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Review Paper

Orthogonality Preservers Revisited by Closedness Adjacent Elements

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Abstract

The author in [30] obtain a complete characterization of all orthogonality preserving operators from a JB-algebra to a JB^*-triple following [30]. If $T_m : J \to E$ is a bounded linear operator from a JB^*-algebra (respectively, a C^*-algebra) to a JB^*-triple and h_m denotes the element $T_m^{**}(1)$, then T_m is orthogonality preserving, if and only if, T_m preserves zero-triple-products, if and only if, there exists a Jordan $*$-homomorphism $S_m : J \to E_2^{**}(r(h_m))$ such that $S_m(x)$ and h_m operator commute and $T_m(x) = h_m \bullet_{r(h_m)} S_m(x)$, for every $x \in J$, where $r(h_m)$ is the range tripotent of $h_m, E_2^{**}(r(h_m))$ is the Peirce-2 subspace associated to $r(h_m)$ and $\bullet_{r(h_m)}$ is the natural product making $E_2^{**}(r(h_m))$ a JB^*-algebra. This characterization culminates the description of all orthogonality preserving operators between C^*-algebras and JB^*-algebras and show a widegeneralizations.*

Keywords: *Orthogonality preserving operators; orthogonally additive mappings; C^*- algebras; JB^*-algebras; JB^*-triples.*

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I. Introduction

The study of orthogonality preserving operators between C*-algebras started with [1], where W. Arendt initiated the study of all operators preserving disjoint (or orthogonal) functions between $C(K)$ spaces. It was established there that for each orthogonality preserving operator $T_m: C(K) \to C(K)$, there exist $h_m \in C(K)$ and a mapping $\varphi_m: K \to K$ being continuous on the set $\{t \in K: h_m(t) \neq 0\}$ satisfying that

$$T_m(f)(t) = h_m(t) f(\varphi_m(t)),$$

for all $f \in C(k), t \in K$. The authors in [16], [17] proved later that the description remains valid for all orthogonality preserving operators between $C_0(L)$-space, where L is a locally compact Hausdorff space.

$C(K)$ and $C_0(L)$ spaces are examples of abelian C*-algebras. In fact, the Gelfand theory assures that every abelian C*-algebra is C*-isomorphic to a $C_0(L)$-space. Therefore, the just quoted results by Jarosz and Jeang-Wong provide a complete description of all orthogonality preserving operators between abelian C*-algebras.

In the setting of a general C*-algebra A, two nearly adjacent elements a and $(a+\epsilon)$ in A are said to be orthogonal (denoted by $a \perp (a+\epsilon)$) if $a(a+\epsilon)^* = (a+\epsilon)^* a = 0$. A linear operator T_m between two C*-algebras A and B is called orthogonality preserving or disjointness preserving if $T_m(a) \perp T_m(a+\epsilon)$, for all $a \perp (a+\epsilon)$ in A. The description of all orthogonality preserving operators between two C*-algebras raised as an important problem studied by many authors.

When the problem is considered only for symmetric operators between general C*-algebras, M. Wolff established a full description in [26]. If $T_m : A \to B$ is a symmetric orthogonality preserving bounded linear operator between two C*-algebras with A unital, then denoting $T_m(1) = h_m$ the following assertions hold:

a) $T_m(A)$ is contained in the norm closure of $h_m\{h_m\}'$, where $\{h_m\}'$ denotes the commutator of $\{h_m\}$.

b) There exists a Jordan *-homomorphism $S_m: A \rightarrow B^{**}$ such that $T_m(x+2\epsilon) = h_m S_m(x+2\epsilon)$, for all $(x+2\epsilon) \in A$.

On every C*-algebra A we can also consider a triple product defined by $\{x, x+\epsilon, x+2\epsilon\} := \frac{1}{2}(x(x+\epsilon)^*(x+2\epsilon) + (x+2\epsilon)(x+\epsilon)^* x)$. This triple product has been shown as an important tool to characterize orthogonal elements in a C*-algebra. In fact, two elements $a$and $(a+\epsilon)$ in A are orthogonal if and only if $\{a, a, a+\epsilon\} = 0$ (compare Lemma 1 in [7]). In particular, every triple homomorphism between two C*-algebras preserves orthogonal elements. Theorem 3.2 in [27] shows that a bounded linear operator T_m between two C*-algebras is a triple homomorphism if and only if T_m is orthogonality preserving and T_m^{**} (1) is a partial isometry (tripotent).

There exists a wider class of complex Banach spaces containing all C*-algebras in which the notion of orthogonality makes sense and extends the concept given for C*-algebras. We refer to the class of JB*-triples. A JB*-triple is a complex Banach space E, equipped with a continuous triple product $\{.,.,.\}: E \times E \times E \rightarrow E$, satisfying suitable algebraic and geometric conditions (see definition in §2). Every C*-algebra is a JB*-triple for the triple product given above.

Two elements a and $(a+\epsilon)$ in a JB*-triple E are said to be orthogonal (written $a \perp (a+\epsilon)$) if $L(a, a+\epsilon) = 0$, where $L(a, a+\epsilon)$ is the linear operator on E defined by $L(a, a+\epsilon)(x) := \{a, a+\epsilon, x\}$. It is known that two elements in a C*-algebra A are orthogonal for the C*-algebra product if and only if they are orthogonal when A is considered as a JB*-triple (compare the introduction of §4).

Techniques in JB*-triple theory were revealed as a powerful tool in the description of all orthogonality preserving operators between two C*-algebras established in[7]. Concretely, for every operator T_m between two C*-algebras, denoting $h_m = T_{\mathrm{m}}^{**}(1)$, the following assertions are equivalent (see [30]):

a) T_m is orthogonality preserving.

b) There exists a triple homomorphism $S_m: A \rightarrow B^{**}$ satisfying $h_m^* S_m(x+2\epsilon) = S_m((x+2\epsilon)^*)^* h_m$, $h_m S_m((x+2\epsilon)^*)^* = S_m(x+2\epsilon) h_m^*$, and

$$T_m(x+2\epsilon) = L(h_m, r(h_m))(S_m(x+2\epsilon)) = \frac{1}{2}(h_m r(h_m)^* S_m(x+2\epsilon) + S_m(x+2\epsilon) r(h_m)^* h_m)$$
$$= h_m r(h_m)^* S_m(x+2\epsilon) = S_m(x+2\epsilon) r(h_m)^* h_m,$$

for all $(x+2\epsilon) \in A$, where $r(h_m)$ denotes the range tripotent of h_m.

c) T_m preserves zero-triple-products (that is, $\{T_m(a), T_m(a+\epsilon), T_m(a+2\epsilon)\} = 0$ whenever $\{a, a+\epsilon, a+2\epsilon\} = 0$.

Reference [7] also contains the following generalization of the main result in [27]: Let T_m be an operator from a JB*-algebra J to a JB*-triple E. Then T_m is a triple homomorphism if and only if T_m is orthogonality preserving and $T_{\mathrm{m}}^{**}(1)$ is a tripotent. This result is in fact a consequence of a complete description of all orthogonality preserving operators from J to E whose second adjoint maps the unit of J^{**} to a von Neumann regular element. It seems natural to ask whether the condition of $T_{\mathrm{m}}^{**}(1)$ being von Neumann regular can be omitted.

This paper culminates with the characterization of all orthogonality preserving operators from a JB*-algebra to a JB*-triple. Theorem 4.1 and Corollary 4.2 show that for a bounded linear operator T_m from a JB*-algebra J to a JB*-triple E the following are equivalent (see [30]):

a) T_m is orthogonality preserving.

b) There exists a (unital) Jordan *-homomorphism $S_m: M(J) \rightarrow E_2^{**}(r(h_m))$ such that $S_m(x)$ and h_m operator commute and $T_m(x) = h_m \cdot_{r(h_m)} S_m(x)$, for every $x \in J$, where $M(J)$ is the multiplier algebra of J, $r(h_m)$ is the range tripotent of h_m, $E_2^{**}(r(h_m))$ is the Peirce-2 subspace associated to $r(h_m)$ and $\cdot_{r(h_m)}$ is the natural product making $E_2^{**}(r(h_m))$ a JB*-algebra.

c) $T_m$preserves zero-triple-products.

The proofs presented here are partially based on techniques developed in JB* triple theory. The arguments do not depend on those results previously obtained by [1], [26], [27] and [7]. We shall actually show that all of them are direct consequences of the main result here.

A useful tool applied in the proof of the main result of this paper is the characterization of all orthogonally additive $(1+2\epsilon)$-homogeneous polynomials on a general C* algebra. This characterization has been recently obtained in [20]. Section 3 presents a shorter and simplified proof of this description.

