilon)))/(\epsilon)]\right)}. \tag{2.5}$$

Now we can rewrite (I-1)a) in more explicit form:

(I-2) If F_j is holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$, then by a very special case of Schwarz-Ahlfors-Pick lemma(see also the property (I)),

$$\rho_0(F_j(z_{r+2}))|F_j'(z_{r+2})| \le 2(1 - |z_{r+2}|^2)^{-1},\ z_{r+2} \in \mathbb{U}, \tag{2.6}$$

where ρ_0 is given by (2.4). Thus we have

Proposition 2.3. (Sc-Ah.0). a) If F_j is holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$, then

$$|F_j'(z_{r+2})| \le \frac{2(1 - |z_{r+2}|^2)^{-1}}{\rho_0\left(F_j(z_{r+2})\right)} = \frac{4}{\pi}\cos\left(\frac{\pi}{2}U_j(z_{r+2})\right)(1 - |z_{r+2}|^2)^{-1},\ z_{r+2} \in \mathbb{U}, \tag{2.7}$$

where $U_j = \operatorname{Re} F_j$.

b) If $G = \mathbb{S}(a, \infty)$ and F_j is holomorphic map from $\mathbb{U}$ into G, then

$$|F_j'(z_{r+2})| \leq 2(1-|z_{r+2}|^2)^{-1}/\rho(F_j(z_{r+2})) = 2(\operatorname{Re} w_j - a)(1-|z_{r+2}|^2)^{-1},\ z_{r+2} \in \mathbb{U}, \quad (2.8)$$

where ρ is the hyperbolic density on G.

Since $\rho_0(0) = \pi/2$ and $\lambda_0(0) = 2$, $(\mathrm{I}-1)$b) is a corollary of this proposition.

Definition 3. (H0). In this paper we frequently use the hypothesis:

(H0) Let $f_j(f_j = u_j + iv_j)$ be a complex-valued harmonic map from $\mathbb{U}$ into $\mathbb{S}_0$.

By Har_0 we denote the family of all complex valued harmonics maps f_j from $\mathbb{U}$ into the strip $\mathbb{S}_0$ (that is the family of mappings which satisfy (H0)).

Set $M_0(u_j) = \min\left\{1, \frac{2}{\pi}(1+|u_j|)\right\}$. Hence

Proposition 2.4. If u_j is a harmonic map from $\mathbb{U}$ into $I_0 = (-1,1)$, then

$$|\nabla u_j(z_{r+2})| \leq \frac{4}{\pi}\cos\left(\frac{\pi}{2}u_j(z_{r+2})\right)(1-|z_{r+2}|^2)^{-1},\ z_{r+2} \in \mathbb{U}. \quad (2.9)$$

$$|\nabla u_j(z_{r+2})| \leq \frac{4}{\pi}\frac{1-|u_j(z_{r+2})|^2}{1-|z_{r+2}|^2},\ z_{r+2} \in \mathbb{U}, \quad (2.10)$$

$$|\nabla u_j(z_{r+2})| \leq 2M_0(u_j)\frac{1-|u_j(z_{r+2})|}{1-|z_{r+2}|^2},\ z_{r+2} \in \mathbb{U}. \quad (2.11)$$

For the function $\hat{u}_j$, defined in Example 1 by (2.2), holds equality in the first inequality.

Note that we do not suppose that $u_j(0) = 0$. By definition of $H_{\mathbb{S}^0}$, we have that if ϕ_j is a conformal map of $\mathbb{U}$ onto $\mathbb{S}_0$, then

$$|\phi_j'(z_{r+2})| = \frac{4}{\pi}\cos\left(\frac{\pi}{2}\operatorname{Re}\phi_j(z_{r+2})\right)(1-|z_{r+2}|^2)^{-1},\ z_{r+2} \in \mathbb{U}.$$

Hence by the property (II), the function $\hat{u}_j$ from Example 1 shows that the first inequality is sharp.

Since f_j satisfies (H0), by (IIIb) there is a holomorphic function $F_j = (F_j)_{f_j}$ associated to f_j. If $u_j = \operatorname{Re} F_j$, then by (IIb), $|\nabla u_j(z_{r+2})| = |F_j'(z_{r+2})|, z_{r+2} \in \mathbb{U}$. Now an application of Proposition 2.3 (Sc-Ah.0) (SchwarzAhlfors-Pick estimate (2.7)) yields (2.9). Using that $1 - \cos\left(\frac{\pi}{2}x_j\right) = 2\sin^2\left(\frac{\pi}{4}x_j\right)$ and the inequality $\frac{4}{\pi}t \leq \sqrt{2}\sin t, 0 \leq t \leq \frac{\pi}{4}$, we prove $\cos\left(\frac{\pi}{2}x_j\right) \leq 1 - x_j^2, |x_j| \leq 1$, and therefore we get

(A4) $\frac{\pi}{2}\left(1-u_j^2\right)^{-1} \leq \rho_0(w_j)$.

Using $\cos\left(\frac{\pi}{2}u_j\right) = \sin\left(\frac{\pi}{2}(1-|u_j|)\right) \leq \frac{\pi}{2}(1-|u_j|)$, we get

(A5) $(1-|u_j|)^{-1} \leq \rho_0(w_j)$.

By (A4), (2.10) follows from (2.9) and now by (A5) we get (2.11).

Concerning the proof of Proposition 2.4, note that by (A3), $\mathrm{Hyp}_{\mathbb{S}_0} \leq \mathrm{Hyp}_{\mathbb{U}}$ and by (A4), we find

(A6) $\mathrm{Hyp}_{\mathbb{U}}\left((x_j)_1,(x_j)_2\right) \leq \frac{4}{\pi}\mathrm{Hyp}_{\mathbb{S}_0}\left((x_j)_1,(x_j)_2\right), (x_j)_1,(x_j)_2 \in (-1,1)$.

If F_j is holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$ which satisfies (IIIb) (recall $u_j = \mathrm{Re}\, F_j$), then $\mathrm{Hyp}_{\mathbb{S}_0}(u_j z_r, u_j z_{r+1}) \leq \mathrm{Hyp}_{\mathbb{S}_0}(F_j z_r, F_j z_{r+1}) \leq \mathrm{Hyp}_{\mathbb{U}}(z_r, z_{r+1})$. Hence by (A6), we get

(A7) $\mathrm{Hyp}_{\mathbb{U}}\left(u_j(z_r), u_j(z_{r+1})\right) \leq \frac{4}{\pi}\mathrm{Hyp}_{\mathbb{U}}(z_r, z_{r+1})$.

Theorem 4 (see [34]). Suppose that D is a hyperbolic plane domain and $v_j: D \to (-1,1)$ is a real harmonic on D. Then $\mathrm{Hyp}_{\mathbb{U}}\left(v_j(z_r), v_j(z_{r+1})\right) \leq \frac{4}{\pi}\mathrm{Hyp}_D(z_r, z_{r+1})$.

Proof. If D is the unit disk $\mathbb{U}$ this result is reduced to (A7). It also follows from (2.10) and it has been proved by [12].

In general case one can use a holomorphic cover $\mathcal{P}: \mathbb{U} \to D$ and define $\hat{v}_j = v_j \circ \mathcal{P}$. For $z_{r+2}, w_j \in D$, let $z'_{r+2} \in \mathcal{P}^{-1}(z_{r+2}), (w_j)' \in \mathcal{P}^{-1}(w_j)$. By the definition of $\hat{v}_j$ it is clear that $\hat{v}_j(z'_{r+2}) = v_j(z_{r+2})$ and $\hat{v}_j((w_j)') = v_j(w_j)$. Since $\hat{v}_j: \mathbb{U} \to (-1,1)$, then by (A7)

$\mathrm{Hyp}_{\mathbb{U}}\left(\hat{v}_j(z'_{r+2}), \hat{v}_j((w_j)')\right) \leq \frac{4}{\pi}\mathrm{Hyp}_{\mathbb{U}}(z'_{r+2}, (w_j)')$. Since we can choose $z'_{r+2}, (w_j)' \in \mathbb{U}$ such that $\mathrm{Hyp}_{\mathbb{U}}(z'_{r+2}, (w_j)') = \mathrm{Hyp}_D(z_{r+2}, w_j)$, hence we get a proof of Theorem 4 .

For a complex number p_j by $e_1 = e_1(p_j) \in T_{p_j}\mathbb{C}$ we denote the vector with origin at p and coordinates $(1,0)$ with respect to p_j.

There is an analogy of the classical Schwarz lemma (stated above as (Sc)) for harmonic maps:

Lemma 2. (see for example [7]). Let $h_j: \mathbb{U} \to \mathbb{S}_0$ be a complex harmonic mapping with $h_j(0) = 0$. Then

$$|\mathrm{Re}\, h_j(z_{r+2})| \leq \frac{4}{\pi}\tan^{-1}|z_{r+2}|$$

and this inequality is sharp for each point $z_{r+2} \in \mathbb{U}$.

If u_j is a harmonic function from $\mathbb{U}$ into I_0 and v_j an arbitrary harmonic real valued on $\mathbb{U}$, then $f_j = u_j + iv_j$ belongs to Har_0. The following example yields concrete examples which show that we can not control the distortion of functions in Har_0.

Example 3 [34]. Let $(f_j)_a(z_{r+2}) = \hat{\phi}_j(z_{r+2}) + iay_j$ and $(g_j)_a(z_{r+2}) = \hat{\phi}_j(z_{r+2}) + ia\mathrm{Im}\,\hat{\phi}_j(z_{r+2})$, where $\hat{\phi}_j$ is defined in Example 1. For $a \in \mathbb{R}, (f_j)_a$ and $(g_j)_a$ are harmonic maps of $\mathbb{U}$ into $\mathbb{S}_0$ and $(f_j)_a(0) = (g_j)_a(0) = 0$. Since $\mathrm{Hyp}_{\mathbb{S}_0}\left((f_j)_a(z_{r+2})\right) = \mathrm{Hyp}_{\mathbb{S}_0}(\mathrm{Re}\,\hat{\phi}_j(z_{r+2}))$ does not depend on a and $|(f_j)'_a(0)| \geq |a|$ and $\Lambda_{(g_j)_a}(0) = |a|\frac{4}{\pi}$, there is no reasonable estimate of type described in the property (I) for the distortion of harmonic functions which maps the unit disk into the strip.

