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Review Paper

A Review on the Fuglede $(1+\epsilon)-$ modulus

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Abstract
In the same analogous behavior we follow [16] showing the properties of extremal functions for the Fuglede's $(1+\epsilon)$–modulus of a measure family Σ. Based on the recent result concerning $(1+\epsilon)$–modulus showing the existence of certain Borel measure n_Σ on Σ, we give an interpretation of n_Σ in some simple cases. We consider the case of disjoint supports and a natural family of measures associated with a double fibration.
Keywords: $(1+\epsilon)$–modulus, Extremal function, Optimal plan, Double fibration

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I. Introduction

B. Fuglede in [8] introduced the $(1+\epsilon)$-modulus of a family of measures, which plays a central role in many aspects of analysis on metric measure spaces. And recently recognized by many authors. The crucial fact noticed by Fuglede is that families of $(1+\epsilon)$-modulus equal to zero, called $(1+\epsilon)$-exceptional, may play the same role as equivalence of functions on sets of measure zero in $L^{1+\epsilon}$ spaces. The second remarkable observation is named the Fuglede lemma, which states that convergence in $L^{1+\epsilon}$ space implies convergence in L^1 with respect to all measures in considered family except for measures in an $(1+\epsilon)$-exceptional family.

The above mentioned results led [13] to introduce the notion of the first Sobolev spaces $N^{1,1+\epsilon}(X)$ on the metric measure space X. Hence, $(1+\epsilon)$-harmonicity may be studied on such spaces [14]. So that the measures considered are the arc length measures on curves. On considering measures associated with hypersurfaces, we get the equivalent notion of $(1+\epsilon)$-capacity of a condenser [15]. Capacity is an important tool in potential theory, partial differential equations and in differential geometry, by its conformal invariance for certain choice of the coefficient $\epsilon > 0$ (see [11], [4] and [7]).

Recently, [1] associated to any family Σ of Borel measures on the Polish space X with the reference measure $\mathfrak{m}$ the unique optimal measure $\mathfrak{n}_\Sigma$ on Σ. This measure is absolutely continuous with respect to the $(1+\epsilon)$-modulus, thus provides weaker condition for negligible sets and considered on the family of measures associated to parametric curves allows to provide alternative definition of $(1+\epsilon)$-week upper gradient (on metric measure spaces).

We describe the measure $\mathfrak{n}_\Sigma$ in some simple cases. We consider a family of measures with disjoint supports and a family naturally associated with a double fibration. The common feature of these cases is existence of a Borel map onto Σ. The push-forward of $\mathfrak{n}_\Sigma$ with respect to this map defines a measure which may be described with a given data (see [16]), hence more precisely, by the sequence of extremal functions $(f_j)_\Sigma$ for the $(1+\epsilon)$-modulus of Σ. The main ingredient, is the improved integral formula derived in [1]

$$\int_\Sigma \sum_j \hat{f}_j d\mathfrak{n}_\Sigma = \mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} \int_X \sum_j (f_j)_\Sigma^\epsilon f_j d\mathfrak{m},$$

where $\mathrm{mod}_{1+\epsilon}(\Sigma)$ is the Fuglede $(1+\epsilon)$-modulus of Σ with respect to a measure $\mathfrak{m}$ and $\hat{f}_j\colon \Sigma \to \mathbb{R}$ is given by $\sum_j \hat{f}_j(\mu) = \int_X \sum_j f_j \, d\mu$.

2. Properties of Fuglede's $(1+\epsilon)$–Modulus

We review the notion and properties of the Fuglede's $(1+\epsilon)$-modulus. For more details see [8], [2]. For X be a metric space and $\mathcal{M}$ the σ-algebra of Borel sets. Fix a reference Borel measure $\mathfrak{m}$ and let Σ be a set of Borel measures on X. Denote by $\mathcal{L}_+^{1+\epsilon}(X,\mathfrak{m})$ the space of all Borel functions $f_j\colon X \to [0,\infty]$ such that $\int_X \sum_j f_j^{1+\epsilon} d\mathfrak{m} < \infty$ and by $\mathcal{L}_{[-\infty,\infty]}^{1+\epsilon}(X,\mathfrak{m})$ the space of all Borel measurable functions taking values in $[-\infty,\infty]$ and such that $\int_X \sum_j f_j^{1+\epsilon} d\mathfrak{m} < \infty$. We define the $(1+\epsilon)$-modulus of Σ by

$$\mathrm{mod}_{1+\epsilon}(\Sigma) = \inf\left\{ \int_X \sum_j (f_j)_X^{1+\epsilon} d\mathfrak{m} \mid f_j \in \mathcal{L}_+^{1+\epsilon}(X,\mathfrak{m}), \int_X \sum_j f_j d\mu \geq 1 \text{ for } \mu \in \Sigma \right\}.$$

A function $f_j \in \mathcal{L}_+^{1+\epsilon}(X,\mathfrak{m})$ such that $\int_X \sum_j f_j d\mu \geq 1$ for all $\mu \in \Sigma$ is called $(1+\epsilon)$-admissible or admissible for the $(1+\epsilon)$-modulus of Σ. $(1+\epsilon)$-modulus has the following properties [8]:

(i) if $T \subset \Sigma$, then $\mathrm{mod}_{1+\epsilon}(T) \leq \mathrm{mod}_{1+\epsilon}(\Sigma)$,

(ii) if $T \subset \bigcup_i \Sigma_i$, then $\mathrm{mod}_{1+\epsilon}(T) \leq \sum_i \mathrm{mod}_{1+\epsilon}(\Sigma_i)$.