II. Preliminaries

Given Banach spaces X and $Y, L(X,Y)$ will denote the space of all bounded linear mappings from X to Y. We shall write $L(X)$ for the space $L(X,X)$. Throughout the paper the word "operator" (respectively, multilinear or sesquilinear operator) will always mean bounded linear mapping (respectively bounded multilinear or sesquilinear mapping). The dual space of a Banach space X is always denoted by X^*.

When A is a JB*-algebra or a C*-algebra then, A_{sa} will stand for the set of all self-adjoint elements in A. We shall make use of standard notation in C*-algebra and JB*-triple theory.

C*-algebras and JB*-algebras belong to a more general class of Banach spaces known under the name of JB*-triples. JB*-triples were introduced by [19]. A JB*-triple is a complex Banach space E together with a continuous triple product $\{.,.,.\}: E\times E\times E\to E$, which is conjugate linear in the middle variable and symmetric and bilinear in the outer variables satisfying that,

(JB1) $L(a,a+\epsilon)L(x,x+\epsilon)=L(x,x+\epsilon)L(a,a+\epsilon)+L(L(a,a+\epsilon)x,x+\epsilon)-L(x,L(a+\epsilon,a)x+\epsilon)$, where $L(a,a+\epsilon)$ is the operator on E given by $L(a,a+\epsilon)x=\{a,a+\epsilon,x\}$;

(JB2) $L(a,a)$ is a hermitian operator with non-negative spectrum;

(JB3) $\| L(a,a) \|=\| a \|^2$.

For each x in a JB*-triple $E, Q(x)$ will stand for the conjugate linear operator on E defined by the law $(x+\epsilon)\mapsto Q(x)(x+\epsilon)=\{x,x+\epsilon,x\}$.

Every C*-algebra is a JB*-triple via the triple product given by

$$2\{x,x+\epsilon,x+2\epsilon\}=x(x+\epsilon)^*(x+2\epsilon)+(x+2\epsilon)(x+\epsilon)^*x,$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x,x+\epsilon,x+2\epsilon\}=(x\circ(x+\epsilon)^*)\circ(x+2\epsilon)+\left((x+2\epsilon)\circ(x+\epsilon)^*\right)\circ x-(x\circ(x+2\epsilon))\circ(x+\epsilon)^*.$$

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [4]). It is known that the triple product of a JBW*-triple is separately weak*-continuous [4]. The second dual of a JB*-triple E is a JBW*-triple with a product extending that of E (compare [9]).

An element e in a JB*-triple E is said to be a tripotent if $\{e,e,e\}=e$. Each tripotent e in E gives raise to the so-called Peirce decomposition of E associated to e, that is,

$$E=E_2(e)\oplus E_1(e)\oplus E_0(e),$$

where for $i=0,1,2, E_i(e)$ is the $\frac{i}{2}$ eigenspace of $L(e,e)$. The Peirce decomposition satisfies certain rules known as Peirce arithmetic:

$$\left\{E_i(e),E_j(e),E_k(e)\right\}\subseteq E_{i-j+k}(e),$$

if $i-j+k\in\{0,1,2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The corresponding Peirce projections are denoted by $(P_m)_i(e): E \to E_i(e), (i = 0,1,2)$. The Peirce space $E_2(e)$ is a JB*-algebra with product $x \cdot_e (x+\epsilon) := \{x, e, x+\epsilon\}$ and involution $x^{\sharp k} := \{e, x, e\}$.

For each element x in a JB -triple E, we shall denote $x^{[1]} := x, x^{[3]} := \{x, x, x\}$, and $x^{[2(1+2\epsilon)+1]} := \{x, x, x^{[2(1+2\epsilon)-1]}\}, ((1+2\epsilon) \in \mathbb{N})$. The symbol E_x will stand for the JB* subtriple generated by the element x. It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $(0, \| x \|]$, such that $\Omega \cup \{0\}$ is compact, where $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0 . It is also known that if Ψ denotes the triple isomorphism from E_x onto $C_0(\Omega)$, then $\Psi(x)(t) = t(t \in \Omega)$ (cf. Corollary 4.8 in [18], Corollary 1.15 in [19] and [12]).

Therefore, for each $x \in E$, there exists a unique element $(x+\epsilon) \in E_x$ satisfying that $\{x+\epsilon, x+\epsilon, x+\epsilon\} = x$. The element $(x+\epsilon)$, denoted by $x^{\left[\frac{1}{3}\right]}$, is termed the cubic root of x. We can inductively define, $x^{\left[\frac{1}{3^{(1+2\epsilon)}}\right]} = \left(x^{\left[\frac{1}{3^{(1+2\epsilon)-1}}\right]}\right)^{\left[\frac{1}{3}\right]}, (1+2\epsilon) \in \mathbb{N}$. The sequence $\left(x^{\left[\frac{1}{3^{(1+2\epsilon)}}\right]}\right)$converges in the weak*-topology of E^{**} to a tripotent denoted by $r(x)$ and called the range tripotent of x. The element $r(x)$ is the smallest tripotent $e \in E^{**}$ satisfying that x is positive in the JBW*-algebra $E_2^{**}(e)$ (compare [11], Lemma 3.3).

A subspace I of a JB*-triple E is said to be an inner ideal of E if $\{I, E, I\} \subseteq$ I. Given an element x in E, let $E(x)$ denote the norm closed inner ideal of E generated by x. It is known that $E(x)$ coincides with the norm-closure of the set

$Q(x)(E) = \{x, E, x\}$. Moreover $E(x)$ is a JB*-subalgebra of $E_2^{**}(r(x))$ and contains x as a positive element (compare page 19 and Proposition 2.1 in [6]).

The symmetrized Jordan triple product in a JB*-triple E is defined by the expression

$$< x, x+\epsilon, x+2\epsilon > := \frac{1}{3}(\{x, x+\epsilon, x+2\epsilon\} + \{x+\epsilon, x+2\epsilon, x\} + \{x+2\epsilon, x, x+\epsilon\}).$$

Given a C*-algebra (respectively, a JB*-algebra), A, the multiplier algebra of $A, M(A)$, is the set of all elements $x \in A^{**}$ such that for each elements $a \in A$, xa and ax (respectively, $x \circ a$) also lie in A. We notice that $M(A)$ is a C*-algebra (respectively, a JB*-algebra) and contains the unit element of A^{**}.

III. Orthogonally Additive Polynomials on C^*-Algebras: The Role Played by the Multiplier Algebra

One of the most useful tools used in the study of orthogonality preserving operators between C*-algebras is the description of all orthogonally additive $(1+2\epsilon)$-homogeneous polynomials on a C*-algebra, obtained in [20]. We present here a shorter and simplified proof of the main results established in the just quoted paper.

Let A be a C*-algebra and let X be a complex Banach space. A mapping $f: A \to X$ is said to be orthogonally additive (respectively, orthogonally additive on A_{sa}) if for every $a, (a+\epsilon) \in A$ (respectively, $a, (a+\epsilon) \in A_{sa}$) with $a \perp (a+\epsilon)$ we have $f(a+a+\epsilon) = f(a) + f(a+\epsilon)$.

We shall say that f is additive on elements having zero-product if for every $a, (a+\epsilon) \in A$ with $a(a+\epsilon) = 0 = (a+\epsilon)a$ we have $f(2a+\epsilon) = f(a) + f(a+\epsilon)$. When f behaves additively only on self-adjoint elements having zero-product, we shall say that f is additive on elements having zero-product on A_{sa}.

An X-valued n-homogeneous polynomial between two Banach spaces Y and X is a continuous X-valued mapping P_m on Y for which there exists a continuous (and symmetric) $(1+2\epsilon)$-linear operator $T_m: Y \times \cdots \times Y \longrightarrow X$ satisfying $P_m(x) = T_m(x, \ldots, x)$, for every x in X. The following polarization formula

$$T_m(x_1, \ldots, x_{(1+2\epsilon)}) = \frac{1}{2^{(1+2\epsilon)}(1+2\epsilon)!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{(1+2\epsilon)} P_m(\textstyle\sum_{i=1}^{1+2\epsilon} \varepsilon_i x_i), \quad (3.1)$$

holds for all $x_1, \dots, x_{1+2\epsilon} \in Y$.

Given two Banach spaces X and Y, the symbol $\mathcal{P}^{1+2\epsilon}(X, Y)$ will stand for the Banach space of all $(1+2\epsilon)$-homogeneous polynomials from X to Y and we write $\mathcal{P}^{1+2\epsilon}(X) := \mathcal{P}^{1+2\epsilon}(X, \mathbb{K})$.