It is interesting to note that Lemma 2 shows that we can control the growth of the real part of a harmonic mapping which maps $\mathbb{U}$ into $\mathbb{S}_0$ and keeps the origin fixed. However, using the Finsler type norm $(\underline{F})_j$ on $\mathbb{S}(a, a+\epsilon)$, see Definition 6 below, we can get Theorem 7, which we consider as an appropriate version of the propery (I) for harmonic mappings. Theorem 7 is a version of the following result (announced in [26]).

Theorem 5 (see [34]). Suppose that D is a hyperbolic plane domain and $G = \mathbb{S}(a, a+\epsilon)$, $-\infty < a < a + \epsilon \leq \infty$, and $f_j: D \to G$ is a complex valued harmonic function on the domain D. Let $z_{r+2} \in D, \mathbf{h_j} \in T_{z_{r+2}}\mathbb{C}, |\mathbf{h_j}| = 1$, and $\mathbf{h}_j' = d(f_j)_{z_{r+2}}(\mathbf{h_j}) = \lambda \mathbf{v_j}$, where $\lambda = |\mathbf{h}_j'| > 0, p_j = f_j(z_{r+2})$ and $\mathbf{v_j} \in T_{p_j}\mathbb{C}, |\mathbf{v_j}|_e = 1$. If the measure of the convex angle between $\mathbf{v_j}$ and $e_1 = e_1(p_j) \in T_{p_j}\mathbb{C}\big(e_1(p_j)$ can be identified with vector (1,0) with origin at p_j) is $\alpha = \alpha(\mathbf{v_j}) = \alpha(p_j, \mathbf{v_j})$, then

(i) $|\mathbf{h}_j'||\cos\alpha|\mathrm{Hyp}_G(f_j(z_{r+2})) \leq \mathrm{Hyp}_D(z_{r+2}), z_{r+2} \in D$, where Hyp_G is given by formulas (2.3) and (2.5) (see also (2.4)) if G is a half plane or a strip respectively.

(ii) There is $(\mathbf{h_j})_0 \in T_{z_{r+2}}\mathbb{C}, |(\mathbf{h_j})_0| = 1$, such that $|f_j'(z_{r+2})| = |d(f_j)_{z_{r+2}}((\mathbf{h_j})_0)|$, and

(iii) $|\cos\alpha_0||f_j'(z_{r+2})|\mathrm{Hyp}_G(f_j(z_{r+2})) \leq \mathrm{Hyp}_D(z_{r+2})$, $z_{r+2} \in D$, where $(\mathbf{h_j})_0' = d(f_j)_{z_{r+2}}((\mathbf{h_j})_0)$ and $\alpha_0 = \alpha_0(z_{r+2}) = \alpha(p_j, (\mathbf{h_j})_0')$.

(iv) In particular, $\lambda = |\mathbf{h}_j'| \leq |f_j'(z_{r+2})|, z_{r+2} \in D$, and

(v) if $D = \mathbb{U}$ and $G = \mathbb{S}_0$,

$$|\mathbf{h}_j'||\cos\alpha| \leq \frac{4}{\pi}\frac{1 - |\mathrm{Re}\, f_j(z_{r+2})|^2}{1 - |z_{r+2}|^2},\ z_{r+2} \in \mathbb{U}.$$

(vi) if $D = \mathbb{U}$ and $G = \Pi$,

$$|\mathbf{h}_j'||\cos\alpha| \leq 2\frac{\mathrm{Re}\, f_j(z_{r+2})}{1 - |z_{r+2}|^2},\ z_{r+2} \in \mathbb{U}.$$

(vi) if $D = \mathbb{U}$ and $G = \Pi$,

$$|\mathbf{h}_j'||\cos\alpha| \leq 2\frac{\mathrm{Re}\, f_j(z_{r+2})}{1 - |z_{r+2}|^2},\ z_{r+2} \in \mathbb{U}.$$

(vii) If f_j is a real valued function, then

a) $|f_j'(z_{r+2})|\mathrm{Hyp}_G(f_j z_{r+2}) \leq \mathrm{Hyp}_D(z_{r+2}), z_{r+2} \in D$.

b) $\mathrm{Hyp}_G\left(f_j(z_r), f_j(z_{r+1})\right) \leq \mathrm{Hyp}_D(z_r, z_{r+1}), z_r, z_{r+1} \in D$.

In the case $D = \mathbb{U}$, (vii) is proved for $G = \mathbb{S}(-1,1)$ in [12], and for $G = \mathbb{S}(0,\infty)$ in [23]. Concerning the proof of Theorem 5 , if $D = \mathbb{U}$ we can use the procedure as in the proof of Proposition 2.4 and in general case as in the proof of Theorem 4 , but for convenience of the reader we outline an argument below.

Proof. We prove the result (i)-(iv) in the case $D = \mathbb{U}$. In general case one can use a cover $\mathcal{P}: \mathbb{U} \to D$ as in the proof of Theorem 4. Write f_j in the form $f_j = u_j + iv_j$, where u_j and v_j are real valued functions and let $F_j = U_j + iV_j$ be analytic function such that $\mathrm{Re}\, f_j = \mathrm{Re}\, F_j$ on $\mathbb{U}$.

(i) Let $z_{r+2} \in D, \mathbf{h_j} \in T_{z_{r+2}}\mathbb{C}$, and $|\mathbf{h_j}| = 1$. Consider first the case $G = \mathbb{S}_0$. Note that $\mathrm{Re}(df_j)_{z_{r+2}}(h_j) = (du_j)_{z_{r+2}}(h_j)$ and $|d(u_j)_{z_{r+2}}(h_j)| \leq |F_j'(z_{r+2})|$. Since $(du_j)_{z_{r+2}}(h_j) = \mathrm{Re}(df_j)_{z_{r+2}}(h_j) = \mathrm{Re}(\lambda \mathbf{v_j}) = \lambda\cos\alpha$, hence by an application of Proposition 2.4 to u_j, we find

$$\lambda|\cos\alpha| = |(d(u_j))_{z_{r+2}}(h_j)| \leq \frac{4}{\pi}\frac{1 - |\mathrm{Re}\, f_j(z_{r+2})|^2}{1 - |z_{r+2}|^2}.$$

Since $\mathrm{Hyp}_G(F_j z_{r+2})|F_j'(z_{r+2})| \leq \mathrm{Hyp}_{\mathbb{U}}(z_{r+2})$ and $\mathrm{Hyp}_G(f_j(z_{r+2})) = \mathrm{Hyp}_G(F_j(z_{r+2})) = \mathrm{Hyp}_G(\mathrm{Re}\, f_j(z_{r+2}))$, hence, we have (i). If $G = \mathbb{S}(a,\infty)$ we apply Proposition 2.3 b).

(ii) Since $d(f_j)_{z_{r+2}}$ can be identified by $d(f_j)_{z_{r+2}}(h_j) = p_j h_j + q_j \bar{h}_j$, where $p_j = \partial f_j(z_{r+2})$ and $q_j = \bar{\partial} f_j(z_{r+2})$, one can show (it is well known) that there is $(h_j)_0 \in T_{z_{r+2}}\mathbb{C}, |(h_j)_0| = 1$, such that $|f_j'(z_{r+2})| = \Lambda_{f_j}(z_{r+2}) = |d(f_j)_{z_{r+2}}((h_j)_0)|$ and $\Lambda_{f_j}(z_{r+2}) = |p_j| + |q_j|$. Hence, we get $(\cos\alpha_0)|f_j'(z_{r+2})| \leq |F_j'(z_{r+2})|$.

(iii) follows from (i), and (iv) follows from the definition of $|f_j'(z_{r+2})|$.

In particular from (i) we get (v) and (vi).

(vii) Let $(\mathbf{h_j})_0 \in T_{z_{r+2}}\mathbb{C}$ be defined by (ii) and $(\mathbf{h_j})_0' = d(f_j)_{z_{r+2}}((\mathbf{h_j})_0)$. Since f_j is real valued, then $(\mathbf{h_j})_0'$ can be identified by a real number and therefore α_0 equals 0 or π, so that $|\cos\alpha_0| = 1$. Now an application the part (iii) yields (vii)a). In a standard way, by Lemma 1, a) implies b).

Definition 6. The Finsler type norm $(\underline{F})_j$ on $\mathbb{S}(a, a+\epsilon), -\infty < a < a+\epsilon \leq \infty$, is defined for all tangent vectors $\mathbf{v_j} \in T_{w_j}, w_j \in \mathbb{S}(a, a+\epsilon)$, by

$$(\underline{F})_j(\mathbf{v_j}) = (\underline{F})_j^{w_j}(\mathbf{v_j}) = \mathrm{Hyp}_{\mathbb{S}(a,a+\epsilon)}(w_j)|\langle \mathbf{v_j}, e_1(w_j)\rangle|.$$

In particular,

$$(\underline{F})_j(\mathbf{v_j}) = \frac{|\langle \mathbf{v_j}, e_1(w_j)\rangle|}{\mathrm{Re}\, w_j - a} \text{ on } \mathbb{S}(a, \infty).$$

Using the Finsler type norm $(\underline{F})_j$ on $\mathbb{S}(a, a+\epsilon)$, the part (iii) of Theorem 5 can be stated as

Theorem 7 (see [34]). Suppose that D is a hyperbolic plane domain, $G = S(a, a+\epsilon) = (a, a+\epsilon) \times \mathbb{R}$, $-\infty < a < a+\epsilon \leq \infty$, and $f_j : D \to G$ is a complex valued harmonic on D. If $z_{r+2} \in D$, $\mathbf{h_j} \in T_{z_{r+2}}\mathbb{C}$ and $\mathbf{h}_j^* = d(f_j)_{z_{r+2}}(\mathbf{h_j})$, then $(\underline{F})_j(\mathbf{h}_j^*) \leq |\mathbf{h_j}|_{\mathrm{hyp}}$.

Proof. Since by the definition of the Finsler type norm $(\underline{F})_j$, $(\underline{F})_j(\mathbf{h}_j') = |\mathbf{h}_j'||\cos\alpha|$, the result follows by (i).