In other words, $(1+\epsilon)$-modulus is an outer measure on the space of measures. Moreover, one can easily show that

$$\mathrm{mod}_{1+\epsilon}(T \cup \Sigma) = \mathrm{mod}_{1+\epsilon}(T) + \mathrm{mod}_{1+\epsilon}(\Sigma),$$

for families T and Σ such that $\mathrm{supp}\, T \cap \mathrm{supp}\, \Sigma = \emptyset$, where

$$\mathrm{supp}\, \Sigma = \bigcup_{\mu \in \Sigma} \mathrm{supp}\, \mu.$$

We say that a family Σ is $(1+\epsilon)$-exceptional if its $(1+\epsilon)$-modulus is equal to zero. Moreover, we say that a certain property (P) holds $(1+\epsilon)$-almost everywhere ($(1+\epsilon)$-a.e., for short) with respect to Σ, if there is a subfamily $T \subset \Sigma$ such that (P) holds for every measure $\mu \in \Sigma \setminus T$ and T is $(1+\epsilon)$-exceptional. A $(1+\epsilon)$-admissible functions $(f_j)_\Sigma$ which realizes the infimum for the $(1+\epsilon)$-modulus of Σ is called extremal for the $(1+\epsilon)$-modulus of Σ. One can show, that up to a subfamily of $(1+\epsilon)$-modulus zero, there is unique extremal function [8]. It can be characterized by the following Badger's criterion [2].

Theorem 1. Let X be a metric space and $\mathfrak{m}$ a Borel measure on X. Assume $\mathrm{mod}_{1+\epsilon}(\Sigma) < \infty$. A $(1+\epsilon)$-admissible function $(f_j)_\Sigma \in \mathcal{L}_+^{1+\epsilon}(X,\mathfrak{m})$ is extremal for the $(1+\epsilon)$-modulus of the family Σ if and only if there is a family Σ_0 such that $\mathrm{mod}_{1+\epsilon}(\Sigma \cup \Sigma_0) = \mathrm{mod}_{1+\epsilon}(\Sigma)$ and the following two conditions hold:

(i) $\int_X \sum_j (f_j)_\Sigma d\nu = 1$ for $\nu \in \Sigma_0$,

(ii) if $\int_X \sum_j f_j \, d\nu \geq 0$ for all $\nu \in \Sigma_0$, where $f_j \in \mathcal{L}_{[-\infty,\infty]}^{1+\epsilon}(X,\mathfrak{m})$, then $\int_X \sum_j (f_j)_\Sigma^\epsilon f_j d\mathfrak{m} \geq 0$.

3. Measures Associated with Families of Borel Measures

All facts are taken from [1]. For X be a Polish space, $\mathcal{M}(X)$ the set of all Borel measures on X and fix a measure $\mathfrak{m} \in \mathcal{M}(X)$. We can endow $\mathcal{M}(X)$ with the topology of $*$-week convergence, i.e., $\mu_n \to \mu$ if

$$\int_X \sum_j f_j d\mu_n \to \int_X \sum_j f_j d\mu$$

for all $f_j \in C_b(X)$ (here, $C_b(X)$ denotes the set of all continuous and bounded functions on X). Now for any Borel functions f_j put

$$\hat{f}_{\mathrm{j}}: \mathcal{M}(X) \to \mathbb{R}, \qquad \sum_j \hat{f}_{\mathrm{j}}(\mu) = \int_X \sum_j f_j d\mu.$$

Notice that if $f_j \in C_b(X)$ then, by the definition, $\hat{f}_j$ is continuous, hence Borel. If, more generally, $f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m})$, then $\hat{f}_j$ is again Borel. This follows by the use of monotone class theorem. In fact, we can find a sequence $\left((f_j)_n\right)$ in $C_b(X)$ converging in $L^1(X,\mu)$ to f_j for any $\mu \in \mathcal{M}(X)$. Since $(\hat{f}_j)_n$ are Borel, the limit $(\hat{f}_j)$ is Borel.