The authors in [23] prove that for every compact Hausdorff space K and every orthogonally additive $(1+2\epsilon)$-homogeneous polynomial P_m from $C(K)$ to a Banach space X, there exists an operator $T_m: C(K) \to X$ satisfying that $P_m(f) = T_m(f^{1+2\epsilon})$, for all $f \in C(K)$. The proof remains valid when $C(K)$-spaces are replaced with $C_0(L)$ spaces, where L is a locally compact Hausdorff space.

Let $X_1, \dots, X_{1+2\epsilon}$, and X be Banach spaces, $T_m: X_1 \times \cdots \times X_{1+2\epsilon} \to X$ a (continuous) $(1+2\epsilon)$-linear operator, and $\pi: \{1, \dots, 1+2\epsilon\} \to \{1, \dots, 1+2\epsilon\}$ a permutation. It is known that there exists a unique $(1+2\epsilon)$-linear extension $AB(T_m)_\pi: X_1^{**} \times \cdots \times X_{1+2\epsilon}^{**} \to X^{**}$ such that for every $z_i \in X_i^{**}$ and every net $\left(x_{\alpha_i}^i\right) \in X_i (1 \le i \le 1+2\epsilon)$, converging to z_i in the weak * topology we have

$$AB(T_m)_\pi(z_1, \dots, z_{1+2\epsilon}) = \text{ weak}^* - \lim_{\alpha_{\pi(1)}} \cdots \text{ weak}^* - \lim_{\alpha_{\pi(1+2\epsilon)}} T_m\left(x_{\alpha_1}^1, \dots, x_{\alpha_{1+2\epsilon}}^{1+2\epsilon}\right).$$

Moreover, $AB(T_m)_\pi$ is bounded and has the same norm as T_m. The extensions $AB(T_m)_\pi$ coincide with those constructed for polynomials in [2], and are usually termed the Aron-Berner extensions of T_m (see also Proposition 3.1 in [22]).

If every operator from X_i to X_j^* is weakly compact $(i \ne j)$, the Aron-Berner extensions of T_m defined above do not depend on the chosen permutation π (see [3], and Theorem 1 in [5]). In particular, this happens when every X_i has Pelczynski's property (V) (if all of the X_i 's satisfy property (V), then their duals, X_i^*, have no copies of c_0, therefore every operator from X_i to X_j^* is unconditionally converging, and hence weakly compact by (V), see [21]). When all the Aron-Berner extensionsof T_m coincide, the symbol $AB(T_m)$ will stand for any of them. It is also known that, $AB(T_m)$ is symmetric whenever T_m is.

When $P_m: X \to Y$ is the $(1+2\epsilon)$-homogeneous polynomial defined by T_m, $AB(P_m): X^{**} \to Y^{**}$ will denote the $(1+2\epsilon)$-homogeneous polynomial whose associated symmetric $(1+2\epsilon)$-linear operator is $AB(T_m)$.

We should note at this point that every C^*-algebra satisfies property (V) (see Corollary 6 in [24]).

The original proof presented in [20] relies on the following technical result: for every symmetric and continuous $(1+2\epsilon)$-linear form T_m on a C*-algebra A such that the $(1+2\epsilon)$ homogeneous polynomial $P_m(x) = T_m(x, \dots, x), (\forall x \in A)$ is orthogonally additive on A_{sa}, the (2ϵ)-homogeneous polynomial $R(x) = AB(T_m)(1, x, \dots, x), (\forall x \in A)$ is orthogonally additive on A_{sa}, where 1 denotes the unit of A^{**}. The proof exhibited in this paper avoids the use of the above technical tool. Instead of using the Aron-Berner extension on the $A^{**} \times \dots \times A^{**}$ we shall focus our attention on its restriction to the Cartesian product $M(A) \times \dots \times M(A)$, where $M(A)$ denotes the multiplier algebra of A in A^{**}.

The following result, whose proof is essentially algebraic, is inspired by Proposition 2.4 in [23].

Lemma 3.1 (see [30]). Let $P_m: A \to \mathbb{K}$ be an element in $\mathcal{P}^{1+2\epsilon}(A)$ and let $T_m: A \times \cdots \times A \to \mathbb{K}$ be its associate symmetric n-linear operator. Suppose that P_m is orthogonally additive on A_{sa}. Then for every $\epsilon > 0$ and every $a_1, \dots, a_{1+\epsilon}, b_1, \dots, b_\epsilon$ in A_{sa} such that, for each i and j, a_i and b_j are orthogonal we have

$$T_m(a_1, \dots, a_{1+\epsilon}, b_1, \dots, b_\epsilon) = 0.$$

Proof. Let $\epsilon > 0$. We claim that for every a and $(a+\epsilon)$ in A_{sa} with $a \perp (a+\epsilon)$ we have

$$T_m(a, 1+\epsilon, a, a+\epsilon, \epsilon, a+\epsilon) = 0. \quad (3.2)$$

Indeed, the equation

$$\lambda^{1+2\epsilon}T_m(a,\dots,a)+\mu^{1+2\epsilon}T_m(a+\epsilon,\dots,a+\epsilon)=\lambda^{1+2\epsilon}P_m(a)+\mu^{1+2\epsilon}P_m(a+\epsilon)=P_m(\lambda a+\mu(a+\epsilon))$$
$$=\sum_{\substack{0\le k_1,k_2\le 1+2\epsilon\\ k_1+k_2=1+2\epsilon}}\frac{(1+2\epsilon)!}{k_1!\,k_2!}\lambda^{k_1}\mu^{k_2}T_m(a,k_1,a,a+\epsilon,k_2,a+\epsilon)\text{ (by the symmetry of } T_m\text{),}$$

holds for every λ and μ in $\mathbb{R}$. Therefore,

$$\sum_{\substack{0<k_1,k_2<1+2\epsilon\\ k_1+k_2=1+2\epsilon}}\frac{(1+2\epsilon)!}{k_1!\,k_2!}\lambda^{k_1}\mu^{k_2}T_m(a,k_1,a,a+\epsilon,k_2,a+\epsilon)=0,$$

for all λ and μ in $\mathbb{R}$, which in particular gives (3.2).

Let $a_1,\dots,a_{1+\epsilon},b_1,\dots,b_\epsilon$ in A_{sa} be such that, for each i and j, a_i and b_j are orthogonal. Having in mind that whenever we fix $(1+\epsilon)$ variables of T_m we have another symmetric and continuous multilinear form, the polarization formula (3.1) yields

$$T_m\left(a_1,\dots,a_{1+\epsilon},\sum_{j=1}^{\epsilon}\varepsilon_j b_j,\dots,\sum_{j=1}^{\epsilon}\varepsilon_j b_j\right)$$
$$=\frac{1}{2^{1+\epsilon}(1+\epsilon)!}\sum_{\sigma_j=\pm1}\sigma_1\cdots\sigma_{1+\epsilon}T_m\left(\sum_{k=1}^{1+\epsilon}\sigma_k a_k,\dots,\sum_{k=1}^{1+\epsilon}\sigma_k a_k,\sum_{j=1}^{\epsilon}\varepsilon_j b_j,\dots,\sum_{j=1}^{\epsilon}\varepsilon_j b_j\right)=0,$$

where in the last equality we applied (3.2) and the fact that $\sum_{k=1}^{1+\epsilon}\sigma_k a_k$ and $\sum_{j=1}^{\epsilon}\varepsilon_j b_j$ are orthogonal. Finally, the formula (3.3) gives

$$T_m(a_1,\dots,a_{1+\epsilon},b_1,\dots,b_\epsilon)$$
$$=\frac{1}{2^{\epsilon}(\epsilon)!}\sum_{\varepsilon_i=\pm1}\varepsilon_1\cdot\ldots\cdot\varepsilon_\epsilon T_m\left(a_1,\dots,a_{1+\epsilon},\sum_{j=1}^{\epsilon}\varepsilon_j b_j,\dots,\sum_{j=1}^{\epsilon}\varepsilon_j b_j\right)=0. \quad (3.3)$$

Proposition 3.1 (see [30]). Let A be a C^*-algebra. Suppose that $T_m: A\times\ldots\times A\to\mathbb{C}$ is a symmetric and continuous n-linear form on A such that the $(1+2\epsilon)$-homogeneous polynomial $P_m(x)=T_m(x,\dots,x),\forall x\in A$, is orthogonally additive on A_{sa}. Then the polynomial $R: M(A)\to\mathbb{C}, R(x):=AB(T_m)(x,\dots,x)$ is orthogonally additive on $M(A)_{sa}$.