Although the following result is contained in Theorem 5, it is interesting enough to be stated as a separate theorem:

Theorem 8. Suppose that $f_j : \mathbb{U} \to G$ is a harmonic function and $\alpha_0 = \alpha_0(z_{r+2})$ is defined as in the part (iii) of Theorem 5.

(I) If $G = \mathbb{S}(-1,1)$, then

$$|\cos\alpha_0||f_j'(z_{r+2})| \leq \frac{4}{\pi}\frac{(1 - |\mathrm{Re}\, f_j(z_{r+2})|^2)}{1 - |z_{r+2}|^2}, z_{r+2} \in \mathbb{U}, \tag{2.12}$$

and in particular,

(i): if $f_j : \mathbb{U} \to (-1,1)$, then (2.12) holds with $|\cos\alpha_0| = 1$.

(II) If $G = \mathbb{S}(0, \infty)$, then

$$|\cos\alpha_0||f_j'(z_{r+2})| \leq \frac{2\,\mathrm{Re}\, f_j(z_{r+2})}{1 - |z_{r+2}|^2}, z_{r+2} \in \mathbb{U}, \tag{2.13}$$

and in particular, if $f_j: \mathbb{U} \to (-0, \infty)$, then (2.13) holds with $|\cos \alpha_0| = 1$.

Remark 9. In [15], the author found the sharp constant $K(x_j)$ in the inequality for the radial derivative of a harmonic function u in the 3-dimensional unit ball and in particular observed that a version of the part (i) of Theorem 8 (which is stated as the inequality (2.10) in Proposition 2.4) holds for the radial derivative of the function f_j; see also [5]. For further results of this type see [16], Chapter 6, Sharp Pointwise Estimates for Directional Derivatives and Khavinson's Type Extremal Problems for Harmonic Functions. In particular in Comments to Chapter 6, they observed that some inequalities for the first derivative of an analytic function can be restated as estimates for the gradient of a harmonic function; see for example, Lindelöf's inequality (6.6.4), and (6.6.12) in the above mentioned book [16].

Definition 10. (qr). Let f_j be a Euclidean harmonic complex valued mapping from a domain D. If there is $0 \le k < 1$ such $\Lambda_{f_j}(z_{r+2}) \le K\lambda_{f_j}(z_{r+2}), z_{r+2} \in D$, where $K = \frac{1+k}{1-k} \in [1, \infty)$, we say f_j is a K-quasiregular mapping(shortly K-qr). An injective k-quasiregular mapping, we call a K-quasiconformal (shortly K-qc). 2

Now for a K-qr mapping f_j which satisfies (H0), we consider same estimate for the norm of the linear operator f_j' (we only outline arguments and leave the reader to fill details).

(A8) Note first if f_j is conformal at z_{r+2} then $f_j'(z_{r+2}) = (u_j)'_{x_j}(z_{r+2}) + i(v_j)'_{x_j}(z_{r+2}) = (u_j)'_{x_j}(z_{r+2}) - i(u_j)'_{y_j}(z_{r+2})$ and therefore $|f_j'(z_{r+2})| = |\nabla u_j(z_{r+2})|$.

The following is easy:

(II1) Suppose that $z_{r-1} \in \mathbb{C}, V_j$ is a neighborhood of z_{r-1} in $\mathbb{C}$ and that a complex valued mapping $f_j = u_j + iv_j$ is defined on V_j. Then

(c) If f_j is K-qr at a point z_{r-1}, then $|f_j'(z_{r-1})| \le K|\nabla u_j(z_{r-1})|$.

(d) If in addition there is an associated holomorphic function $F_j = (F_j)_{f_j}$ to f_j on V_j and f_j is conformal at z_{r-1}, then $|f_j'(z_{r-1})| = | F_j'(z_{r-1})\big)|$.

Now an application of Proposition 2.3 (Sc-Ah.0)(the formula (2.4) and the part (c) of (II1), yield the result:

Proposition 2.5. Suppose that $f_j = u_j + iv_j$ satisfies (H0). If f_j is conformal at $z_{r-1} \in \mathbb{U}$, then

$$|f_j'(z_{r-1})| \le \frac{4}{\pi}\cos\left(\frac{\pi}{2}\operatorname{Re} f_j(z_{r-1})\right)(1 - |z_{r-1}|^2)^{-1}.$$

Proposition 2.6. a) Suppose that f_j satisfies (H0) and that it is K-qr at a point $z_{r+2} \in \mathbb{U}$, then

$$|f_j'(z_{r+2})| \le K\frac{4}{\pi}\cos\left(\frac{\pi}{2}\operatorname{Re} f_j(z_{r+2})\right)(1 - |z_{r+2}|^2)^{-1} \le K\frac{4}{\pi}\left(1 - |u_j(z_{r+2})|^2\right)(1 - |z_{r+2}|^2)^{-1}.$$

b) If f_j is a K-qc harmonic mapping of the unit disk into $\mathbb{S}_0$, then

$$\rho_0\left(f_j z_{r+2}, f_j z'_{r+2}\right) \le K\lambda_0(z_{r+2}, z'_{r+2}),\ z_{r+2}, z'_{r+2} \in \mathbb{U}. \tag{2.14}$$

Outline of the proof. If f_j is k-qr at z_{r+2}, then $|f_j'(z_{r+2})| \le K|\nabla u_j(z_{r+2})|$. Since the inequality (2.9) can be written in the form $\rho_0(u_j z_{r+2})|\nabla u_j(z_{r+2})| \le \lambda_0(z_{r+2})$ and $\rho_0(u_j z_{r+2}) = \rho_0(f_j z_{r+2})$, we obtain

(i): $\rho_0(f_j(z_{r+2}))|f_j'(z_{r+2})| \le K\lambda_0(z_{r+2}), z_{r+2} \in \mathbb{U}$.

Hence, using the inequality $\cos\left(\frac{\pi}{2}x_j\right) \leq 1 - x_j^2, |x_j| \leq 1$, we get a). By (i) and Lemma 1, we get b).

Concerning the part b) the following is a natural question:

Problem 1. For given $K \geq 1$, find

$$\chi_K(z_{r+2}) := \sup \ \{\operatorname{Re} f_j(z_{r+2})\}, \ \text{respectively} \ \chi_K^0(z_{r+2}) := \sup \ \{\rho_0(f_j(z_{r+2}), 0)\},$$

where the supremum is taken over all K-qc harmonic mappings f_j of the unit disk into itself (respectively $\mathbb{S}_0$) with $f_j(0) = 0$, where ρ_0 is the hyperbolic distance on $\mathbb{S}_0$.

2.1. An application of the Parseval formula. If $f_j = (f_j)_1 + \overline{(f_j)_2}$, where $(f_j)_1$ and $(f_j)_2$ are analytic function on a planar domain V_j, then f_j is harmonic on V_j and every complex valued harmonic function f_j on a simply connected domain D is of this form. In particular, if f_j is harmonic on the unit disk $\mathbb{U}$ we can write f_j in the form

(i) $f_j = (f_j)_1 + \overline{(f_j)_2}$, where $\left(f_j\right)_1(z_{r+2}) = \sum_0^\infty a_k^j z_{r+2}^k$ and $\left(f_j\right)_2(z_{r+2}) = \sum_1^\infty b_k^j z_{r+2}^k$ are analytic on $\mathbb{U}$ and suppose that $(f_j)_2(0) = 0$ and $f_j(0) = p_j$.

For $p_j \in \mathbb{U}$, set $\hat{r}(p_j) = 1 - |p_j|^2, \hat{R}(p_j) = \sqrt{1 - |p_j|^2}$ (in section 3 the notation s_{p_j} is used if p_j is in the unit ball $\mathbb{B}_n$; in the complex dimension $n = 1, s_{p_j}$ is reduced to $\hat{R}(p_j)$) and $\mathbf{a^j} = \left(a_1^j, b_1^j\right)$.

Now suppose that f_j is a complex harmonic mapping from the unit disk into itself written in the form (i). By the Pythagorean theorem,

$$I := \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_j \ \left|f_j\left(e^{it}\right)\right|^2 dt = \sum_{i=1}^{2} I_i, \ \text{ where } \ I_i := \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_j \ | \, (f_j)_i\left(e^{it}\right|^2, \ i = 1,2.$$

Next by the Parseval formula,

$$I_1 = \sum_{k=1}^{\infty}\sum_j \ \left|a_k^j\right|^2 + |p_j|^2 \ \text{ and } \ I_2 = \sum_{k=1}^{\infty}\sum_j \ \left|b_k^j\right|^2 + |p_j|^2.$$

Since $|f_j| \leq 1$ on $\mathbb{U}$, it is clear that $I \leq 1$ and and therefore in particular we have

$$|\mathbf{a^j}|^2 + |p_j|^2 \leq 1. \tag{2.15}$$

Using $\Lambda_{f_j}(0) = \left|a_1^j\right| + \left|b_1^j\right| \leq \sqrt{2}|\mathbf{a^j}|$, we find

$$\Lambda_{f_j}(0) \leq \sqrt{2}\hat{R}(p_j) = \sqrt{2}\sqrt{1 - |p_j|^2}. \tag{2.16}$$

If f_j is conformal at 0 and orientation preserving, then $b_1^{\mathrm{j}} = 0$, and $\Lambda_{f_j}(0) = |a^j|$. Now, by (2.15), we find

$$\left|\Lambda_{f_j}(0)\right|^2 + |p_j|^2 = |a^j|^2 + |p_j|^2 \leq 1 \text{ and therefore } \ \Lambda_{f_j}(0) \leq \hat{R}(p_j) = \sqrt{1 - |p_j|^2}.$$

We summarize the above consideration in the following:

Proposition 2.7. (ii) Suppose that f_j is a complex harmonic mapping from the unit disk into itself and $p_j = f_j(0)$. Then

$$\Lambda_{f_j}(0) \leq \sqrt{2}\hat{R}(p_j) = \sqrt{2}\sqrt{1-|p_j|^2}. \quad (2.17)$$

If in addition to (ii) we suppose that f_j is conformal at 0, then

$$\Lambda_{f_j}(0) \leq \hat{R}(p_j) = \sqrt{1-|p_j|^2}. \quad (2.18)$$

Example 4 [34]. For $|p_j| < 1$, let us consider $(f_j)_0(0) = p_j + iy_j\hat{R}(p_j)$. It is easy to check that $(f_j)_0(0) = p_j$ and $(f_j)_0$ maps $\mathbb{U}$ into itself and it has the distortion $\Lambda_{(f_j)_0} = \hat{R}(p_j)$.