Let $\Sigma \subset \mathcal{M}(X)$. [1] show, assuming Σ is Suslin, existence of a Borel probability measure $\mathfrak{n} = \mathfrak{n}_\Sigma$ on $\mathcal{M}(X)$ concentrated on Σ such that

$$\int_{\mathcal{M}(X)} \sum_j \hat{f}_j d\mathfrak{n} \le c(\mathfrak{n}) \sum_j \parallel f_j \parallel_{1+\epsilon},\ f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m}), \qquad (1)$$

for some nonnegative constant $c(\mathfrak{n})$, which we choose smallest possible. Any measure $\mathfrak{n}$ on $\mathcal{M}(X)$ satisfying (1) is called a plan with barycenter in $L^{\frac{1+\epsilon}{\epsilon}}(X, \mathfrak{m})$, $\epsilon \ge 0$. Assume now that $\mathrm{mod}_{1+\epsilon}(\Sigma) > 0$ and $\sup_\Sigma \mu(X) < \infty$ and let Σ be a Suslin set in $\mathcal{M}(X)$. Recall, that a set S is Suslin if it is an image of a Polish space under continuous map. Among all plans with barycenter (not necessary probability measures) in $L^{\frac{1+\epsilon}{\epsilon}}(X, \mathfrak{m})$ there is one optimal, what is explained in a proposition below (see [16]).

Proposition 2. [1], Let Σ be a Suslin set such that $\mathrm{mod}_{1+\epsilon}(\Sigma) > 0$ and $\sup_\Sigma \mu(X) < \infty$. Put

$$C_{1+\epsilon,\mathfrak{m}} := \sup_{c(\mathfrak{n})>0} \frac{\mathfrak{n}(\Sigma)}{c(\mathfrak{n})}. \qquad (2)$$

Then there exists an optimal plan $\mathfrak{n}_\Sigma$ with barycenter in $L^{\frac{1+\epsilon}{\epsilon}}(X, \mathfrak{m})$ in (2). Moreover, $(\mathfrak{n}_\Sigma) = C^{-1}_{1+\epsilon,\mathfrak{m}} = \mathrm{mod}_{1+\epsilon}(\Sigma)^{-\frac{1}{1+\epsilon}}$. In addition, there is an extremal function $(f_j)_\Sigma$ for the $(1+\epsilon)$-modulus of Σ. It satisfies

$$\left(\hat{f}_{\mathrm{j}}\right)_\Sigma = 1\ \mathfrak{n}_\Sigma -\ \text{a.e. on } \Sigma. \qquad (3)$$

We recall the alternative definition of a plan with barycenter in $L^{\frac{1+\epsilon}{\epsilon}}(X, \mathfrak{m})$ [1]. Let $\mathfrak{n}$ be a plan with barycenter in $L^{\frac{1+\epsilon}{\epsilon}}(X, \mathfrak{m})$. Define a Borel measure $\underline{\mathfrak{n}}$ on X by

$$\underline{\mathfrak{n}}(A_j) = \int_{\mathcal{M}(X)} \sum_j \mu(A_j) d\mathfrak{n}(\mu) = \int_{\mathcal{M}(X)} \sum_j \widehat{\chi_{A_j}} d\mathfrak{n},\ A_j \in \mathcal{B}(X).$$

Thus by (1)

$$\underline{\mathfrak{n}}(A_j) \le c\mathfrak{m}(A_j)^{\frac{1}{1+\epsilon}},\ A_j \in \mathcal{B}(X).$$

By Radon-Nikodym theorem there is a density $\rho \in L^1(X, \mathfrak{m})$ such that $\underline{\mathfrak{n}} = \rho\mathfrak{m}$. By argument of approximation by simple functions we get that

$$\int_X \sum_j f_j d\underline{\mathfrak{n}} = \int_{\mathcal{M}(X)} \sum_j \hat{f}_j d\mathfrak{n},\ f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m}). \qquad (4)$$

Hence,

$$\int_X \sum_j f_j \rho d\mathfrak{m} \le c \sum_j \| f_j \|_{1+\epsilon},\ f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m}).$$

By duality of Lebesgue spaces $L^{1+\epsilon}(X, \mathfrak{m})$ and $L^{\frac{1+\epsilon}{\epsilon}}(X, \mathfrak{m})^*$ it follows that $\rho \in L^{\frac{1+\epsilon}{\epsilon}}(X, \mathfrak{m})$ and the smallest possible constant c equals $\| \rho \|_{\frac{1+\epsilon}{\epsilon}}$.

Notice, that for a plan $\mathfrak{n}_\Sigma$ we have

$$\rho = \mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} (f_j)_\Sigma^\epsilon.$$

In other words, the following formula holds

$$\int_\Sigma \sum_j \hat{f}_j d\mathfrak{n}_\Sigma = \mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} \int_X \sum_j (f_j)_\Sigma^\epsilon f_j d\mathfrak{m}, \qquad f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m}). \quad (5)$$

4. Some results

We provide a formula for a plan $\mathfrak{n}_\Sigma$ assuming there is a Borel surjective map $\pi: X \to \Sigma$.