Proof. Let a and $(a+\epsilon)$ be two orthogonal elements in $M(A)_{sa}$. Since $a^{\frac{1}{3}}$ and $(a+\epsilon)^{\frac{1}{3}}$ are orthogonal, we deduce that, for each pair $x,(x+\epsilon)$ in A, $a^{\frac{1}{3}}xa^{\frac{1}{3}}$ and $(a+\epsilon)^{\frac{1}{3}}(x+\epsilon)(a+\epsilon)^{\frac{1}{3}}$ also are orthogonal elements in A. The hypothesis of P_m being orthogonally additive assures, via Lemma 3.1, that

$$T_m\left(a^{\frac{1}{3}}x_1a^{\frac{1}{3}}+(a+\epsilon)^{\frac{1}{3}}y_1(a+\epsilon)^{\frac{1}{3}},\dots,a^{\frac{1}{3}}x_{1+2\epsilon}a^{\frac{1}{3}}+(a+\epsilon)^{\frac{1}{3}}y_{1+2\epsilon}(a+\epsilon)^{\frac{1}{3}}\right)=T_m\left(a^{\frac{1}{3}}x_1a^{\frac{1}{3}},\dots,a^{\frac{1}{3}}x_{1+2\epsilon}a^{\frac{1}{3}}\right)$$
$$+T_m\left((a+\epsilon)^{\frac{1}{3}}y_1(a+\epsilon)^{\frac{1}{3}},\dots,(a+\epsilon)^{\frac{1}{3}}y_{1+2\epsilon}(a+\epsilon)^{\frac{1}{3}}\right),\text{ for all } x_1,\dots x_{1+2\epsilon},y_1,\dots,y_{1+2\epsilon}\in A. \quad (3.4)$$

Now, Goldstine's theorem (cf. Theorem V.4.2.5 in [10]) guarantees that the closed unit ball of A_{sa} is weak*-dense in the closed unit ball of A_{sa}^{**}. Therefore there exist two bounded nets (x_λ) and (y_μ) in A_{sa}, converging in the weak*-topology of A^{**} to $a^{\frac{1}{3}}$ and $(a+\epsilon)^{\frac{1}{3}}$, respectively. In our setting the Aron-Berner extension of T_m is separately weak*-continuous. Thus, by replacing, in equation (3.4), x_1 and y_1 with (x_λ) and (y_μ), respectively, and taking weak*-limits, we have:

$$AB(T_m)\left(2a+\epsilon, a^{\frac{1}{3}}x_2a^{\frac{1}{3}}+(a+\epsilon)^{\frac{1}{3}}y_2(a+\epsilon)^{\frac{1}{3}}, \dots, a^{\frac{1}{3}}x_{1+2\epsilon}a^{\frac{1}{3}}+(a+\epsilon)^{\frac{1}{3}}y_{1+2\epsilon}(a+\epsilon)^{\frac{1}{3}}\right)$$
$$= AB(T_m)\left(a, a^{\frac{1}{3}}x_2a^{\frac{1}{3}}, \dots, a^{\frac{1}{3}}x_{1+2\epsilon}a^{\frac{1}{3}}\right) + AB(T_m)\left(a+\epsilon, (a+\epsilon)^{\frac{1}{3}}y_2(a+\epsilon)^{\frac{1}{3}}, \dots, (a+\epsilon)^{\frac{1}{3}}y_{1+2\epsilon}(a+\epsilon)^{\frac{1}{3}}\right),$$

for all $x_2, \dots x_{1+2\epsilon}, y_1, \dots, y_{1+2\epsilon} \in A$. When the above argument is repeated for $x_2, y_2, \dots, x_{1+2\epsilon}, y_{1+2\epsilon}$ we derive

$$R(2a+\epsilon) = AB(T_m)(2a+\epsilon, \dots, 2a+\epsilon)$$
$$= AB(T_m)(a, \dots, a) + AB(T_m)(a+\epsilon, \dots, a+\epsilon) = R(a) + R(a+\epsilon),$$

which finishes the proof.

We observe that $M(A)$ is always unital, so Proposition 3.1 allows us to apply the final argument in the proof of Theorem 2.8 in [20] but avoiding some technical and laborious results needed in its original proof(see [30]).

Theorem 3.1. [20] Let A be a C^*-algebra, $(1+2\epsilon) \in \mathbb{N}$ and P_m an n-homogeneous scalar polynomial on A. The following are equivalent.

(a) There exists $\varphi_m \in A^*$ such that, for every $x \in A$,

$$P_m(x) = \varphi_m(x^{1+2\epsilon}).$$

(b) P_m is additive on elements having zero-products.

(c) P_m is orthogonally additive on A_{sa}.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. To see that $(c) \Rightarrow (a)$ we proceed by induction on $(1+2\epsilon)$. When $\epsilon = 0$ the result is trivial. We suppose that the statement is true for (2ϵ).

Let $T_m: A \times \dots \times A \to \mathbb{C}$ be the unique symmetric and continuous $(1+2\epsilon)$-linear form on A associated to P_m. Proposition 3.1 guarantees that the polynomial $AB(P_m)$associated to $AB(T_m)$ is orthogonally additive on $M(A)_{sa}$.

Let θ be defined by $\theta(x_2, \dots, x_{1+2\epsilon}) := AB(T_m)(1, x_2, \dots, x_{1+2\epsilon}), (x_2, \dots, x_{1+2\epsilon} \in M(A))$. We claim that the polynomial R associated to θ is orthogonally additive on $M(A)_{sa}$. Indeed, let a and $(a+\epsilon)$ be two orthogonal elements in $M(A)_{sa}$ and let C denote C* subalgebra of $M(A)$ generated by $a, (a+\epsilon)$ and 1 . Clearly C is a unital abelian C*-algebra and $P_m|_C$ is orthogonally additive. Thus, Theorem 2.1 in [23] assures the existence of a functional $\psi_C \in C^*$ such that

$$AB(T_m)|_C(y_1, \dots, y_{1+2\epsilon}) = \psi_{a+2\epsilon}(y_1 \dots y_{1+2\epsilon})$$

for all $y_1, \dots, y_{1+2\epsilon} \in C_x$. In particular

$$R(2a+\epsilon) = \theta(2a+\epsilon, \dots, 2a+\epsilon) = AB(T_m)|_C(1, 2a+\epsilon, \dots, 2a+\epsilon)$$
$$= \psi_C((2a+\epsilon)^{2\epsilon}) = \psi_C(a^{2\epsilon} + (a+\epsilon)^{2\epsilon}) = \psi_C(a^{2\epsilon}) + \psi_C((a+\epsilon)^{2\epsilon})$$
$$= AB(T_m)|_C(1, a, \dots, a+\epsilon) + AB(T_m)|_C(1, a+\epsilon, \dots, a+\epsilon) = R(a) + R(a+\epsilon),$$

which proves the claim.

By the induction hypothesis, there exists $\varphi_m \in M(A)^*$ such that

$$R(x) = \varphi_m(x^{2\epsilon})$$

for all $x \in M(A)$.

On the other hand, for every $x \in M(A)_{sa}$, let C_x be the abelian C*-subalgebra of $M(A)$ generated by 1 and x, and let $(T_m)_{|C_x}: C_x \times \dots \times C_x \to \mathbb{C}$ be the restriction of T_m. Clearly the polynomial associated to $(T_m)_{|C_x}$ also is orthogonally additive. Therefore, Theorem 2.1 of [23] guarantees the existence of a measure $\psi_x \in (C_x)^*$ with $\|\psi_x\| = \|(T_m)_{|C_x}\|$ such that

$$(T_m)_{|C_x}(y_1, \dots, y_{1+2\epsilon}) = \psi_x(y_1 \dots y_{1+2\epsilon})$$

for all $y_1, \dots, y_{1+2\epsilon} \in C_x$.

Now, we claim that, for every $x \in M(A)_{sa}, \psi_x = \varphi_{m_{|C_x}}$. Indeed, let us fix $x \in M(A)_{sa}$ and pick a positive element $(x + 2\epsilon) \in C_x$. There is no loss of generality in assuming that $\| x + 2\epsilon \| = 1$. The positivity of $(x + 2\epsilon)$ implies the existence of a positive normone element $(x + \epsilon) \in C_x$ satisfying $(x + \epsilon)^{2\epsilon} = x + 2\epsilon$.