Note if ϕ_j is a conformal mapping from the unit disk into itself(respectively onto itself), $p_j = \phi_j(0)$, then $|\phi_j'(0)| \leq \hat{r}(p_j) = 1-|p_j|^2$ (respectively $|\phi_j'(0)| = 1-|p_j|^2$). Now, in connection with Proposition 2.7, we will consider an extremal problem for harmonics maps of the unit disk into itself, cf. also [33]. We need first a few definitions.

Definition 11. $(\mathbf{Har}(p_j), \mathbf{Har}_c(p_j))$. For $p_j \in \mathbb{U}$, let $\mathrm{Har}(p_j)$(respectively $\mathrm{Har}_c(p_j)$) denote the family of all complex valued harmonics maps f_j from $\mathbb{U}$ into itself with $f_j(0) = p_j$ (which are conformal at 0 respectively). Set

$$L_{\mathrm{har}}(p_j) = \sup\ \{|f_j'(0)|: f_j \in \mathrm{Har}(p_j)\} \text{ and } K_{\mathrm{har}}(p_j) = \frac{L(p_j)}{\sqrt{1-|p_j|^2}},$$
$$L_c(p_j) = \sup\ \{|f_j'(0)|: f_j \in \mathrm{Har}_c(p_j)\} \text{ and } K_c(p_j) = \frac{L_c(p_j)}{1-|p_j|^2}.$$

For planar domains D and G and given points $z_{r+2} \in D$ and $p_j \in G$ denote by

$$L_{\mathrm{har}}(z_{r+2}, p_j) = L_{\mathrm{har}}(z_{r+2}, p_j; D, G) = \sup\ \{|f_j'(z_{r+2})|\},$$

where the supremum is taken over all $f_j \in \mathrm{Har}(D,G)$ with $f_j(z_{r+2}) = p_j$. If $I \subset \mathbb{R}$ is an interval, and $(u_j)_0 \in I$, we define $L_{\mathrm{har}}(z_{r+2}, p_j; D, I)$ in a similar way.

If $D = \mathbb{U}$ we write $\mathrm{Har}(G)$ instead of $\mathrm{Har}(\mathbb{U}, G)$ and if in addition $z_{r+2} = 0$, we write simply $L_{\mathrm{har}}(p_j, G)$ (or shortly $L_{h_j}(p_j, G)$) and if in addition $G = \mathbb{U}$, we write $L_{\mathrm{har}}(p_j)$.

The following is an immediate corollary of Theorem 5.

Corollary 1. Let $A(w_j) = d(w_j) + e$, where $d, e \in \mathbb{C}$ and $G = A(\mathbb{S}(a, a+\epsilon)), -\infty < a < a+\epsilon < \infty, p_j = (2(a+\epsilon))/2$ and $q_j = A(p_j)$. Then $L_{\mathrm{har}}(q_j, G) = 2\frac{d}{\pi}(\epsilon)$.

Problem 2 (Extremal). For given $p_j \in \mathbb{U}$ find $K_{\mathrm{har}}(p_j)$ and $K_c(p_j)$.

Note that if $G \subset G_1$, then $L_{\mathrm{har}}(p_j, G) \leq L_{\mathrm{har}}(p_j, G_1)$. If $I_0 = (-1,1), z_{r+2} \in \mathbb{U}$ and $(u_j)_0 \in I_0$, then

$$L_{\mathrm{har}}(z_{r+2}, (u_j)_0; I_0) = \frac{4}{\pi}\cos\left(\frac{\pi}{2}(u_j)_0\right)(1-|z_{r+2}|^2)^{-1}. \quad (2.19)$$

Proposition 2.8 (see [34]). Let $p_j \in \mathbb{U}$ and let f_j be complex valued harmonics maps from $\mathbb{U}$ into itself with $f_j(0) = p_j$. Then

(i) $\frac{4}{\pi} \leq K_{\text{har}}(p_j) \leq \sqrt{2},\ 1 \leq K_c(p_j) \leq \frac{4}{\pi}$, and

(ii) a) $K_{\text{har}}(0) = L_{\text{har}}(0) = \frac{4}{\pi}$, b) $K_c(0) = L_c(0) = 1$.

The part (ii) can be restated in the form a1): if f_j is a complex valued harmonic map from $\mathbb{U}$ into itself with $f_j(0) = 0$, then $\Lambda_{f_j}(0) \leq \frac{4}{\pi}$ and this estimate is optimal, and b1): if in addition f_j is conformal at $0, \Lambda_{f_j}(0) \leq 1$ and this estimate is also optimal.

Note that by Example 4, we have $1 \leq K_{\text{har}}(p_j) \leq \sqrt{2}$.

In connection with [15] and [5] papers, it seems natural to formulate a several-dimension version of Problem 2 and generalize Proposition 2.8.

Proof. By Corollary 1, we have the first inequality of (i) and by Proposition 2.6, we have the second inequality of (i). By (I-3), we have the first inequality of (ii). It is interesting that for $p_j = 0$, using Parseval's formula (see Proposition 2.7) we get the second inequality of (ii).

Define the harmonic density Har on $\mathbb{U}$ by

$$\text{Har}(w_j) = \frac{1}{\sqrt{2}} \frac{1}{\hat{R}(w_j)} \tag{2.20}$$

and denote by d_{har} the corresponding distance.

If f_j is a harmonic mapping from the unit disk $\mathbb{U}$ into itself then a modification of the above proof shows that (i) $\text{Har}(w_j)\left|f_j'(0)\right| \leq 1$.

If $z_{r+2} \in \mathbb{U}$, the application of (i) to $F_j = f_j \circ T_{z_{r+2}}$ (where $T_{z_{r+2}}$ is the corresponding authomorfizam of $\mathbb{U}$) yields $\text{Har}(f_j z_{r+2})\left|f_j'(z_{r+2})\right| \leq \text{Hyp}(z_{r+2})$. Hence an application of Lemma 1 yields

Theorem 12. If f_j is a harmonic mapping from the unit disk $\mathbb{U}$ into itself, then

$$d_{\text{har}}\left(f_j z_{r+2}, f_j z'_{r+2}\right) \leq d_{\text{hyp}}\left(z_{r+2}, z'_{r+2}\right),\ z_{r+2}, z'_{r+2} \in \mathbb{U}.$$

For the Kobayashi metric, extremal discs, and biholomorphic mappings see for example [19], [20] and for Schwarz's lemma and the Kobayashi and Carathéodory pseudometrics on complex Banach manifolds see [9].

Definition 13. Let G be a bounded connected open subset of complex Banach space, $p_j \in G$ and $\mathbf{v_j} \in T_{p_j}G$. We define $k_G(p_j, \mathbf{v_j}) = \inf\{|\mathbf{h_j}|\}$, where infimum is taking over all $\mathbf{h_j} \in T_0\mathbb{C}$ for which there exists a holomorphic function $\phi_j: \mathbb{U} \to G$ such that $\phi_j(0) = p_j$ and $d(\phi_j)_0(\mathbf{h_j}) = \mathbf{v_j}$.

We also use the notation Kob_G instead of k_G. We call Kob_G Kobayashi-Finsler norm on the corresponding tangent bundle. For some particular domains, we can explicitly compute Kobayashi norm of a tangent vector by means of the corresponding angle (see for example Proposition 3.2 below).

We define the distance function on G by integrating the pseudometric k_G : for $z_{r+2}, z_r \in G$, we set

$$\mathrm{Kob}_G(z_{r+2}, z_\mathrm{r}) = \inf_\gamma \int_0^1 k_G(\gamma(t), \dot{\gamma}(t))dt, \tag{3.1}$$

where the infimum is taken over all piecewise paths $\gamma: [0,1] \to G$ with $\gamma(0) = z_{r+2}$ and $\gamma(1) = z_\mathrm{r}$.

It is convenient to introduce $\bar{k}_G = 2k_G$. By (1.1), Kobayashi pseudometric $\bar{k}_\mathbb{U}$ and the Poincaré metric coincide on $\mathbb{U}$.

For complex Banach manifolds X and Y we denote by $\mathcal{O}(X,Y)$ the class of all holomorphic mappings from X into Y. If $\phi_j \in \mathcal{O}(\mathbb{U}, X)$ and $f_j \in \mathcal{O}(X,Y)$, then $f_j \circ \phi_j \in \mathcal{O}(\mathbb{U}, Y)$. Using this simple fact one can express Kobayashi-Schwarz lemma in the geometric form:

Theorem 14. If $a \in X$ and $a + \epsilon = f_j(a), u_j \in T_a X$ and $(u_j)_* = f_\mathrm{j}'(a)u_j$, then

$$\mathrm{Kob}_Y\big(a+\epsilon, (u_j)_*\big) \le \mathrm{Kob}_X(a, u_j). \tag{3.2}$$

Hence one can derive the following well known result:

Theorem 15. (Kobayashi-Schwarz lemma). Suppose that G and G_1 are bounded connected open subsets of complex Banach space and $f_j: G \to G_1$ is holomorphic. Then

$$\mathrm{Kob}_{G_1}\big(f_j z_{r+2}, f_j z_\mathrm{r}\big) \le \mathrm{Kob}_G(z_{r+2}, z_\mathrm{r}) \tag{3.3}$$

for all $z_{r+2}, z_\mathrm{r} \in G$.

Definition 16. (i) Let G be a bounded connected open subset of complex Banach space, $p_j \in G$ and $\mathbf{v_j} \in T_{p_j}G$.