We start with some observations. Let $\Sigma \subset \mathcal{M}(X)$ be Suslin and assume $\mathrm{mod}_{1+\epsilon}(\Sigma) > 0$. Then there is an optimal plan $\mathfrak{n}_\Sigma$ with barycenter in $L^{\frac{1+\epsilon}{\epsilon}}$. Assume there is a Borel surjective map $\pi: X \to \Sigma$. Put

$$X_\mu = \pi^{-1}(\mu). \quad (6)$$

We may push-forward a measure $\mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} (f_j)_\Sigma^{1+\epsilon} \mathfrak{m}$ with respect to π to obtain a measure $\mathfrak{n}'_\Sigma$ on Σ,

$$\mathfrak{n}'_\Sigma = \pi_\sharp \left(\mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} \left(f_j\right)_\Sigma^{1+\epsilon} \mathfrak{m} \right). \quad (7)$$

Moreover, denote by $\bar{f}_j$ a composition $\hat{f}_j \circ \pi$ for $f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m})$. Then $\bar{f}_j$ is a Borel function on X. By the definition we have

$$\int_\Sigma \sum_j \hat{f}_j d\mathfrak{n}'_\Sigma = \mathrm{mod}_{1+\epsilon}(\Sigma')^{-1} \int_X \sum_j (f_j)_\Sigma^{1+\epsilon} \bar{f}_j d\mathfrak{m}, \qquad f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m}). \quad (8)$$

4.1. Case of Disjoint Supports

Assume that support of μ is contained in X_μ for any $\mu \in \Sigma$. We say in this case that a family Σ is separate.

Lemma 3 ([16]). For the $(1+\epsilon)$-modulus of separate family Σ we have $(\hat{f}_j)_\Sigma = 1$ $(1+\epsilon)$-a.e. on Σ.

Proof. Suppose $(\hat{f}_j)_\Sigma(\mu) > 1$. Then we could replace $(f_j)_\Sigma$ by $\frac{1}{(f_j)_\Sigma(\mu)} (f_j)_\Sigma$ on X_μ to get a function with the smaller $L^{1+\epsilon}(X, \mathfrak{m})$-norm. This contradicts the minimality of $(f_j)_\Sigma$.

Lemma 4 (see [16]). For any Suslin subset Π of a separate family $\Sigma (f_j)_\Sigma$ restricted to $\bigcup_{\mu \in \Pi} X_\mu$ and zero elsewhere is extremal for the $(1+\epsilon)$-modulus of family Π.

Proof. Suppose there exist $(f_j)_\Pi$, which is admissible for Π and such that $\left\| \sum_j (f_j)_\Pi \right\|_{1+\epsilon} < \sum_j \left\| (f_j)_\Sigma \right\|_{1+\epsilon}$ on $\bigcup_{\mu \in \Pi} X_\mu$. Put

$$\sum_j f_j(x) = \begin{cases} \sum_j (f_j)_\Pi(x) & \text{for } x \in \bigcup_{\mu \in \Pi} X_\mu \\ \sum_j (f_j)_\Sigma(x) & \text{for remaining } x \end{cases}$$

Then f_j is $(1+\epsilon)$-admissible for Σ and $\| \sum_j f_j \|_{1+\epsilon} < \sum_j \|(f_j)_\Sigma\|_{1+\epsilon}$, which contradicts extremality of $\sum_j (f_j)_\Sigma$.

For any $f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m})$ by Lemma 3 we have

$$\sum_j \widehat{(f_j)_\Sigma(\bar{f}_j)}(\mu) = \int_{X_\mu} \sum_j (f_j)_\Sigma \bar{f}_j d\mu = \sum_j \hat{f}_j(\mu)(\hat{f}_j)_\Sigma(\mu) = \sum_j \hat{f}_j(\mu),\ \mu \in \Sigma.$$

Hence, by (8) and (5) we get

$$\int_\Sigma \sum_j \hat{f}_j d\mathfrak{n}'_\Sigma = \int_\Sigma \sum_j \widehat{(f_j)_\Sigma \bar{f}_j} d\mathfrak{n}_\Sigma = \int_\Sigma \sum_j \hat{f}_j d\mathfrak{n}_\Sigma.$$

Choose a Suslin subset $\Pi \subset \Sigma$. By Lemmas 3 and 4 we see that $\mathfrak{n}'_\Sigma(\Pi) = \mathfrak{n}_\Sigma(\Pi)$, hence these two measures agree on Suslin sets. Moreover, taking $f_j = (f_j)_\Pi$ in (5) we get

$$\mathfrak{n}_\Sigma(\Pi) = \mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} \int_X \sum_j (f_j)_\Sigma^\epsilon (f_j)_\Pi d\mathfrak{m} = \mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} \int_X \sum_j (f_j)_\Pi^{1+\epsilon} d\mathfrak{m}$$
$$= \frac{\mathrm{mod}_{1+\epsilon}(\Pi)}{\mathrm{mod}_{1+\epsilon}(\Sigma)},$$

Concluding, we have the following proposition (see [16]).