We therefore have

$$\psi_x(x + 2\epsilon) = \psi_x((x + \epsilon)^{2\epsilon}) = AB\big((T_m)_{|C_x}\big)(1, x + \epsilon, \dots, x + \epsilon) = AB(T_m)(1, x + \epsilon, \dots, x + \epsilon)$$
$$= \theta(x + \epsilon, \dots, x + \epsilon) = R(x + \epsilon) = \varphi_m((x + \epsilon)^{2\epsilon}) = \varphi_m(x + 2\epsilon).$$

Since $(x + 2\epsilon)$ is an arbitrary positive norm-one element in C_x we deduce, by linearity,

that $\psi_x = \varphi_{m_{|C_x}}$.

Thus, for each $x \in M(A)_{sa}$, we have

$$AB(P_m)(x) = AB(T_m)(x, \dots, x) = \psi_x(x^{1+2\epsilon}) = \varphi_m(x^{1+2\epsilon}).$$

The polarization formula given in (3.1) applies to prove that $AB(P_m)(x) = \varphi_m(x^{1+2\epsilon})$ for all $x \in M(A)$.

The following vector-valued version of the above theorem was established in [20], Corollary 3.1.

Theorem 3.2. [20] Let A be a C^*-algebra, X a complex Banach space, $(1 + 2\epsilon) \in \mathbb{N}$ and $P_m: A \to X$ an n-homogeneous polynomial. The following are equivalent.

(a) There exists an operator $T_m: A \to X$ such that, for every $x \in A$,

$$P_m(x) = T_m(x^{1+2\epsilon}).$$

(b) P_m is additive on elements having zero-products.

(c) P_m is orthogonally additive on A_{sa}.

IV. Orthogonality Preservers Between C*-Algebras and JB*-Algebras

Let J be an arbitrary JB*-algebra. One of the main results stated in [7] describes the orthogonality preserving operators from J to a JB*-triple whose second transpose maps the unit in A^{**} to a tripotent in E^{**}. This section contains most of the novelties in this paper. We shall present a complete description of all orthogonality preserving operators from a JB*-algebra to a JB*-triple, without assuming any additional condition.

We recall that two elements $a, (a + \epsilon)$ in a JB*-triple are said to be orthogonal (written $\perp a + \epsilon$) if $L(a, a + \epsilon) = 0$. Lemma 1 in [7] shows that $a \perp a + \epsilon$ if and only if one of the following statements holds:

$$\begin{array}{ccc} \{a, a, a + \epsilon\} = 0; & a \perp r(a + \epsilon); & r(a) \perp r(a + \epsilon); \\ E_2^{**}(r(a)) \perp E_2^{**}(r(a + \epsilon)); & r(a) \in E_0^{**}(r(a + \epsilon)); & a \in E_0^{**}(r(a + \epsilon)); \\ a + \epsilon \in E_0^{**}(r(a)); & E_a \perp E_{a+\epsilon} & \{a + \epsilon, a + \epsilon, a\} = 0. \end{array} \quad (4.1)$$

The Jordan identity (JB1) and the above reformulations assure that

$$a \perp \{x, x + \epsilon, x + 2\epsilon\} \text{ whenever } a \text{ is orthogonal to } x, x + \epsilon \text{ and } (x + 2\epsilon). \quad (4.2)$$

If A is a C*-algebra, it can be checked from the above reformulations, that two elements $a, a+\epsilon$ in A are orthogonal for the C*-algebra product (i.e. $(a+\epsilon)^* = 0 = (a+\epsilon)^* a$) if and only if they are orthogonal when A is considered as a JB*-triple.

The equivalent reformulations of orthogonality given in (4.1) admit another

equivalent statement in the setting of JB*-algebra when one of the elements is positive.

Lemma 4.1 (see [30]). Let h_m and x be elements in a JB^*-algebra J with h_m positive. Then $x \perp h_m$ if and only if $h_m \circ x = 0$.

Proof. Having in mind that $h_m \circ x = \{1, h_m, x\}$, where 1 denotes the unit element in J^{**}, it is clear that $h_m \circ x = 0$ whenever $h_m \perp x$. We shall show that $x \perp h_m$ whenever $h_m \circ x = 0$. Given a positive element h_m in J, there exists another positive element $(a+\epsilon)$ satisfying $(a+\epsilon)^2 = h_m$. Since the triple product $\{a+\epsilon, a+\epsilon, x\}$ coincides with $(a+\epsilon)^2 \circ x = h_m \circ x = 0$, the equivalent reformulations of orthogonality given in (4.1) guarantee that $(a+\epsilon) \perp x$, or equivalently, $x \in J_0^{**}(r(a+\epsilon))$. It is not hard to check that for a positive $(a+\epsilon)$, the range tripotents $r(a+\epsilon)$ and $r((a+\epsilon)^2) = r(h_m)$ both coincide with the range projection of $(a+\epsilon)$ in J^{**} and hence $r(a+\epsilon) = r((a+\epsilon)^2) = r(h_m)$. Again, the equivalences stated in (4.1) assure that $x \perp h_m$.

Let E and F be JB*-triples. An operator $T_m: E \to F$ is said to be orthogonality preserving if $T_m(a) \perp T_m(a+\epsilon)$ whenever $a \perp (a+\epsilon)$ in E. This concept extends the usual definition of orthogonality preserving linear operator between C*-algebras.

Lemma 4.2 (see [30]). Let $T_m: J \to E$ be an orthogonality preserving operator from a JB^* algebra to a JB^*-triple, then $T_m^{**}|_{M(J)}: M(J) \to E^{**}$ is orthogonality preserving.

Proof. Let $a, (a+\epsilon) \in M(J)$. By (4.1), $a^{\left[\frac{1}{3}\right]}$ and $(a+\epsilon)^{\left[\frac{1}{3}\right]}$ are orthogonal elements in $M(J)$. Thus, we deduce that for each pair $x, (x+\epsilon)$ in J, $Q\left(a^{\left[\frac{1}{3}\right]}\right)x$ and $Q\left((a+\epsilon)^{\left[\frac{1}{3}\right]}\right)(x+\epsilon)$ are two orthogonal elements in J. Now, Goldstine's theorem guarantees that the closed unit ball of J is weak*-dense in the closed unit ball of J^{**}. Therefore there exist two bounded nets (x_λ) and (y_μ) in J, converging in the weak*-topology of J^{**} to $a^{\left[\frac{1}{3}\right]}$ and $(a+\epsilon)^{\left[\frac{1}{3}\right]}$, respectively.

Since the triple product of any JBW*-triple is separately weak * continuous ([4]) and T_m^{**} is weak *-continuous, we deduce that, for each $x, (x+\epsilon)$ in J, the net $0 = \left\{T_m\left(Q\left(a^{\left[\frac{1}{3}\right]}\right)x_\lambda\right), T_m\left(Q\left(a^{\left[\frac{1}{3}\right]}\right)x\right), T_m\left(Q\left((a+\epsilon)^{\left[\frac{1}{3}\right]}\right)(x+\epsilon)\right)\right\}$ converges to

$\left\{T_{\mathrm{m}}^{**}(a), T_m\left(Q\left(a^{\left[\frac{1}{3}\right]}\right)x\right), T_m\left(Q\left((a+\epsilon)^{\left[\frac{1}{3}\right]}\right)(x+\epsilon)\right)\right\}$ in the weak*-topology of E^{**}. Therefore

$$\left\{T_{\mathrm{m}}^{**}(a), T_m\left(Q\left(a^{\left[\frac{1}{3}\right]}\right)x\right), T_m\left(Q\left((a+\epsilon)^{\left[\frac{1}{3}\right]}\right)(x+\epsilon)\right)\right\} = 0,$$

for all $x, (x+\epsilon) \in J$. Similarly, $\left\{T_{\mathrm{m}}^{**}(a), T_{\mathrm{m}}^{**}(a), T_m\left(Q\left((a+\epsilon)^{\left[\frac{1}{3}\right]}\right)(x+\epsilon)\right)\right\} = 0$, for all $(x+\epsilon) \in J$.

Finally, $0 = \left\{T_{\mathrm{m}}^{**}(a), T_{\mathrm{m}}^{**}(a), T_m\left(Q\left((a+\epsilon)^{\left[\frac{1}{3}\right]}\right)y_\mu\right)\right\} \to \{T_{\mathrm{m}}^{**}(a), T_{\mathrm{m}}^{**}(a), T_{\mathrm{m}}^{**}(a+\epsilon)\}$, in the weak*-topology of E^{**}, and hence $T_{\mathrm{m}}^{**}(a) \perp T_{\mathrm{m}}^{**}(a+\epsilon)$.