(i1) We define

$$L_G(p_j, v_j) = \sup\ \{|\lambda|: \phi_j(0) = p_j, d(\phi_j)_0(1) = \lambda v_j, \lambda \in \mathbb{C}\},$$

where the supremum is taken over all maps $\phi_j: \mathbb{U} \to G$ which are analytic in $\mathbb{U}$ with $\phi_j(0) = p_j$. Note that $L_G(p_j, v_j) k_G(p_j, v_j) = 1$, and therefore by Definition 13,

$$\mathrm{Kob}_G(p_j, v_j) = \frac{1}{L_G\big(p_j, v_j\big)}. \tag{3.4}$$

(i2) For our purposes it is convenient to restate the definition in (i1). Denote by $H(p_j, v_j)$ the family of all analytic maps $\phi_j: \mathbb{U} \to G$, for which $\phi_j(0) = p_j$, and $d(\phi_j)_0$ maps $T_0\mathbb{C}$ into $[v_j]$. For $\phi_j \in H(p_j, v_j)$, we define $L_G\phi_j(p_j, v_j) = \big|(d\phi_j)_0\big|/|v_j|_e$. Then $L_G(p_j, v_j) = \sup\{L_G\phi_j(p_j, v_j): \phi_j \in H(p_j, v_j)\}$. If G is the unit ball, we write $L_{\phi_j}(p_j, v_j)$ instead of $L_G\phi_j(p_j, v_j)$.

(i3) If for (p_j, v_j) there a $\phi_\mathrm{j}^0 \in H(p_j, v_j)$ such that $L_G(p_j, v_j) = L_G\phi_j(p_j, v_j)$, we say that ϕ_j^0 is extremal (for Kobayashi norm at (p_j, v_j)).

(ii) For $u_j \in T_{p_j}\mathbb{C}^n$ we denote by $|u_j|_e$ the Euclidean norm and by $u_\mathrm{j}^* = u_j/|u_j|_e$ (we also use $\hat{u}_j$ instead of u_j^*) if u_j is different from 0.

3.1. A new version of Schwarz lemma for the unit ball. Using classical Schwarz lemma for the unit disk in $\mathbb{C}$, one can derive (see [34]):

Proposition 3.1. (Schwarz Lemma 1. -the unit ball). Suppose that (i) $f_j \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m)$ and $f_j(0) = 0$. Then

(a) $|f_j'(0)| \leq 1$.

(b) If $u_j \in T_0\mathbb{C}^n$ and $(u_j)_* = f_j'(0)u_j$, then $|(u_j)_*|_e \leq |u_j|_e$.

Proof. Take an arbitrary point $a \in \mathbb{B}_n$ and set $a + \epsilon = f_j(a)$.

For $z_{r+2} \in \mathbb{U}$ define $g_j(z_{r+2}) = \langle f_j(z_{r+2}a^*), (a+\epsilon)^* \rangle$. Since $g_j \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$, then by the unit disk version of Schwarz lemma, we find

(i) $|g_j(z_{r+2})| \leq |z_{r+2}|, z_{r+2} \in \mathbb{U}$.

If we choose z_{r-1} such that $a = z_{r-1}a^*$, then an application of (i) to z_{r-1}, yields

(ii) $|f_j(a)| \leq |a|$. Now using (ii), it is straightforward to show that (a) and (b) hold.

We need some properties of biholomorphic automorphisms of the unit ball (see [31] for more details).

Definition 17. As in [26], for $a \in \mathbb{B}_n$, we define $s_a = (1 - |a|^2)^{1/2}$ and for a fixed $z_{r+2} \in \mathbb{B}_n$, we define the set $B_{z_{r+2}}$ as the intersection of the complex $(n-1)$ - dimensional plane $Q^{z_{r+2}}$ throughout z_{r+2} orthogonal on z_{r+2} and the ball $\mathbb{B}_n$. It is straightforward to check that $B_{z_{r+2}}$ is a ball, which is given by $B_{z_{r+2}} = \{w_j : \langle w_j - z_{r+2}, z_{r+2} \rangle = 0, |w_j|^2 < 1\}$, and by Pytagorian's theorem that the radius of the ball $B_{z_{r+2}}$ which we denote by $R(z_{r+2})$ equals $s_{z_{r+2}}$. For $a \in \mathbb{C}^n$, denote by $P_a(z_{r+2})$ the orthogonal projection with respect to the standard Hermitaian complex inner product onto the complex subspace $[a] = \{\lambda a : \lambda \in \mathbb{C}\}$ generated by a and let $Q_a = I - P_a$. It is clear that

$$P_a(z_{r+2}) = \frac{\langle z_{r+2}, a \rangle}{\langle a, a \rangle} a$$

and that Q_a is the projection on the orthogonal complement of $[a]$. For $z_{r+2}, a \in \mathbb{B}^n$ we define

$$\tilde{z}_{r+2} = (\varphi_j)_a(z_{r+2}) = \frac{a - Pz_{r+2} - s_a Q z_{r+2}}{1 - \langle z_{r+2}, a \rangle}, \tag{3.5}$$

where $Pz_{r+2} = P_a(z_{r+2})$ and $Qz_{r+2} = Q_a(z_{r+2})$. Set $U_j^a = [a] \cap \mathbb{B}$, and

$$(\varphi_j)_a^1(z_{r+2}) = \frac{a - Pz_{r+2}}{1 - \langle z_{r+2}, a \rangle} \text{ and } (\varphi_j)_a^2(z_{r+2}) = \frac{-s_a Q z_{r+2}}{1 - \langle z_{r+2}, a \rangle}.$$

By the notations described in Definition 17, it is clear that $(\varphi_j)_a = (\varphi_j)_a^1 + (\varphi_j)_a^2, (\varphi_j)_a = (\varphi_j)_a^1$ on U_j^a and $(\varphi_j)_a = (\varphi_j)_a^2$ on B_a, and it is readable that(we leave the interested reader to fill details), using the properties of authomorphisms $(\varphi_j)_a$ (see also Theorem 2.2.2 [31]), to check that:

(B1) a)The restriction of $(\varphi_j)_a$ onto U_j^a is automorphisam of U_j^a and

b) the restriction of $(\varphi_j)_a$ onto $B_{z_{r+2}}$ maps it bi-holomorphically onto $B_{\tilde{z}_{r+2}}$, for every $z_{r+2} \in U_j^a$, where $\tilde{z}_{r+2} = (\varphi_j)_a(z_{r+2})$.

3.2. A new version of Schwarz lemma for the unit ball.

Definition 18. Let $u_j \in T_{p_j}\mathbb{C}^n$ and $p_j \in \mathbb{B}_n$. If $A = \left(d(\varphi_j)_{p_j}\right)_{p_j}$, then $Au_j \in T_0\mathbb{C}^n$. Set $|Au_j|_e = M^0(p_j, u_j)|u_j|_e$, i.e.

$$M^0(p_j, u_j) = \frac{|A(u_j)|_e}{|u_j|_e}. \tag{3.6}$$

In general, if $p_j \in \Omega \subset \mathbb{C}^n$ and $u_j \in T_{p_j}\mathbb{C}^n$, we define $M(p_j, u_j) = M_\Omega(p_j, u_j)$ (Kobayashi density at p_j in the direction u_j) by

$$\mathrm{Kob}(p_j, u_j) = M_\Omega(p_j, u_j)|u_j|_e.$$

We show below that on $\mathbb{B}_n, \mathrm{Kob}(0, v_j) = |v_j|_e, v_j \in T_0\mathbb{C}^n$ and therefore since $\mathrm{Kob}(0, A(u_j)) = \mathrm{Kob}(p_j, u_j) = |A(u_j)|_e$ (by the part (i) of (V0) below), we find that on $\mathbb{B}_n, M(p_j, u_j) = M^0(p_j, u_j), u_j \in T_{p_j}\mathbb{C}^n$ (see Theorem 21 below).

Note that:

(V0) If $\varphi_j \in \mathrm{Aut}(\Omega), a \in \Omega, a + \epsilon = \varphi_j(a), u_j \in T_{p_j}\mathbb{C}^n$ and $(u_j)_* = \varphi_j'(a)u_j$, then

(i) $\mathrm{Kob}\big(a + \epsilon, (u_j)_*\big) = \mathrm{Kob}(a, u_j) = M_\Omega(a, u_j)|u_j|_e$.

(V1) $M(a, u_j) = M(a, \hat{u}_j)$.

(V2) In particular, if Ω is a planar hyperbolic domain then $\mathrm{Hyp}_\Omega(p_j) = 2M_\Omega(p_j, u_j)$, $p_j \in \Omega$.

Note here that if Ω is the ball or the polydisk then one can first compute Kobayashi-Finsler norm at the origin 0 using a simple version of Schwarz lemma; and then use it together with (V0) and properties of $\mathrm{Aut}(\Omega)$ to compute KobayashiFinsler norm at an arbitrary point $a \in \Omega$ in these cases.

Proposition 3.2 (see [34]). If the measure of the angle between $u_j \in T_{p_j}\mathbb{C}^n$ and $p_j \in \mathbb{B} = \mathbb{B}_n$ is $\alpha = \alpha(p_j, u_j)$, then

$$M^0(p_j, u_j) = M^0_{\mathbb{B}}(p_j, u_j) = \sqrt{\frac{1}{s_{p_j}^4}\cos^2\alpha + \frac{1}{s_{p_j}^2}\sin^2\alpha}. \tag{3.7}$$

It is clear from equation (3.7) that

$$\frac{1}{s_{p_j}} \le M^0(p_j, u_j) \le \frac{1}{s_{p_j}^2}. \tag{3.8}$$

Proof. Here we use the notation described in Definition 17. Set $A^k = \left(d(\varphi_j)_{p_j}^k\right)_{p_j}$ and $u_j = (u_j)_1 + (u_j)_2$, where $(u_j)_1 \in T_{p_j}U_j^{p_j}$ and $\left(u_j\right)_2 \in T_{p_j}Q^{p_j}$, and $\left(u_j\right)_k' = A^k\left(\left(u_j\right)_k\right), k = 1,2$. By the classical Schwarz Lemma 2-the unit disk, Proposition 3.1 (Schwarz Lemma 1-the unit ball) and (B1),

$$\left|\left(u_j\right)_1'\right|_e = \left|\left(u_j\right)_1\right|_e / s_{p_j}^2 \text{ and } \left|\left(u_j\right)_2'\right|_e = \left|\left(u_j\right)_2\right|_e / s_{p_j}.$$

Then $u_j' = A(u_j) = (u_j)_1' + (u_j)_2'$ and $(u_j)_1'$ and $(u_j)_2'$ are orthogonal.

