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Review Paper

A descent improved three-term derivative-free spectral gradient projection approach for solving monotone nonlinear equations with convex constraints

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Abstract*: In this paper, based on projection technique Solodove and Svaiter (Reformulation: Nonsmooth, piecewise smooth, semismooth and smoothing methods; 355-369,1995), we propose a decent improved three-term derivative free approach for solving nonlinear monotone equations with convex constraints. The algorithm combines the spectral gradient parameter with a newly PRPlike CG coefficient in the search direction. The global convergence of the proposed approach established under standard conditions. Additionally, using some benchmark problems, the numerical results highlight the outstanding performance of this approach compared to popular conjugate gradient methods. The experiments also show its effectiveness in solving large-scale nonlinear equations with convex constraints.*

Keywords: *Sufficient descent, Monotone, Nonlinear equations, Derivative free, Spectral Gradient, Projection method.*
MSC: 65K05, 90C52, 90C56, 94A08. 1.

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I. Introduction

Numerous applications of the nonlinear monotone equations in addressing problems in the field of science and technology such as chemical equilibrium [9], signal processing [2,21] and financial forecasting [8] make it one of the most importance areas of the applied mathematics.
Let Q be a nonempty closed convex subset of R^n, where R^n is an-dimension Euclidean space. A convex constrained equation has its form:

$$J(a) = 0,\ a \in Q, \tag{1}$$

where J is a nonlinear mapping from R^n to R^n.

Definition 1.1 *$J : R^n \to R^n$ is monotone if*

$$\langle J(a) - J(b), a - b\rangle \geq 0,\ \forall\ a, b \in R^n. \tag{2}$$

The mapping J is called a monotone system of nonlinear equations if (1) *equips* (2).

In an attempt to solve (1), a number of numerical methods have been developed. Newton's methods, quasi-Newton's methods and scale trust-method are among ancient numerical techniques for solving (1). see for example [3–5, 13]. These methods posses nice properties including super-linear and local quadratic convergences, on the other hand, they are not suitable in solving a large-scale problems as they deal with a Jacobian matrix in each iteration step which is computationally expansive. Conjugate gradient (CG) is an another method used to optimize any version of (1). This method utilizes the following iterative procedure,

$$a_{r+1} = a_r + \alpha_r d_r,\ r = 0, 1, 2, ..., \tag{3}$$

where a_{r+1}, a_r donate the current and previous iterates, α_r is a stepsize obtain via a proper line search, d_r is a search direction which undergoes updates through

$$d_r = -J_r + \beta_r d_{r-1}, \tag{4}$$

where the scalar β_r is the CG coefficient. However, the search direction d_r is required to satisfies the steepest descent property to be discussed in the subsequent section of this paper. Different proposals of β_r lead to different CG methods. The Four well-known traditional two-term CG methods are Hestenese-Stiefel (HS) [14], Flether-Reeves (FR) [11], Polak-Ribiere and Polyak (PRP) [19] and Dai-Yuan (DY) [7], they employ the following CG coefficients.

$$\beta_r^{HS} = \frac{g_r^T y_{r-1}}{d_r^T y_{r-1}}, \quad \beta_r^{FR} = \frac{\|g_r\|^2}{\|g_{r-1}\|^2},$$

$$\beta_r^{PRP} = \frac{g_r^T y_{r-1}}{\|g_{r-1}\|^2}, \quad \beta_r^{DY} = \frac{\|g_r\|^2}{d_{r-1}^T y_{r-1}},$$

where $y_{r-1} = g_r - g_{r-1}$ and $g_r = g(a_r)$. HS and PRP have adequate numerical performance, their convergence properties are not efficient, wheres FR and DY methods have certain good convergence features, but their numerical results are not effective [10]. More recently, mathematicians and engineers give more consideration to the CG methods due their nice convergence property, low memory requirement and easy implementation, as a result, several CG and spectral gradient methods for solving unconstrained optimization have been extended to address system of nonlinear equations. An affine-scaling trust-region approach was proposed by Kanzow and Klug [15] to solve semismooth systems of equations with box constraints. Systems of equations that are semismooth or continuously differentiable can be solved using this technique. Under appropriate assumptions, they have demonstrated good local and global convergence properties. In [25] Yu extended the classical PRP scheme for unconstrained minimization problems and proposed a spectral gradient method for solving large-scale nonlinear system of equations. The method has some pleasant attributes, such as well-defined iterations in all cases, regardless of whether the search direction is sufficiently decent or not, which caused the algorithm to converge globally.