Proposition 5. The plan $\mathfrak{n}_\Sigma$ on separate family Σ is equal to the conditional probability with respect to $(1+\epsilon)$-modulus for Suslin subsets. Moreover,

$$\mathfrak{n}_\Sigma = \mathfrak{n}'_\Sigma = \pi_\sharp\big(\mathrm{mod}_{1+\epsilon}(\Sigma)(f_j)_\Sigma^{1+\epsilon}\mathfrak{m}\big).$$

By above proposition we have the following integral formula

$$\int_X \sum_j (f_j)_\Sigma^{1+\epsilon} \bar{f}_j d\mathfrak{m} = \int_X \sum_j (f_j)_\Sigma^\epsilon f_j d\mathfrak{m},\ f_j \in \mathcal{L}^{1+\epsilon}(X, \mathfrak{m}). \quad (9)$$

Remark 1. Notice that the integral formula (9) was obtained in the case of Lebesgue measures associated with a foliation on a Riemannian manifold [5,6].

4.2. Connections with Disintegration Theorem

For Σ be a Suslin family of Borel measures on a space X such that $\mathrm{mod}_{1+\epsilon}(\Sigma) > 0$ with respect to a fixed Borel measure $\mathfrak{m}$ is positive. Assume there is a Borel map $\pi: X \to \Sigma$. Then $\mathfrak{n}_{\Sigma'}$, defined by (7), is a Borel measure on Σ. By Disintegration Theorem (see [3]) applied to a map π and a measure $\mathfrak{n}'_\Sigma$, there is a family of Borel measures $\Sigma' = \{\nu_\mu\}_{\mu\in\Sigma}$ on X such that

$$\mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} \int_X \sum_j (f_j)_\Sigma^\epsilon f_j d\mathfrak{m} = \int_\Sigma \int_{\pi^{-1}(\mu)} \sum_j f_j d\nu_\mu d\mathfrak{n}_{\Sigma'}(\mu). \quad (10)$$

We compute the $(1+\epsilon)$-modulus of Σ'. If f_j is admissible for the $(1+\epsilon)$-modulus of Σ' then by (10) it follows that

$$1 \le \int_\Sigma 1 d\mathfrak{n}_{\Sigma'} \le \int_\Sigma \int_{\pi^{-1}(\mu)} \sum_j f_j d\nu_\mu d\mathfrak{n}_{\Sigma'}(\mu) \le \mathrm{mod}_{1+\epsilon}(\Sigma)^{-\frac{1}{1+\epsilon}} \sum_j \| f_j \|_{1+\epsilon}.$$

Thus

$$\mathrm{mod}_{1+\epsilon}(\Sigma') \ge \mathrm{mod}_{1+\epsilon}(\Sigma). \quad (11)$$

Moreover, above last inequality can be stated as

$$\int_\Sigma \sum_j \hat{f}_j d\mathfrak{n}_{\Sigma'} \le \mathrm{mod}_{1+\epsilon}(\Sigma)^{-\frac{1}{1+\epsilon}} \sum_j \| f_j \|_{1+\epsilon},$$

where $\hat{f}_j$ is taken with respect to Σ'. This shows that $\pi_{\Sigma'}$ is a plan with barycenter in $L^{\frac{1+\epsilon}{\epsilon}}(X,\mathfrak{m})$ associated with Σ' and that $c(\pi_{\Sigma'}) \le \mathrm{mod}_{1+\epsilon}(\Sigma)$. Notice that measures $\nu_\mu, \mu \in \Sigma$, have disjoint supports.

It would be interesting to check when there is equality in (11). In the example below, we show that it may happen (see [16]).

Example 1. Let $X = [0,1]$ and consider a Lebesgue measure $\mathfrak{m}$ on X. Let $\mu_{1-\epsilon}, 0 \le \epsilon < 1$, be a normalized Lebesgue measure on $[0,1-\epsilon]$, i.e. $\mu_{1-\epsilon} = \frac{1}{1-\epsilon}\mathfrak{m}\Big|_{[0,1-\epsilon]}$ and let $\mu_0 = \delta_0$ be the Dirac measure at 0. Put $\Sigma = \{\mu_{1-\epsilon}\}_{\epsilon\in[0,1]}$. A map $\pi: X \to \Sigma$ given by $\pi(1-\epsilon) = \mu_{1-\epsilon}$ is continuous. An admissible function $(f_j)_\Sigma$ is extremal for the $(1+\epsilon)$-modulus of Σ if

$$\frac{1}{1-\epsilon}\int_0^{1-\epsilon}\sum_j (f_j)_\Sigma(s)ds = 1$$

for all $1-\epsilon$. Differentiating with respect to $1-\epsilon$, we get that $(f_j)_\Sigma = 1$. Thus $\mathrm{mod}_{1+\epsilon}(\Sigma) = 1$. We will derive the formula for the $(1+\epsilon)$-modulus of a family Σ' of measures $\nu_{1-\epsilon}$ on X, which existence follows by Disintegration Theorem. We know that $\nu_{1-\epsilon}$ is concentrated on a set $X_{1-\epsilon} = \pi^{-1}(\mu_{1-\epsilon}) = \{1-\epsilon\}$. Hence $\nu_{1-\epsilon} = c_{1-\epsilon}\delta_{1-\epsilon}$ for some positive $c_{1-\epsilon}$. Thus, by Disintegration Theorem