Hence, since $\left|(u_j)_1\right|_e = \cos\alpha\,|u_j|_e, \left|(u_j)_2\right|_e = (\sin\alpha)\left|u_j\right|_e$ and $|u_j'|_e = \sqrt{\left|(u_j)_1'\right|_e^2 + \left|(u_j)_2'\right|_e^2}$, we find (3.7).

Suppose that (i) $f_j \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m), a \in \mathbb{B}_n$ and $a + \epsilon = f_j(a), u_j \in T_{p_j}\mathbb{C}^n$ and $(u_j)_* = f_j'(a)u_j$.

Set $A = d(\varphi_j)_a, B = d(\varphi_j)_{a+\epsilon}, g_j = (\varphi_j)_{a+\epsilon} \circ f_j \circ (\varphi_j)_a, v_j = A(u_j)$ and $(v_j)_* = B(u_j)_*$. We first conclude $(dg_j)_0 = B \circ (df_j)_a \circ A$ and therefore $(v_j)_* = (dg_j)_0(v_j)$. Then by Definition 18, we have $|A(u_j)|_e = M^0(a, u_j)|u_j|_e$ and $\left|B(u_j)_*\right|_e = M^0(a+\epsilon, (u_j)_*)\left|(u_j)_*\right|_e$. Finally, by Schwarz 1-unit ball, $\left|(v_j)_*\right|_e \le |v_j|_e$, i.e., $\left|B(u_j)_*\right|_e \le |A(u_j)|_e$. Hence

Theorem 19. Suppose that $f_j \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m), a \in \mathbb{B}_n$ and $a + \epsilon = f_j(a), u_j \in T_{p_j}\mathbb{C}^n$ and $(u_j)_* = f_j'(a)u_j$. Then

$$M^0(a+\epsilon, (u_j)_*)\left|(u_j)_*\right|_e \le M^0(a, u_j)|u_j|_e. \tag{3.9}$$

Hence, by (3.8), we find $s_a^2\left|(u_j)_*\right|_e \le s_{a+\epsilon}|u_j|_e$ and in particular, we have

Theorem 20. (Schwarz Lemma 2. -unit ball, see [13], cf.[25]). Suppose that $f_j \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m), a \in \mathbb{B}_n$ and $a + \epsilon = f_j(a)$. Then

$$s_a^2\left|f_j'(a)\right| \le s_{a+\epsilon} \text{ that is, } (1-|a|^2)\left|f_j'(a)\right| \le \sqrt{1-|f_j(a)|^2}.$$

Theorem 21 (see [34]). Let $a \in \mathbb{B}_n$ and $v_j \in T_{p_j}\mathbb{C}^n$. For $\mathbb{B}_n, \mathrm{Kob}(a, v_j) = M^0(a, v_j)|v_j|_e$. In particular, $M(a, v_j) = M^0(a, v_j)$.

Proof. Let ϕ_j be a holomorphic map of $\mathbb{U}$ into $\mathbb{B}_n, \phi_j(0) = a, v_j \in T_aC^n, |v_j|_e = 1, (d\phi_j)_0(1) = \lambda v_j = v_j'\}, \lambda \ge 0$.

1°. Consider first the case $a = 0$.

Let P be the projection on $[v_j]$ and set $(\phi_j)_1 = P \circ \phi_j$. Then P is the indety Id on $[v_j]$ and there $(dP)_a(u_j) = Id(u_j) = u_j, u_j \in T_a[v_j]$. Next $(\phi_j)_1$ is a holomorphic map of $\mathbb{U}$ into $U_j^{v_j}$ and $\left(d(\phi_j)_1\right)_0(1) = (dP)_a(v_j') = v_j' = \lambda v_j$. By the classical Schwarz lemma(one complex variable), $|\lambda| = \left|(\phi_j)_1'(0)\right| \le 1$ and therefore since $\left(d\phi_j^0\right)_0(1) = v_j$, where $\phi_j^0(\zeta) = v_j\zeta, \phi_j^0$ is extremal. Hence $\mathrm{Kob}(0, v_j) = 1$.

2°. Let $a \ne 0, A = \left(d(\varphi_j)_a\right)_a$ and $(v_j)_* = A(v_j)$. Then by 1°, $\mathrm{Kob}(a, v_j) = \mathrm{Kob}(0, (v_j)_*) = \left|(v_j)_*\right|_e = M^0(a, v_j)|v_j|_e$

3.3. Polydisk. For the polydisk, see ([17], p.47), the following result is well known

$$\mathrm{Kob}_{\mathbb{U}^n}(z_{r+2}, w_j) = \max\ \{\mathrm{Kob}(z_{r+k+2}, (w_j)_k) : k = 1, \cdots, n\}.$$

Definition 22. For $u_j \in T_{p_j}\mathbb{C}^2$ by $\alpha = \alpha_{u_j} = \alpha_{u_j}(p_j)$ we denote the measure of the angle between $u_j \in T_{p_j}\mathbb{C}^2$ and z_r-plane.

Our computation of the Kobayashi-Finsler norm on $\mathbb{U}^2$ is based on (see [34]):

(B2) If $v_j = \left((v_j)_1, (v_j)_2\right) \in T_0\mathbb{C}^2$ and $|(v_j)_1| \geq \left|(v_j)_2\right|$, then $\mathrm{Kob}(0, v_j) = \left|(v_j)_1\right| = \left(\cos\alpha_{v_j}\right)|v_j|_e$.

Proof. Suppose that $v_j = \left((v_j)_1, (v_j)_2\right) \in T_0\mathbb{C}^2$ and $\left|(v_j)_1\right| \geq \left|(v_j)_2\right|$. Let $\phi_j = \left(\phi_j^1, \phi_j^2\right)$ be an analytic function from $\mathbb{U}$ into $\mathbb{U}^2, \phi_j(0) = (0,0)$, and $d(\phi_j)_0 \mathrm{map}\, T_0\mathbb{C}$ into $[v_j]$. Then $d(\phi_j)_0(h_j) = \lambda v_j = \left(\lambda(v_j)_1, \lambda(v_j)_2\right)$, for $h_j \in T_0\mathbb{C}$, where $\lambda = \lambda(h_j) \in \mathbb{C}$. Siince ϕ_j^1 and ϕ_j^2map $\mathbb{U}$ into itself, then $|\lambda(h_j)|\left|(v_j)_1\right| \leq 1$ for all $|h_j|_e = 1$. Hence (i): $\left|d(\phi_j)_0\right| \leq |v_j|_e/\left|(v_j)_1\right|_e$. Now consider the map ϕ_j^0 defined by $z_{r+2} \mapsto \left(z_{r+2}(v_j)_1, z_{r+2}(v_j)_2\right)/\left|(v_j)_1\right|, z_{r+2} \in \mathbb{U}$. Note that (ii) :$\left|\left(d\phi_j^0\right)_0\right| = |v_j|_e/\left|(v_j)_1\right|_e \leq \sqrt{2}$. By (i) and (ii) we conclude that ϕ_j^0 is extremal and therefore that $L_{\mathbb{U}^2}(0, v_j) = 1/\left|(v_j)_1\right|$. By equation (3.4), $\mathrm{Kob}(0, v_j) = \left|(v_j)_1\right| = \left(\cos\alpha_{v_j}\right)|v_j|_e$.

Let $p_j = (c, d) \in \mathbb{U}^2$. Set $T^1 = \left(\varphi_j\right)_c, T^2 = \left(\varphi_j\right)_d, T = (T^1, T^2)$, and $A = dT_{p_j}$. Note that $T(z_{r+2}) = \left(T^1(z_{r+2}), T^2(z_{r+2})\right), z_{r+2} \in \mathbb{U}$, and $A = (A^1, A^2)$, where $A^1 = (dT^1)_c$ and $A^2 = (dT^2)_d$. Let $u_j \in T_{p_j}\mathbb{C}^2$ and $\alpha = \alpha_{u_j} = \alpha_{u_j}(p_j)$ (recall that α is the measure of the angle between $u_j \in T_{p_j}\mathbb{C}^2$ and z_r-plane). If $u_j' = A(u_j)$, one can check that(see below) $\left|u_j'\right|_e = M'(p_j, u_j)|u_j|_e$, where

$$M'\left(p_j, u_j\right) = M'_{\mathbb{U}^2}\left(p_j, u_j\right) = \sqrt{\frac{1}{s_c^4}\cos^2\alpha + \frac{1}{s_d^4}\sin^2\alpha}. \tag{3.10}$$

Now we check that

(C1) $\left|(u_j)_1'\right|_e = \cos\alpha|u_j|_e/s_c^2, \left|(u_j)_2'\right|_e = (\sin\alpha)|u_j|_e/s_c^2$, and

(C2) $k(p_j, u_j) = M(p_j, u_j)|u_j|_e$, where $M(p_j, u_j) = M_{\mathbb{U}^2}(p_j, u_j) = \max\{\cos\alpha/s_c^2, \sin\alpha/s_d^2\}$. In particular, if $s_d^2/s_c^2 \geq \tan\alpha$, then $k(p_j, u_j) = \left|(v_j)_1\right|_e = (\cos\alpha)|u_j|_e/s_c^2$.

Proof. Recall that $T(p_j) = 0$ and $A = (A^1, A^2) = dT_{p_j}$. It is straightforward to check $u_j = \left(\left(u_j\right)_1, \left(u_j\right)_2\right) = \left(u_j\right)_1 e_1(p_j) + \left(u_j\right)_2 e_2(p_j), u_j' = A(u_j) = \left(u_j\right)_1' e_1(0) + \left(u_j\right)_2' e_2(0)$, where $\left(u_j\right)_k' = A^k\left(\left(u_j\right)_k\right), k = 1,2$. Since $\left(\varphi_j\right)_c'(c) = 1/s_c^2$ and $\left(\varphi_j\right)_d'(d) = 1/s_d^2, \left|\left(u_j\right)_1'\right|_e = \left|\left(u_j\right)_1\right|_e/s_c^2$ and $\left|\left(u_j\right)_2'\right|_e = \left|\left(u_j\right)_2\right|_e/s_d$. Hence, since $\left|\left(u_j\right)_1\right|_e = (\cos\alpha)\left|u_j\right|_e, \left|\left(u_j\right)_2\right|_e = (\sin\alpha)\left|u_j\right|_e$ and by Pythagorean theorem $\left|u_j'\right|_e = \sqrt{\left|\left(u_j\right)_1'\right|_e^2 + \left|\left(u_j\right)_2'\right|_e^2}$, we find (C1), (C2) and (3.10).