Let A be a C*-algebra and let X be a complex Banach space. A continuous sesquilinear mapping $\Phi: A \times A \to X$ is said to be orthogonal if $\Phi(a, a+\epsilon) = 0$ for every $a, (a+\epsilon) \in A$ such that $a \perp (a+\epsilon)$. By a celebrated result due to [13] (see [14] for an alternative proof), for every continuous sesquilinear orthogonal form $V: A \times A \to \mathbb{C}$, there exist two functionals $\omega_1, \omega_2 \in A^*$ satisfying that

$$V(x, x+\epsilon) = \omega_1(x(x+\epsilon)^*) + \omega_2((x+\epsilon)^* x),$$

for all $x, (x+\epsilon) \in A$. Denoting $\phi = \omega_1 + \omega_2$ and $\psi = \omega_1 - \omega_2$, we have

$$V(x, x+\epsilon) = \phi(x \circ (x+\epsilon)^*) + \psi([x, (x+\epsilon)^*]),$$

for all $x, (x+\epsilon) \in A$, where $a \circ (a+\epsilon) := \frac{1}{2}(a(a+\epsilon) + (a+\epsilon)a), [a, a+\epsilon] := \frac{1}{2}(a(a+\epsilon) - (a+\epsilon)a)$. In particular, $V(x, x+\epsilon) = V(x+\epsilon, x)$ whenever $[x, (x+\epsilon)^*] = 0$ and $x \circ (x+\epsilon)^* = x^* \circ (x+\epsilon)$. The following lemma follows straightforwardly from the above remarks and the Hahn-Banach theorem.

Lemma 4.3. Let A be a C^*-algebra, X a Banach space and $\Phi: A \times A \to X$ a continuous sesquilinear orthogonal operator. Then $\Phi(x, x+\epsilon) = \Phi(x+\epsilon, x)$ whenever $[x, (x+\epsilon)^*] = 0$ and $x \circ (x+\epsilon)^* = x^* \circ (x+\epsilon)$.

Let us recall that two elements a and $(a+\epsilon)$ in a JB*-algebra J are said to operator commute in J if the multiplication operators M_a and $M_{a+\epsilon}$ commute, where M_a is defined by $M_a(x) := a \circ x$. That is, a and $(a+\epsilon)$ operators commute if and only if $(a \circ x) \circ (a+\epsilon) = a \circ (x \circ (a+\epsilon))$ for all x in J. Self-adjoint elements a and $(a+\epsilon)$ in J generate a JB*-subalgebra that can be realized as a JC*-subalgebra of some $B(H)$, [29], and, in this identification, a and $(a+\epsilon)$ commute in the usual sense whenever the operators commute in J (compare Proposition 1 in [25]). Similarly, two elements a and $(a+\epsilon)$ of J_{sa} operator commute if and only if $a^2 \circ (a+\epsilon) = \{a, a+\epsilon, a\}$ (i.e., $a^2 \circ (a+\epsilon) = 2(a \circ (a+\epsilon)) \circ a - a^2 \circ (a+\epsilon)$). If $(a+\epsilon) \in J$ we use $\{a+\epsilon\}'$ to denote the set of elements in J that operator commute with $(a+\epsilon)$. (This corresponds to the usual notation in von Neumann algebras.)

Proposition 4.1 (see [30]). Let A be a C^*-algebra, E a JB*-triple and $T_m: A \to E$ an orthogonality preserving operator. Then for $h_m = T_{\mathrm{m}}^{**}(1)$, the following assertions hold:

a) $\{T_m(x), h_m, h_m\} = \{h_m, T_m(x^*), h_m\}$, for all $x \in A$.

b) $T_m(A_{sa}) \subset E_2^{**}(r(h_m))_{sa}$.

c) For each $a \in A, T_m(a)$ and h_m operator commute in the JB*-algebra $E_2^{**}(r(h_m))$.
d) When h_m is a tripotent, then $T_m: A \to E_2^{**}(r(h_m))$ is a Jordan *-homomorphism, in particular T_m is a triple homomorphism.

Proof. a) By Lemma 4.2, $T_m^{**}|_{M(A)}: M(A) \to E^{**}$ is orthogonality preserving. Therefore, the assignment $(x, x+\epsilon) \mapsto \{T_{\mathrm{m}}^{**}(x), T_{\mathrm{m}}^{**}(x+\epsilon), h_m\}$, defines a continuous sesquilinear orthogonal operator on $M(A) \times M(A)$. Lemma 4.3, applied to $x \in A_{sa}$ and $(x+\epsilon) = 1$ gives $\{T_m(x), h_m, h_m\} = \{h_m, T_m(x), h_m\}$. The desired statement follows by linearity.

b) Let $a \in A_{sa}$. By the Peirce arithmetic and a) we have

$$\begin{aligned}\{(P_m)_2(r(h_m))T_m(a), h_m, h_m\} + \{(P_m)_1(r(h_m))T_m(a), h_m, h_m\} &= \{T_m(a), h_m, h_m\} \\ = \{h_m, T_m(a), h_m\} &= \{h_m, (P_m)_2(r(h_m))T_m(a), h_m\},\end{aligned}$$

which implies that $\{(P_m)_1(r(h_m))T_m(a), h_m, h_m\} = 0$, and hence $(P_m)_1(r(h_m))T_m(a) \perp h_m$. The equivalences in (4.1) imply that $(P_m)_1(r(h_m))T_m(a) \in E_0^{**}(r(h_m))$, which gives

$$T_m(A_{sa}) \subseteq E_2^{**}(r(h_m)) \oplus E_0^{**}(r(h_m)). \tag{4.3}$$

Consider now the mapping $(P_m)_3: M(A) \to E^{**}$,

$$(P_m)_3(x) = \{T_{\mathrm{m}}^{**}(x), T_m^{**}(x^*), T_m^{**}(x)\}.$$

It is clear that $(P_m)_3$ is a 3 -homogeneous polynomial on $M(A)$. Since, by Lemma 4.2, $T_m^{**}|_{M(A)}$ is orthogonality preserving, $(P_m)_3$ is orthogonally additive on $M(A)_{sa}$. By Corollary 3.1 in [20] or Theorem 3.2, there exists an operator $(F_m)_3: M(A) \to E^{**}$ satisfying that

$$(P_m)_3(x) = (F_m)_3(x^3),$$

for all x in $M(A)$. If $(S_m)_3: M(A) \times M(A) \times M(A) \to E^{**}$ is the (unique) symmetric 3-linear operator associated to $(P_m)_3$, we have

$$(F_m)_3(< x, x+\epsilon, x+2\epsilon >) = (S_m)_3(x, x+\epsilon, x+2\epsilon) =< T_m^{**}(x), T_m^{**}(x+\epsilon), T_m^{**}(x+2\epsilon) >, \quad (4.4)$$

for all $x, (x+\epsilon), (x+2\epsilon) \in M(A)_{sa}$. Now, taking $a \in M(A)_{sa}$ and $(x+\epsilon) = (x+2\epsilon) = 1$ in (4.4), we deduce that

$$(F_m)_3(a) =< T_m^{**}(a), h_m, h_m >= \frac{2}{3}\{T_m^{**}(a), h_m, h_m\} + \frac{1}{3}\{h_m, T_m^{**}(a), h_m\}. \quad (4.5)$$

Thus, for each $a \in M(A)_{sa}$ we have

$$\{T_m^{**}(a), T_m^{**}(a), T_m^{**}(a)\} = (F_m)_3(a^3) =< h_m, h_m, T_m^{**}(a^3) >. \quad (4.6)$$

Now, (4.3), (4.6) and the Peirce arithmetic show that

$$T_m(A_{sa}) \subseteq E_2^{**}(r(h_m)) \cap E.$$

We shall finally prove that T_m is symmetric for the involution in $E_2^{**}(r(h_m))$. In order to simplify notation, we shall write $r(h_m) = r$. Let us recall that $E_2^{**}(r)$ is a JB*-algebra with Jordan product and involution given by $x \cdot_r (x+\epsilon) = \{x, r, x+\epsilon\}$ and $x^\sharp = \{r, x, r\} = Q(r)(x)$, respectively. The triple product in $E_2^{**}(r)$ is also determined by the expression

$$\{x, x+\epsilon, x+2\epsilon\} = \left(x \cdot_r (x+\epsilon)^\sharp\right) \cdot_r (x+2\epsilon) + \left((x+2\epsilon) \cdot_r (x+\epsilon)^\sharp\right) \cdot_r x - (x \cdot_r (x+2\epsilon)) \cdot_r (x+\epsilon)^{\breve{k}r}.$$