Inspired by the projection method of Solodov and Svaiter [22], several researchers developed interest in monotone system of nonlinear equation with convex constraints. For instance, the method in [24] solved a monotone nonlinear system of equations and is proved to be globally convergent with linear rate of convergence, the key attribute of this method involved solving a system of linear equations iteratively, beginning with an initial approximation. After the initialization, an approximate solution is computed to obtain a trial point. This trial point is then refined using a projection method in [22] to generate the next iteration. However, Wang and Wang [23] modified the Wang's approach [24] to enhances the efficiency and performance of the iterative process, leading to a superlinear convergence. Ma et al. [18] modified inertial three-term conjugate gradient projection method for solving(1), by employing an inertial extrapolation step into the search direction, the method ensures a sufficient descent property without relying on line search rules. The algorithm's global convergence and Q-linear convergence rate are proven under standard conditions. Numerical experiments show that it outperforms existing methods. Finally, Yuan [26] incorporated projection method in [22] and extended CG DESCENT algorithm, traditionally used for large-scale unconstrained minimization, to solve large-scale nonlinear convex constrained monotone equations(1), ℓ_1-norm regularized problems and also proved the global convergence of the method under some mild conditions.

Therefore, inspired by Projection technique in [22] and the above mentioned contributions, the aim of this paper is to propose a descent modified three-term derivative free spectral gradient projection approach for solving monotone nonlinear equation with convex constraints. The approach can also be seen as the extension of the work in [27] by Zhang et al. wheres we integrate the search direction by combining a spectral gradient parameter with a newly define CG coefficient. The major contributions of this paper are:

- The integrated search direction satisfied an in important attribute, $J_r^T d_r \leq -\lambda \|J_r\|^2, \ \forall r \geq 0,$ where λ is a positive constant.
- The global convergence of the proposed approach has been established under some standard conditions.
- Numerical experiment has been conducted to asses the approach's efficiency in comparison with some existing methods.

We structured the remaining parts of the paper as follows: we describe the algorithm of our proposed approach in section 2, in section 3, we prove the global convergence of the proposed approach, whereas, in section 4, we conduct the numerical experiment to highlight the efficiency of the new approach in comparison with the existing methods.

II. THE APPROACH

We open this section by recalling the a nonlinear unconstrained optimization problem

$$minf(x),\ x \in R^n, \tag{5}$$

where $f : R^n \to R$ is a nonlinear mapping. A descent modified Polak-Ribiere and Polyak conjugate gradient method and its global convergence proposed by Zhang et al. [27] utilizes the iterative procedure in (3) to generate sequence of solutions $\{x_r\}$ with stepsize α_r and search direction d_r defined by

$$d_{r+1} = \begin{cases} -g_{r+1}, & \text{if } r = 0, \\ -g_r + \beta_r d_r - d_r^T \Gamma_r, & \text{if } r \geq 1, \end{cases} \tag{6}$$

where $g_r = \nabla f(x_r)$ is gradient at point x_r and

$$\beta_r = \frac{g_{r+1}^T y_r}{g_r^T g_r}, \quad \Gamma_r = \frac{g_{r+1} y_r}{g_r^T g_r}. \tag{7}$$

$y_r = g_{r+1} - g_r$. They have shown that the search direction(6) has satisfied the sufficient decent property. We introduce a new algorithm, namely, a descent improved three-term derivative free spectral gradient projection approach for solving monotone nonlinear equations with convex constraints.