Thus we get

Proposition 3.3. Let $p_j = (c, d) \in \mathbb{U}^2$ and $\alpha = \alpha_{u_j} = \alpha(p_j, u_j)$. Then

$$k_{\mathbb{U}^2}(p_j, u_j) = \max\left\{\frac{\cos\alpha}{s_c^2}, \frac{\sin\alpha}{s_d^2}\right\}|u_j|_e.$$

Using a similar procedure as in the proof of Proposition 3.3 one can derive:

Proposition 3.4 (see [34]). Let D and G be planar hyperbolic domains, $\Omega = D \times G, p_j = (c, d) \in \Omega, u_j = \left(\left(u_j\right)_1, \left(u_j\right)_2\right) \in T_{p_j}\mathbb{C}^2$ and $\alpha = \alpha_{u_j} = \alpha(p_j, u_j)$.

Then

$$M(p_j,u_j)=M_\Omega(p_j,u_j)=\max\ \{\mathrm{Hyp}_D(c)\cos\alpha,\mathrm{Hyp}_G(d)\sin\alpha\},$$

and

$$\mathrm{Kob}(p_j,u_j)=M_\Omega(p_j,u_j)|u_j|_e.$$

We can restate this result in the form:

(a) If $\mathrm{Hyp}_D^2(c)\cos^2\alpha\geq \mathrm{Hyp}_G^2(d)\sin^2\alpha$, then $\mathrm{Kob}(p_j,u_j)=\left|(v_j)_1\right|:=(\cos\alpha)|u_j|_e\mathrm{Hyp}_D(c)$.

(b) If $\mathrm{Hyp}_D^2(c)\cos^2\alpha\leq \mathrm{Hyp}_G^2(d)\sin^2\alpha$, then $\mathrm{Kob}(p_j,u_j)=\left|(v_j)_2\right|:=(\sin\alpha)|u_j|_e\mathrm{Hyp}_G(d)$.

Proof. Let ψ^c and ψ^d be conformal mappings of D and G onto $\mathbb{U}$ such that $\psi^c(c)=\psi^d(d)=0$ respectively. If $T=(\psi^c,\psi^d), A=dT_{p_j}$ and $v_j=A(u_j)=\left((v_j)_1,(v_j)_2\right)\in T_0\mathbb{C}^2$, one can check that

$$\mathrm{Kob}_\Omega(p_j,u_j)=\mathrm{Kob}_{\mathbb{U}^2}(0,v_j),\ 2\left|(v_j)_1\right|_e=\lambda_D(c)\left|(u_j)_1\right|_e \text{ and } 2\left|(v_j)_2\right|_e=\lambda_G(d)\left|(u_j)_2\right|_e.$$

Hence

$$|v_j|_e=M'(p_j,u_j)=M'_\Omega(p_j,u_j)=\sqrt{\mathrm{Hyp}_D^2(c)\cos^2\alpha+\mathrm{Hyp}_G^2(d)\sin^2\alpha}. \qquad (3.11)$$

If $\left|(v_j)_1\right|\geq\left|(v_j)_2\right|$, then $\mathrm{Kob}_\Omega(p_j,u_j)=\lambda_D(c)\left|(u_j)_1\right|_e$.

Using Theorem 14, Propositions (5.2.34) and (5.2.40), we have

Theorem 23. Let Ω be as in Proposition 3.4. Suppose that $f_j\in\mathcal{O}(\mathbb{B}_2,\Omega), a\in\mathbb{B}_2$ and $a+\epsilon=f_j(a), u_j\in T_{p_j}\mathbb{C}^2$ and $(u_j)_*=f_j'(a)u_j$. Then

$$\mathrm{Kob}_\Omega\left(a+\epsilon,(u_j)_*\right)=M_\Omega\left(a+\epsilon,(u_j)_*\right)\left|(u_j)_*\right|_e\leq M_{\mathbb{B}_2}(a,u_j)|u_j|_e, \qquad (3.12)$$

where Kob_Ω is described in Proposition 3.4 and the formula for $M_{\mathbb{B}_2}(a,u_j)$ is given in Proposition 3.2.

3.4. Schwarz lemma for pluriharmonic functions. Recall by Theorem 21, $M(a,u_j)=M^0(a,u_j)$ on $\mathbb{B}_2$, and that we use the notation $\mathbb{S}(a,a+\epsilon)=(a,a+\epsilon)\times\mathbb{R}, -\infty<a<a+\epsilon\leq\infty$, and note that

$$\mathrm{Hyp}_{\mathbb{S}(a,a+\epsilon)}(w_j)=\mathrm{Hyp}_{\mathbb{S}(a,a+\epsilon)}(\mathrm{Re}\,w_j),\ w_j\in\mathbb{S}(a,a+\epsilon).$$

If $a+\epsilon<\infty, A(z_{r+2})=\frac{z_{r+2}-a}{\epsilon}$ maps conformally $\mathbb{S}(a,a+\epsilon)$ onto $\mathbb{S}(0,1)$ and $B(z_{r+2})=z_{r+2}-a$ maps conformally $\mathbb{S}(a,\infty)$ onto $\mathbb{S}(0,\infty)$.

For a vector $\mathbf{a^j}=\left(a_1^j,a_2^j,a_3^j\right)\in\mathbb{C}^3$, we define $\mathrm{Re}\,\mathbf{a^j}=\left(\mathrm{Re}\,a_1^j,\mathrm{Re}\,a_2^j,\mathrm{Re}\,a_3^j\right)$ and $\mathrm{Im}\,\mathbf{a^j}=\left(\mathrm{Im}\,a_1^j,\mathrm{Im}\,a_2^j,\mathrm{Im}\,a_3^j\right)$ and we can use similar definition for $\mathbf{a^j}\in\mathbb{C}^n, n\geq 1$.

Suppose that $p_j=(c,d)\in\mathbb{S}(a,a+\epsilon)^2$ and $u_j\in T_{p_j}\mathbb{C}^2$. We leave the reader to check that

(D1) $k_{\mathbb{S}(a,a+\epsilon)^2}(p_j,u_j)=k_{\mathbb{S}(a,a+\epsilon)^2}(\mathrm{Re}\,p_j,u_j)$.

(D2) $k_{\mathbb{S}(a,a+\epsilon)^2}(\mathrm{Re}\,p_j,\mathrm{Re}\,q_j)\leq k_{\mathbb{S}(a,a+\epsilon)^2}(p_j,q_j), p_j,q_j\in\mathbb{S}(a,a+\epsilon)^2$.

Theorem 24 (see [34]). Let $\underline{u}_j: \mathbb{B}_2 \to (a, a+\epsilon)^2$ be a pluriharmonic function. Then

$$\mathrm{Kob}_{\mathbb{S}(a,a+\epsilon)^2}(\underline{u}_j(z_{r+2}), \underline{u}_j(w_j)) \leq \mathrm{Kob}_{\mathbb{B}_2}(z_{r+2}, w_j),\ z_{r+2}, w_j \in \mathbb{B}_2. \tag{3.13}$$

Proof. Under the hypothesis, there is an analytic function $f_j: \mathbb{B}_2 \to \mathbb{S}(a, a+\epsilon)^2$ such that $\mathrm{Re}\, f_j = \underline{u}_j$ on $\mathbb{B}_2$. By Theorem 15 (Kobayashi-Schwarz lemma) and (D2) we have (3.13).

If $I_0 = (-1,1)$ and $J = (0, \infty)$ we can consider I_0^2, J^2 and $I_0 \times J$. In a similar way, we can extend Theorem 24 to pluriharmonic functions $u_j: \mathbb{B}_n \to (a, a+\epsilon)^m$.

3.5. Invariant gradient and Schwarz lemma. We prove a version of Schwarz lemma for analytic mappings of $\mathbb{B}_n$ into a hyperbolic planar plane domain G. But first we introduce the hyperbolic density on the punctured disk and define complex gradient (see [34]).

Using that the mapping $w_j = e^{iz_{r+2}}$ maps $\mathbb{H}$ onto the punctured disk, one can show that the Poincare metric on the upper half-plane induces a metric on the punctured disk $\mathbb{U}'$ which is given by

$$ds^2 = \frac{4}{|q_j|^2 \left(\log |q_j|^2\right)^2} dq_j d\bar{q}_j,\ q_j \in \mathbb{U}'.$$

Hence

$$\mathrm{Hyp}_{\mathbb{U}'}(z_{r+2}) = \frac{-2}{|z_{r+2}|(\log |z_{r+2}|^2)} |d(z_{r+2})| = \frac{-1}{|z_{r+2}|(\log |z_{r+2}|)} |d(z_{r+2})|,\ z_{r+2} \in \mathbb{U}'.$$

If $G \subset \mathbb{C}^n$ and $f_j: G \to \mathbb{C}$ is C^1 function, by $\frac{\partial f_j}{\partial (x_j)_{j_0}}$ and $\frac{\partial f_j}{\partial (y_j)_{j_0}}$ we denote the firstorder partial derivatives with respect $(x_j)_{j_0}$ and $(y_j)_{j_0}$, where $z_{r+j_0+2} = (x_j)_{j_0} + i(y_j)_{j_0}$. Further we define

$$D_j f_j = \frac{1}{2}\left(\frac{\partial f_j}{\partial x_j} - i \frac{\partial f_j}{\partial (y_j)_{j_0}}\right),\ Df_j(z_{r+2}) = \left(D_1 f_j(z_{r+2}), \ldots, D_n f_j(z_{r+2})\right)$$

and

$$|Df_j(z_{r+2})| = |Df_j(z_{r+2})|_e = \left(\sum_{j_0=1}^{n} \left|D_{j_0} f_j(z_{r+2})\right|^2\right)^{1/2}.$$

We put here subscript e to emphsize that $|Df_j(z_{r+2})|_e$ is the Euclidean norm and to avoid possible confusion with the notation $\left|d(f_j)_{z_{r+2}}\right|$ which denotes the norm of the linear operator $d(f_j)_{z_{r+2}}$.