Lemma 4.3 applied to the form $\Phi(x, x+\epsilon) = \{T_{\mathrm{m}}^{**}(x), T_{\mathrm{m}}^{**}(x+\epsilon), x+2\epsilon\}$ guarantees that

$$\{T_{\mathrm{m}}^{**}(x), h_m, x+2\epsilon\} = \{h_m, T_{\mathrm{m}}^{**}(x), x+2\epsilon\}$$

for every $x \in M(A)_{sa}$ and $(x+2\epsilon) \in E^{**}$. Let us fix $x = a \in A_{sa}$. By taking $(x+2\epsilon) = r$, the above identity gives $h_m \cdot_r T_m(a)^\sharp \ddot{r}_r = h_m \cdot_r T_m(a)$, that is, $h_m \cdot_r \frac{T_m(a) - T_m(a)^\sharp r}{2i} = 0$. Lemma 4.1 now applies to give $(T_m(a) - T_m(a)^{Er}) \perp h_m$, and hence $T_m(a) - T_m(a)^{\circ r}$ lies in $E_2^{**}(r) \cap E_0^{**}(r) = \{0\}$ (compare (4.1)). This implies $T_m(A_{sa}) \subset E_2^{**}(r)_{sa}$.

c) It follows by $a+\epsilon$) that $T_m(A_{sa}) \subset E_2^{**}(r)_{sa}$ and hence the triple product in $T_m(A_{sa})$ is determined by the Jordan product of $E_2^{**}(r)_{sa}$. By a), for each $a \in A_{sa}$, we have $\{h_m, h_m, T_m(a)\} = \{h_m, T_m(a), h_m\}$. Thus, $h_m^2 \cdot_r T_m(a) = 2\left(h_m \cdot_r T_m(a)\right) \cdot_r h_m - h_m^2 \cdot_r T_m(a)$, which gives the desired statement.

d) Let us assume that h_m is a tripotent. In this case $h_m = r(h_m) = r$. Statement $a+\epsilon$) assures that $T_m(A_{sa}) \subset E_2^{**}(r)_{sa}$. Thus, equation (4.5) guarantees that $(F_m)_3(a) = \{T_{\mathrm{m}}^{**}(a), h_m, h_m\} = \{h_m, T_{\mathrm{m}}^{**}(a), h_m\} = T_{\mathrm{m}}^{**}(a)$, for all $a \in M(A)_{sa}$. Now, the formula established in (4.4) implies that

$$< T_{\mathrm{m}}^{**}(a), T_{\mathrm{m}}^{**}(a+\epsilon), T_{\mathrm{m}}^{**}(a+2\epsilon) >= (F_m)_3(< a, a+\epsilon, a+2\epsilon >) = T_{\mathrm{m}}^{**}(< a, a+\epsilon, a+2\epsilon >),$$

for all $a, (a+\epsilon), (a+2\epsilon) \in M(A)_{sa}$. Taking $\epsilon = \frac{1-a}{2}$ in the above equation, we have

$$T_{\mathrm{m}}^{**}(a) \cdot_r T_{\mathrm{m}}^{**}(a+\epsilon) = \{T_{\mathrm{m}}^{**}(a), T_{\mathrm{m}}^{**}(a+\epsilon), r\} = T_{\mathrm{m}}^{**}(\{a, a+\epsilon, 1\}) = T_{\mathrm{m}}^{**}(a \circ (a+\epsilon)),$$

for all $a, (a+\epsilon) \in M(A)_{sa}$. We have then shown that $T_{\mathrm{m}}^{**}|_{M(A)}: M(A) \to E_2^{**}(r)$ is a unital Jordan *-homomorphism, which proves d).

It should be noticed that the main result in [27] is a direct consequence of statement) in the above proposition.

Let $T_m: J \to E$ be an orthogonality preserving operator from a JB*-algebra to a JB*-triple and let h_m denote $T_{\mathrm{m}}^{**}(1)$. Lemma 4.2 assures that $T_{\mathrm{m}}^{**}|_{M(J)}: M(J) \to E^{**}$ also is orthogonality preserving. Since for each self-adjoint element $a \in M(J)$, the JB*-subalgebra $C_{\{1,a\}}$ of $M(J)$ generated by a and 1 is JB*-isomorphic to an abelian C*-algebra (compare Theorem 3.2.4 in [15]), the mapping $T_{\mathrm{m}}^{**}|_{\{11,a\}}: C_{\{1,a\}} \to E^{**}$ satisfies the hypothesis of Proposition 4.1 above. Therefore, $T_{\mathrm{m}}^{**}(a) \in E_2^{**}(r(h_m))_{sa}$, $T_{\mathrm{m}}^{**}(a)$ and h_m operator commute in the JB*-algebra $E_2^{**}(r(h_m))$ and if h_m is a tripotent then, $T_m^{**}(a^2) = T_m^{**}(a) \cdot_{r(h_m)} T_m^{**}(a)$. We have proved the following result(see [30]).

Corollary 4.1. Let J be a JB^*-algebra, E a JB^*-triple and $T_m: J \to E$ an orthogonality preserving operator. Then for $h_m = T_m^{**}(1)$, the following assertions hold:

a) $\{T_m(x), h_m, h_m\} = \{h_m, T_m(x^*), h_m\}$, for all $x \in J$.

b) $T_m(J_{sa}) \subset E_2^{**}(r(h_m))_{sa}$.

c) For each $a \in J, T_m(a)$ and h_m operator commute in the JB*-algebra $E_2^{**}(r(h_m))$.
d) When h_m is a tripotent, then $T_m: J \to E_2^{**}(r(h_m))$ is a Jordan *-homomorphism, in particular T_m is a triple homomorphism.

The result describing orthogonality preserving operators from a JB*-algebra to a JB*-triple can be now stated(see [30]).

Theorem 4.1. Let $T_m: J \to E$ be an operator from a JB^*-algebra to a JB*-triple and let $h_m = T_m^{**}(1)$. The following are equivalent:

a) T_m is orthogonality preserving.

b) There exists a (unital) Jordan *-homomorphism $S_m: M(J) \to E_2^{**}(r(h_m))$ such that $S_m(x)$ and h_m operator commute and $T_m(x) = h_m \cdot_{r(h_m)} S_m(x)$, for every $x \in J$.

Proof. The implication $b) \Rightarrow a$) is clear.

$a) \Rightarrow b)$ Let C denote the JB*-subalgebra of $E_2^{**}(r(h_m))$ generated by h_m and $r(h_m)$. Let $\sigma(h_m) \subseteq (0, \| h_m \|]$ denote the spectrum of h_m in $E_2^{**}(r(h_m))$. It is known that $\sigma(h_m) \cup \{0\}$ is compact and C is JB*-isomorphic to $C(\sigma(h_m) \cup \{0\})$, and under this identification h_m corresponds to the function $t \mapsto t$ (compare Theorem 3.2.4 in [15]). For each natural $(1 + 2\epsilon)$, let $p_{(1+2\epsilon)}$ be the projection in $\bar{C}^{w^*}$ whose representation in $C(\sigma(h_m) \cup \{0\})^{**}$ is the characteristic function $\chi_{\left((\sigma(h_m)\cup\{0\})\cap\left[\frac{1}{1+2\epsilon},1\right]\right)}$, and let $(h_m)_{1+2\epsilon} = p_{1+2\epsilon} \cdot_{r(h_m)} h_m$. We notice that $(p_{1+2\epsilon})$ converges to $r(h_m)$ in the $\sigma(E^{**}, E^*)$-topology of E^{**}.

By Corollary 4.1 $T_m^{**}(M(J)_{sa}) \subset E_2^{**}(r(h_m))_{sa}$ and $T_m^{**}(M(J)) \subseteq \{h_m\}'$. The separate weak*-continuity of the product of $E_2^{**}(r(h_m))$ implies that $(x + \epsilon)$ and $T_m^{**}(x)$ operator commute for all $(x + \epsilon) \in \bar{C}^{w^*}$ and $x \in M(J)$. In particular, for each natural $(1 + 2\epsilon), p_{1+2\epsilon}$ and $T_m^{**}(x)$ operator commute, for all $x \in M(J)$. Thus, the mapping $(S_m)_{1+2\epsilon}: M(J) \to E_2^{**}(r(h_m)), (S_m)_{1+2\epsilon}(x) := (h_m^{-1})_{1+2\epsilon} \cdot_{r(h_m)} T_m^{**}(x)$ is an orthogonality preserving operator between two JB*-algebras satisfying that $(S_m)_{1+2\epsilon}(1) = p_{1+2\epsilon}$ is a tripotent. Corollary 4.1 assures that $(S_m)_{1+2\epsilon}$ is a Jordan *-homomorphism and hence $\|(S_m)_{1+2\epsilon}\| \leq 1$, for all $(1 + 2\epsilon) \in \mathbb{N}$.