$$\int_0^1 \sum_j f_j(1-\epsilon)d(1-\epsilon) = \int_0^1 \sum_j c_{1-\epsilon}f_j(1-\epsilon)d(1-\epsilon),$$

for any f_j. Hence $c_{1-\epsilon} = 1-\epsilon$ and $\nu_{1-\epsilon} = \delta_{1-\epsilon}$ is a Dirac measure. We thus have $\mathrm{mod}_{1+\epsilon}(\Sigma') = 1 = \mathrm{mod}_{1+\epsilon}(\Sigma)$.

Moreover, let us determine the plan π_Σ. By a definition we have

$$\int_\Sigma \frac{1}{1-\epsilon}\int_0^{1-\epsilon}\sum_j f_j(s)ds\,d\pi_\Sigma(\mu_{1-\epsilon}) = \int_0^1 \sum_j f_j(1-\epsilon)d(1-\epsilon)$$

for any continuous f_j. Taking $f_j = F'$ such that $F(0) = 0$ we obtain

$$\int_\Sigma \frac{F(1-\epsilon)}{1-\epsilon}d\pi_\Sigma(\mu_{1-\epsilon}) = F(1).$$

Now it easily follows that π_Σ is a Dirac measure at μ_1.

5. Double Fibration

For G be a locally compact group, H_X and $H_{\mathcal{A}}$ closed subgroups of G. Define two left coset spaces

$$X = G/H_X,\ \mathcal{A} = G/H_{\mathcal{A}}.$$

Moreover, consider the following assumptions:

(i) the groups $G, H_X, H_{\mathcal{A}}$ and $H = H_X \cap H_{\mathcal{A}}$ are unimodular, i.e., left-invariant Haar measure are also right-invariant,

(ii) the set $H_X H_{\mathcal{A}}$ is closed in G,

(iii) if $hH_{\mathcal{A}} \subset H_{\mathcal{A}}H_X$, then $h \in H_X$. If $hH_X \subset H_X H_{\mathcal{A}}$, then $h \in H_{\mathcal{A}}$.

We say that two elements $x \in X$ and $\xi \in \mathcal{A}$ are incident if as cosets in G they intersect. The set of elements in $\mathcal{A}$ which are incident with $x \in X$ is denoted by $\check{x}$. Analogously, the set of points in X which are incident with $\xi \in \mathcal{A}$ is denoted by $\hat{\xi}$. Put

$$F = \{(x,\xi) \in X \times \mathcal{A} \mid x \text{ and } \xi \text{ are incident }\}.$$

Then we have two fibrations $F \mapsto X$ and $F \mapsto \mathcal{A}$, this we speak about a double fibration.

Let us recall important results concerning existence of invariant measures for a double fibration [9]. Let $x_0 = \{H_X\}$ and $\xi_0 = \{H_{\mathcal{A}}\}$. There is a unique H_X-invariant measure μ_0 on $\check{x}_0$ and a unique $H_{\mathcal{A}}$-invariant measure $\mathfrak{n}_0$ on $\hat{\xi}_0$. Moreover, there exists a nonzero measure on each $\check{x}$ and on $\hat{\xi}$, such that they coincide with μ_0 on $\check{x}_0$ and $\mathfrak{n}_0$, respectively. For these measures, if $\check{x}_1$ corresponds to $\check{x}_2$ by $g \in G$, then μ_{x_1} corresponds to μ_{x_2} by g. Analogous correspondence holds for elements in $\mathcal{A}$. Denote by $\mathfrak{m}$ and $\mathfrak{n}$ the (normalized) G-invariant measures on X and $\mathcal{A}$, respectively, induced from Haar measure on G. Fix a smooth positive and bounded function ρ on F and consider the following Radon and its dual transformations

$$\sum_j \hat{f}_j(\xi) = \int_{\hat{\xi}} \sum_j f_j(x)\rho(x,\xi)d\mu_\xi(x),$$

$$\sum_j \check{\varphi}_j(x) = \int_{\check{x}} \sum_j \varphi_j(\xi)\rho(x,\xi)d\mathfrak{n}_x(\xi)$$

for Borel and non-negative functions f_j and φ_j. Then it is not hard to see that (compare [12], [9] in the continuous or $L^{1+\epsilon}$ category; see also [8] for similar and general approach)

$$\int_X \sum_j f_j(x)\check{\varphi}_j(x)d\mathfrak{m}(x) = \int_{\mathcal{A}} \sum_j \hat{f}_j(\xi)\varphi_j(\xi)d\mathfrak{n}(\xi) \tag{12}$$

for any non-negative Borel functions f_j and φ_j.