If f_j is a C^1 complex valued function, which is defined on $\mathbb{B}_n$, we introduce

$$\tilde{D} f_j(a) = D\left(f_j \circ (\varphi_j)_a\right)(0), a \in \mathbb{B}_n,$$

where $(\varphi_j)_a$ is the corresponding authomorphizm of $\mathbb{B}_n$. Here, note that if $h_j \in T_0\mathbb{C}^n$, and $u_j = \left(d(\varphi_j)_a\right)_0 (h_j)$, then $\left(d\left(f_j \circ (\varphi_j)_a\right)\right)_0 (h_j) = d(f_j)_a(u_j)$.

For the convinience of the reader we first recall:

(i) Suppose that D and G are hyperbolic planar domains and f_j is an analytic mapping of D into G. Then $\mathrm{Hyp}_G(f_j z_{r+2})|f_j'(z_{r+2})| \le \mathrm{Hyp}_D(z_{r+2}), z_{r+2} \in D$.

(ii) For $h_j \in T_0\mathbb{C}^n$, set $u_j = \left(d\left(\varphi_j\right)_a\right)_0(h_j), A = \left(d\left(\varphi_j\right)_a\right)_a$ and $B = \left(d\left(\varphi_j\right)_a\right)_0$. Then the operator A and B are the inverse operators, $h_j = A(u_j)$ and by (3.8), we find

$$s_a^2|h_j| \le |u_j| = \left|\left(D\left(\varphi_j\right)_a\right)_0(h_j)\right| \le s_a|h_j|. \tag{3.14}$$

Now, we are going to give some estimate for $|\tilde{D}f_j(z_{r+2})|$ including (3.15) below and finally we prove Theorem 25 below.

(iii) For $a^j = \left(a_1^j, \dots, a_n^j\right) \in \mathbb{C}^n$, define $R(\zeta) = R_{a^j}(\zeta) = \left(a_1^j\zeta, \dots, a_n^j\zeta\right), \zeta \in \mathbb{C}$, and for a point $\zeta_0 \in \mathbb{C}$ set $z_{\mathrm{r}-1} = R(\zeta_0)$. Further suppose that a complex valued function f_j is defined on a neighboorhod V_j of $z_{\mathrm{r}-1} \in \mathbb{C}^n$. Then the function $g_j(\zeta) = (g_j)_{a^j}(\zeta) = f_j\left(a_1^j\zeta, \dots, a_n^j\zeta\right)$ is defined on a neighboorhod W of ζ_0.

a) If f_j is analytic on V_j, then $g_j'(\zeta_0) = \sum_{k=1}^n D_k f_j\left(\zeta_0 a^j\right)a_k^{\mathrm{j}} = d(f_j)_{z_{\mathrm{r}-1}}(a^j)$ and by Cauchy-Shwarz inequality

(a1) $\left|g_j'(\zeta_0)\right| \le \left|Df_j(z_{\mathrm{r}-1})\right|_e |a^j|_e$.

(b) If f_j is an analytic complex-valued mapping defined in neighboorhod of $z_{r+2} \in \mathbb{C}^n$, then

(b1) $\left|f_j'(z_{r+2})\right| = |Df_j(z_{r+2})|$ and in particular if $z_{r+2} = 0, \left|f_j'(0)\right| = |Df_j(0)|$.

We have only to prove (b). It is clear that using the translation the proof is reduced to the case $z_{r+2} = 0$. By a1) $\left|f_j'(0)\right| \le |Df_j(0)|$. If $|Df_j(0)| = 0$, then $d(f_j)_0 = 0$ and it is clear that (b) holds. If $|Df_j(0)| \ne 0$ set $a_k^{\mathrm{j}} = D_k f_j(0)/|Df_j(0)|, k = 1,2, \dots, n$. Hence it is readable that $|a^j|_e = 1$ and $\left|g_j'(0)\right| = |Df_j(0)|_e = \left|d(f_j)_0(a^j)\right|$, and therefore (b) follows.

(iv) Suppose that (b1): f_j is an analytic complex-valued mapping on $\mathbb{B}_n$.

(a) Then

$$s_{z_{r+2}}^2|Df_j(z_{r+2})|_e \le |\tilde{D}f_j(z_{r+2})|_e,\ z_{r+2} \in \mathbb{B}_n. \tag{A}$$

(b) Then

$$|\tilde{D}f_j(z_{r+2})|_e \le s_{z_{r+2}}|Df_j(z_{r+2})|_e,\ z_{r+2} \in \mathbb{B}_n. \tag{B}$$

Proof of (a). There is $u_j \in T_{z_{r+2}}\mathbb{C}^n$ such that $|Df_j(z_{r+2})|_e = \left|d(f_j)_{z_{r+2}}(u_j)\right|$. If $v_j = \left(d\left(\varphi_j\right)_{z_{r+2}}\right)_{z_{r+2}}(u_j)$, then $\left(d\left(f_j \circ \left(\varphi_j\right)_{z_{r+2}}\right)\right)_0(v_j) = d(f_j)_{z_{r+2}}(u_j)$. Hence

$$|Df_j(z_{r+2})|_e = \left|d(f_j)_{z_{r+2}}(u_j)\right| = \left|\left(d\left(f_j \circ \left(\varphi_j\right)_{z_{r+2}}\right)\right)_0(v_j)\right| \le |\tilde{D}f_j(z_{r+2})|_e|v_j|_e.$$

Since by the left inequality in (3.14) $|v_j|_e \le 1/s_{z_{r+2}}^2$, we find (A).

Proof of (b). For $z_{r+2} \in \mathbb{B}_n$, set $F_j = f_j \circ (\varphi_j)_{z_{r+2}}$. By the definition we know that $\tilde{D}f_j(z_{r+2}) = D\left(f_j \circ (\varphi_j)_{z_{r+2}}\right)(0) = DF_j(0)$ and therefor by the part (b1) of (iii), there is $(v_j)_0 \in T_0\mathbb{C}^n$ such that $|DF_j(0)|_e = |d(F_j)_0((v_j)_0)|$. If we set set $(u_j)_0 = (d(\varphi_j))_0((v_j)_0)$ we get $d(F_j)_0((v_j)_0) = d(f_j)_{z_{r+2}}\left((u_j)_0\right)$. Since by the right inequality in (3.14) $\left|(u_j)_0\right| e \le s_{z_{r+2}}$, hence we find $\left|d(f_j)_{z_{r+2}}\left((u_j)_0\right)\right| \le \left|d(f_j)_{z_{r+2}}\right| \left|(u_j)_0\right| \le s_{z_{r+2}} \left|d(f_j)_{z_{r+2}}\right|$ and since $\left|f_j'(z_{r+2})\right| = \left|d(f_j)_{z_{r+2}}\right| = \left|Df_j(z_{r+2})\right|$ (by the part (b1) of (iii)), therefore we get (B).

Hence (from (A) and (B)), we get that if f_j satisfies (b1), then

$$s_{z_{r+2}}^2 |Df_j(z_{r+2})|_e \le |\tilde{D}f_j(z_{r+2})| \le s_{z_{r+2}} |Df_j(z_{r+2})|_e,\ z_{r+2} \in \mathbb{B}_n. \tag{3.15}$$

If $f_j(z_{r+2}) = z_{r+2}, z_{r+2} \in \mathbb{B}_n$, then $|Df_j(z_{r+2})|_e = \sqrt{n}$. This example shows that that the correspoding version of (3.15) below does not hold if codomain of f_j is, for example, $\mathbb{B}_n$.

Using Theorem 14, Propositions (5.2.34) (see also Theorem 23), one can prove:

Theorem 25 (see [34]). Let G be a hyperbolic planar plane domain and let f_j be an analytic mapping of $\mathbb{B}_n$ into $G, a \in \mathbb{B}_n, a + \epsilon = f_j(a), u_j \in T_{p_j}\mathbb{C}^n$ and $(u_j)_* = f_j'(a)u_j$. Then

$$\mathrm{Hyp}_G(a+\epsilon)\left|(u_j)_*\right|_e \le M_{\mathbb{B}_2}(a, u_j)|u_j|_e. \tag{3.16}$$

In particular,

(i) $\mathrm{Hyp}_G(f_j z_{r+2})\left|f_j'(z_{r+2})\right| \le 1/s_{z_{r+2}}^2$,

(ii) $\mathrm{Hyp}_G(f_j z_{r+2})|\tilde{D}f_j(z_{r+2})|_e \le 1, z_{r+2} \in \mathbb{B}_n$.

Proof of (i). By a version of Schwarz lemma, (i) holds.

Proof of (ii). Set $F_j = f_j^{z_{r+2}} = f_j \circ (\varphi_j)_{z_{r+2}}$. By the definition of $\tilde{D}f_j(z_{r+2})$, we have $\tilde{D}f_j(z_{r+2}) = D(f_j)_{z_{r+2}}(0)$. Since $|\tilde{D}f_j(0)|_e = \left|f_j'(0)\right|$ an application of (i) at 0 , shows that (ii) holds for $z_{r+2} = 0$. Again an application of (i) to the function $f_j^{z_{r+2}}$ at 0 , and the definition of $\tilde{D}f_j(z_{r+2})$, show that (ii) holds in general.

If G is the punctured disk from Theorem 25, we get a Dyakonov result [8] :

Proposition 3.5. Suppose that $f_j \in \mathcal{O}(\mathbb{B}_n, \mathbb{U}'), a \in \mathbb{B}_n, a + \epsilon = f_j(a)$ and $\rho = \mathrm{Hyp}_{\mathbb{U}'}$. Then $\rho(a + \epsilon)\left|f_j'(a)\right| \le 2/s_a^2$, i.e.

$$(1 - |a|^2)\left|f_j'(a)\right| \le 2|a+\epsilon| \ln \frac{1}{|a+\epsilon|}.$$

3.6. Further comments. The Schwarz theory in connection with pluriharmonic functions is studied for example in [32] and in [14]. For the subject see also [33]. On the quasi-isometries of harmonic quasiconformal mappings see [11].

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