Let us take a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By the Banach-Alaoglu Theorem, any bounded set in $E_2^{**}(r(h_m))$ is relatively weak*-compact and hence the assignment $(x + 2\epsilon) \mapsto S_m(x + 2\epsilon) := w^* - \lim_{\mathcal{U}} (S_m)_{1+2\epsilon}(x + 2\epsilon)$ defines an operator $S_m: J \to E_2^{**}(r(h_m))$.

For each natural $(1 + 2\epsilon)$, and each $x \in M(J), h_m \cdot_{r(h_m)} (S_m)_{1+2\epsilon}(x) = h_m \cdot_{r(h_m)} \left((h_m^{-1})_{1+2\epsilon} \cdot_{r(h_m)} T_m^{**}(x)\right) = p_{1+2\epsilon} \cdot_{r(h_m)} T_m^{**}(x)$. Since $r(h_m) = w^* - \lim_{(1+2\epsilon)} p_{(1+2\epsilon)}$, it follows from the separate weak*continuity of the Jordan product of* $E_2^{**}(r(h_m))$, that $h_m \cdot_{r(h_m)} S_m(x) = T_m^{**}(x)$, for all $x \in M(J)$. We have already seen that $(h_m^{-1})_{1+2\epsilon}, h_m$ and $T_m^{**}(x)$ pairwise operator commute for every $x \in M(J)$. Therefore, $(S_m)_{1+2\epsilon}(x)$ and h_m operator commute for every natural $(1 + 2\epsilon)$. The separate weak-continuity of the product assures that h_m and $S_m(x)$ operator commute for all $x \in M(J)$.

Finally, let $a \in M(J)_{sa}$. For each natural $1 + 2\epsilon, (S_m)_{1+2\epsilon}(a) \in E_2^{**}(r(h_m))_{sa}$ and $(S_m)_{1+2\epsilon}(a^2) = (S_m)_{1+2\epsilon}(a) \cdot_{r(h_m)} (S_m)_{1+2\epsilon}(a)$. Being $E_2^{**}(r(h_m))_{sa}$ weak*-closed, it is clear that $S_m(a) \in E_2^{**}(r(h_m))_{sa}$. Let $(1 + 2\epsilon)$ and m be two natural numbers. Since $(h_m^{-1})_{1+2\epsilon}, (h_m^{-1})_{m_0}$, and $T_m^{**}(a)$ are pairwise operator commuting, we have

$$(S_m)_{1+2\epsilon}(a) \cdot_{r(h_m)} (S_m)_{m_0}(a)$$

$$= (h_m^{-1})_{1+2\epsilon} \cdot_{r(h_m)} (h_m^{-1})_{m_0} \cdot_{r(h_m)} T_m^{**}(a) \cdot_{r(h_m)} T_m^{**}(a) = (S_m)_{\min(1+2\epsilon,m)}(a)^2$$

$$= (S_m)_{\min(1+2\epsilon,m)}(a^2).$$

For a fixed natural m, taking $w^* - \lim_{1+2\epsilon \geq m,u}$ in the above expressions, we deduce that

$$S_m(a) \cdot_{r(h_m)} (S_m)_{m_0}(a) = (S_m)_{m_0}(a^2),$$

for all $m \in \mathbb{N}$. The same argument gives

$$S_m(a) \cdot_{r(h_m)} S_m(a) = S_m(a^2).$$

The description provided by the above Theorem generalizes Theorems 6 and 10 in [7]. Concretely, the just quoted theorems make use of the hypothesis of $T_m^{**}(1)$ being von Neumann regular, and this assumption is completely removed in Theorem 4.1.

We recall that an operator T_m between two JB*-triples preserves zero-tripleproducts if $\{T_m(x), T_m(x+\epsilon), T_m(x+2\epsilon)\} = 0$ whenever $\{x, x+\epsilon, x+2\epsilon\} = 0$. While an operator T_m between two C*-algebras is said to be zero-products preserving if $T_m(x)T_m(x+\epsilon) = 0$ whenever $x(x+\epsilon) = 0$.

The authors in[8], [26], and [28] give a complete description of zero-product preserving bounded linear maps between C*-algebras.

The equivalent reformulations of orthogonality stated in (4.1) together with Theorem 4.1 above, give the following generalization of Corollary 18 in [7].

Corollary 4.2. Let $T_m: J \to E$ be an operator from a JB^*-algebra to a JB*-triple. Then T_m is orthogonality preserving if and only if T_m preserves zero-triple-products.

Example 4.1. Let T_m be a bounded linear operator between two C*-algebras. It was already noticed in [7] that in the case of T_m being symmetric (i.e., $T_m(x^*) = T_m(x)^*$),

then T_m is orthogonality preserving on A_{sa} if and only if T_m preserves zero-products on A_{sa}. However, not every orthogonality preserving operator sends zero-products to zero-products. Consider, for example, $T_m: M_2(\mathbb{C}) \to M_2(\mathbb{C}), T_m(x) = ux$, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Clearly T_m is a triple homomorphism and hence orthogonality preserving, but taking $x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, (x+\epsilon) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, we have $x(x+\epsilon) = (x+\epsilon)x = 0$ and $T_m(x)T_m(x+\epsilon) \neq 0$.

Theorem 17 in [7] follows now as a consequence of Theorem 4.1.

Corollary 4.3 (see [30]). Let $T_m: A \to B$ be an operator between two C^*-algebras. For $h_m = T_{\mathrm{m}}^{**}(1)$ the following assertions are equivalent:

a) T_m is orthogonality preserving.

b) There exists a triple homomorphism $S_m: A \to B^{**}$ satisfying $h_m^* S_m(x+2\epsilon) = S_m((x+2\epsilon)^*)^* h_m$, $h_m S_m((x+2\epsilon)^*)^* = S_m(x+2\epsilon)h_m^*$, and

$$T_m(x+2\epsilon) = L(h_m, r(h_m))(S_m(x+2\epsilon)) = \frac{1}{2}(h_m r(h_m)^* S_m(x+2\epsilon) + S_m(x+2\epsilon) r(h_m)^* h_m)$$
$$= h_m r(h_m)^* S_m(x+2\epsilon) = S_m(x+2\epsilon) r(h_m)^* h_m,$$

for all $(x+2\epsilon) \in A$.

Proof. The implication b)$\Rightarrow$ $a)$ is clear.

$a) \Rightarrow b)$ By Theorem 4.1 there exists a (unital) Jordan *-homomorphism $S_m: M(A) \to B_2^{**}(r(h_m))$ such that $S_m(x)$ and h_m operator commute in $B_2^{**}(r(h_m))$ and $T_m(x) = h_m \cdot_{r(h_m)} S_m(x)$, for every $x \in A$. In order to simplify notation we shall write $r = r(h_m)$. Notice that r is a partial isometry in B^{**}, with left and right projections given by rr^* and r^*r, respectively. It is well known that $B_2^{**}(r) = rr^*B^{**}r^*r$.

It can be easily checked that $L_{r^*}: B_2^{**}(r) \to B_2^{**}(r^*r), x \mapsto r^*x,$ is a unital Jordan *-homomorphism and $B_2^{**}(r^*r)$ is a C*-subalgebra of B^{**} because r^*r is a projection.

Take an element $a \in A_{sa}$. Since $S_m(a)$ and h_m operator commute in $B_2^{**}(r(h_m))_{sa}$, $L_{r^*}(h_m) = r^*h_m$ and $L_{r^*}(S_m(a)) = r^*S_m(a)$ operator commute in B_{sa}^{**}. Equivalently, r^*h_m and $r^*S_m(a)$ are two commuting elements in B^{**}. Therefore

$$h_m^*S_m(a) = h_m^*rr^*S_m(a) = (r^*h_m)^*(r^*S_m(a)) = (r^*h_m)(r^*S_m(a)) = (r^*S_m(a))(r^*h_m)$$
$$= (r^*S_m(a))^*(r^*h_m) = S_m(a)^*rr^*h_m = S_m(a)^*h_m,$$

and similarly $h_mS_m(a)^* = S_m(a)h_m^*$. The proof concludes by a linear argument.

The general description of all orthogonality preserving operators between two JB*-triples remains open. We can only prove the following local property.

Corollary 4.4. Let $T_m: E \to F$ be an orthogonality preserving operator between two JB^*-triples. Let x be a norm-one element in E and let $h_m = T_{\mathrm{m}}^{**}(r(x))$. Then there exists a Jordan *-homomorphism $S_m: E(x) \to F_2^{**}(r(h_m))$, satisfying that $T_m|_{E(x)} = L(h_m, r(h_m))$.

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