Put

$$\Sigma = \{\rho(\cdot,\xi)\mu_\xi \mid \xi \in \mathcal{A}\}.$$

Each μ_ξ is supported on $\hat{\xi} \subset X$, whereas each $\mathfrak{n}_x$ is supported on $\check{x} \subset \mathcal{A}$. We seek for the extremal function $(f_j)_\Sigma$ for the family Σ.

Proposition 6 (see [16]). Assume that there is an admissible function $(f_j)_\Sigma$ for Σ such that the following conditions hold:

(i) $(\hat{f}_j)_\Sigma = 1$,

(ii) there exists Borel positive function $(\varphi_j)_\Sigma$ on $\mathcal{A}$ such that $(f_j)_\Sigma^\epsilon = (\check{\varphi}_j)_\Sigma$.

Then $(f_j)_\Sigma$ is extremal for the $(1+\epsilon)$-modulus of Σ.

Proof. The proof follows by using standard methods. Let f_j be admissible for Σ. By (12) and Hölder inequality,

$$\begin{aligned} 0 &\le \int_{\mathcal{A}} \sum_j \widehat{f_j - (f_j)_\Sigma}(\varphi_j)_\Sigma d\mathfrak{m} = \int_X \sum_j (f_j)_\Sigma^\epsilon f_j d\mathfrak{m} - \int_X \sum_j (f_j)_\Sigma^{1+\epsilon} d\mathfrak{m} \\ &\le \sum_j \left(\int_X (f_j)_\Sigma^{1+\epsilon} d\mathfrak{m}\right)^{\frac{\epsilon}{1+\epsilon}} \left(\int_X f_j^{1+\epsilon} d\mathfrak{m}\right)^{\frac{1}{1+\epsilon}} - \int_X \sum_j (f_j)_\Sigma^{1+\epsilon} d\mathfrak{m}, \end{aligned}$$

where $\frac{1+\epsilon}{\epsilon}$ is a coefficient conjugate to $1+\epsilon$. Thus

$$\int_X \sum_j (f_j)_\Sigma^{1+\epsilon} d\mathfrak{m} \le \int_X \sum_j f_j^{1+\epsilon} d\mathfrak{m}.$$

Hence $(f_j)_\Sigma$ is extremal.

Remark 2. Let us remark on the similarity of above theorem to the Badger's criterion (Theorem 1). In this case, Σ_0 is just Σ. Thus, condition (1) of Proposition 6 and Theorem 1 coincide. By the 'Plancherel' formula (12) for any positive Borel functions f_j on X such that $\hat{f}_j \geq 0$ on Σ we have

$$\int_X \sum_j (f_j)_\Sigma^\epsilon f_j d\mathfrak{m} = \int_{\mathcal{A}} \sum_j \varphi_j \hat{f}_j d\mathfrak{n} \geq 0$$

assuming (2) of Proposition 6.

Remark 3. If the Radon transform is injective, i.e. the map $f_j \to \hat{f}_j$ is on the suitable class injective with the constant functions in the range of this correspondence, then the condition (1) of Proposition 6 characterizes $(f_j)_\Sigma$ completely. In such case, the second assumption is useless.

We will justify above theorem on the concrete examples. The classical example in this theory is the Radon transform on $X = \mathbb{R}^n$ with the dual homogeneous space $\mathcal{A}$ being the manifold of all hyperplanes in $\mathbb{R}^n$. By well-known results extremal function should be harmonic and bounded, hence constant. Since it integrates to 1 on each element of $\mathcal{A}$, we conclude that there is no extremal function in this case and, consequently, $(1+\epsilon)$-modulus of considered family Σ is zero for any $\epsilon > 0$. Hence, Proposition 6 does not apply in this situation.

Let us begin with the 'elementary' example (see [16]).

Example 2. Consider a $(n-1)$-dimensional unit sphere S^{n-1} centered at the origin in $\mathbb{R}^n$. Then $S^{n-1} = SO(n)/SO(n-1)$, where $SO(n-1)$ is considered as a subgroup fixing the north pole x_0. Consider a double fibration with fibrations $SO(n) \mapsto S^{n-1}$ and a trivial one $SO(n) \mapsto SO(n)$, where we consider $SO(n)$ as a trivial quotient $SO(n) = SO(n)/\{e\}$. Then a coset $gSO(n-1)$ intersects coset $\tilde{g}\{e\}$ if and only if $g\tilde{g}^{-1} \in SO(n-1)$, hence $x \in M$ is incident to a transformation $g \in SO(n)$ if and only if $gx_0 = x$. In other words

$$\check{x} = gSO(n-1) \subset SO(n),\ \hat{g} = \{x\} \subset M,$$

where $x = gx_0$. Then μ_g is a Dirac measure δ_x at x and $\mathfrak{n}_x$ is a Haar measure on $gSO(n-1)$. Hence, for any Borel non-negative functions f_j and φ_j we have (we choose $\rho = 1$)

$$\sum_j \hat{f}_j(g) = \sum_j f_j(x), \qquad \sum_j \check{\varphi}_j(x) = \int_{gSO(n-1)} \sum_j \varphi_j(h) d\mathfrak{n}_x(h),\ gx_0 = x.$$

Thus, any admissible function f_j for $\Sigma = \{\mu_x \mid x \in S^{n-1}\}$ satisfies $f_j \geq 1$. Now, it is obvious that the extremal function $(f_j)_\Sigma$ is identically 1. Let us apply Proposition 6. Clearly, $(f_j)_\Sigma = 1$ satisfies condition (1) of Proposition 6. Taking $(\varphi_j)_\Sigma = 1$ also the second condition holds. Notice, that formula (12) may be obtained straightforward by the coarea formula $(gx_0 = x)$

$$\begin{aligned}
\int_{SO(n)} \sum_j \hat{f}_j(g)\varphi_j(g) d\mathfrak{n}(g) &= \int_{SO(n)} \sum_j f_j(gx_0)\varphi_j(g) d\mathfrak{n}(g) \\
&= \int_M \int_{SO(n-1)} \sum_j f_j(hx_0)\varphi_j(h) d\mathfrak{n}_{x_0}(h) d\mathfrak{m}(x) \\
&= \int_M f_j(x) \int_{SO(n-1)} \sum_j \varphi_j(h) d\mathfrak{n}_{x_0}(h) d\mathfrak{m}(x) \\
&= \int_M f_j(x) \int_{gSO(n-1)} \sum_j \varphi_j(g^{-1}h) d\mathfrak{n}_{x_0}(h) d\mathfrak{m}(x) \\
&= \int_M \sum_j f_j(x)\check{\varphi}_j(x) d\mathfrak{m}(x),
\end{aligned}$$

since $hx_0 = x_0$ for any $h \in SO(n-1)$ and by invariance of $\mathfrak{n}_x$.

The second example concerns the Funk transform on a sphere (see [16]).

Example 3. Let $X = S^2$ be the unit 2-dimensional sphere in $\mathbb{R}^3, S^2 = O(3)/O(2)$. Let $\mathcal{A}$ be a set of all great circles in S^2, i.e., $\mathcal{A} = O(3)/O(2) \times \mathbb{Z}_2$, where $O(2) \times \mathbb{Z}_2$ is a subgroup fixing a line through a north pole. Then, for $x \in S^2, \hat{x}$ is a set of all great circles passing through x and for $\xi \in \mathcal{A}, \check{\xi}$ is just ξ considered as a great circle in S^2. Consider a family Σ of all Lebesgue measures on great circles. By an inversion formula for a Funk transform [10] it follows that the extremal function for the $(1+\epsilon)$-modulus of Σ equals $(f_j)_\Sigma = \frac{1}{2\pi}$. Notice that in this case we can apply Proposition 6. In fact, $(f_j)_\Sigma$ satisfies (1) and putting $(\varphi_j)_\Sigma$ to be a constant such that $\check{\varphi}_j = (2\pi)^{-\epsilon}$ it satisfies (2) of Proposition 6.

To finish this section, we show the relation with the plan $\mathfrak{n}_\Sigma$ on Σ. Firstly, we have a bijective map $\Phi: \mathcal{A} \to \Sigma, \Phi(\xi) = \rho(\cdot,\xi)\mu_\xi$. This map is continuous, since it is equivalent to the continuity of $\hat{f}_j$ for any $f_j \in C_b(X)$. Hence, is Borel. Consider a measure $\mathfrak{n}'_\Sigma$ given by the relation

$$\mathfrak{n}'_\Sigma = \Phi_\#\big((\varphi_j)_\Sigma \mathfrak{n}\big).$$

Then $\mathfrak{n}'_\Sigma$ is a Borel measure on Σ. Denote also by $\hat{f}_j$ a function on Σ corresponding by Φ to $\hat{f}_j$, i.e., $\hat{f}_j\big(\rho(\cdot,\xi)\mu_\xi\big) \equiv \hat{f}_j(\xi)$. By (5) and (12),

$$\int_\Sigma \sum_j \hat{f}_j d\mathfrak{n}'_\Sigma = \int_{\mathcal{A}} \sum_j \hat{f}_j (\varphi_j)_\Sigma d\mathfrak{n} = \int_X \sum_j f_j (\check{\varphi}_j)_\Sigma dm = \int_X \sum_j (f_j)_\Sigma^\epsilon f_j dm$$
$$= \mathrm{mod}_{1+\epsilon}(\Sigma)^{-1} \int_\Sigma \sum_j \hat{f}_j d\mathfrak{n}_\Sigma$$

for any $f_j \in \mathcal{L}^{1+\epsilon}(X, m)$. Thus

$$\mathrm{mod}_{1+\epsilon}(\Sigma)\mathfrak{n}'_\Sigma = \mathfrak{n}_\Sigma.$$

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