Algorithm 1:

Input. Given $a_0 \in R^n$, $\rho \in (0,1)$, $c, \eta, \sigma > 0$, $tol > 0$. Set $r := 0$.
Step 1. If $\|J_r\| \leq tol$, pause. Else go to **Step 2**.
Step 2. Calculate d_r by

$$d_r = \begin{cases} -J_r, & \text{if } r = 0 \\ -\theta_r J_r + \bar{\beta}_r s_{r-1} - \Phi_r J_{r-1}, & \text{if } r \geq 1. \end{cases} \tag{8}$$

where

$$\bar{\beta}_r = \frac{J_r^T J_{r-1}}{J_{r-1}^T J_{r-1}}, \quad \Phi_r = \frac{J_r^T s_{r-1}}{J_{r-1}^T J_{r-1}}., \tag{9}$$

$$\theta_r = \frac{s_{r-1}^T s_{r-1}}{s_{r-1}^T t_{r-1}}, \tag{10}$$

and J_r is the gradient of J at a_r, $t_{r-1} = b_{r-1} + c s_{r-1}$, $b_r = J_r - J_{r-1}$ and $s_{r-1} = a_r - a_{r-1}$.
Step 3. Calculate the stepsize $\alpha_r = \eta \rho^i$, for $i = 0, 1, \ldots$, where i is the smallest positive integer satisfying

$$-J(z_r)^T d_r \geq \sigma \alpha_r \|d_r\|^2. \tag{11}$$

Step 4. Set $z_r = a_r + \alpha_r d_r$, if $z_r \in Q$ and $\|J(z_r)\| \leq tol$, then stop , otherwise compute

$$a_{r+1} = P_Q[a_r - \mu_r J(z_r)], \tag{12}$$

where

$$\eta_r = \frac{J(z_r)^T (a_r - z_r)}{\|J(z_r)\|^2},\ J(z_r) \neq 0. \tag{13}$$

Step 5. Let $r = r + 1$ and switch to **Step 1**.

Definition 2: Let Q be a nonempty closed convex set. $\forall a \in R^n$, we define it's orthogonal projection onto Q by

$$P_Q = argmin\|a - b\| : b \in Q. \tag{14}$$

A mapping $P_Q(.) : R^n \to Q$ is called a projection Mapping. It has an important nonexpansive property such that;

$$\|P_Q(a) - P_Q(b)\| \leq \|a - b\|,\ \forall a, b \in R^n. \tag{15}$$

Also from (15), we can deduct

- $\|P_Q(a) - b\|^2 \leq \|a - b\|^2 - \|a - P_Q(a)\|,\ \forall a, b \in R^n,$
- $\|P_Q(a) - (b)\| \leq \|a - b\|,\ \forall b \in Q.$

We consider the following assumptions on the mapping J

- (S_1) The solution set of (1) represented by $\bar{Q}$ is nonempty.
- (S_2) The mapping $J : R^n \to R^n$ is uniformly monotone, i.e, $\forall a, b \in R^n$, there exist $u \geq 1$, such that

$$(J(a) - J(b))^T - (a - b) \geq u\|a - b\|^2. \tag{16}$$

- (S_3) The mapping $J : R^n \to R^n$ is Lipschitz continuous, i.e, $\forall a, b \in R^n$, there exists a positive constant L such that

$$\|J(a) - J(b)\| \leq L\|a - b\|. \tag{17}$$

We state the following remarks:

- From the definition of J_{r-1} we obtain

$$J_{r-1}^T J_{r-1} = \|J_{r-1}\|^2 > 0, \tag{18}$$

which verifies that the parameters $\bar{\beta}_r$ and Φ_r given by (9) are well defined.

- From assumption (S_2) and the definition of t_{r-1}, we get

$$s_{r-1}^T t_{r-1} = s_{r-1}^T (b_{r-1} + c s_{r-1}) = s_{r-1}^T (b_{r-1}) + c s_{r-1}^T s_{r-1} \geq c\|s_{r-1}\|^2 > 0. \tag{19}$$

This shows that, the parameter θ_r in (9) is well defined.
Also, by (18) and assumption (S_3) we have

$$t_{r-1}^T s_{r-1} = (J_r - J_{r-1})^T (a_r - a_{r-1}) + c\|s_{r-1}\|^2 \leq (L + c)\|s_{r-1}\|^2. \tag{20}$$

Therefore, by (19) and (20) we get $\frac{\|s_{r-1}\|^2}{s_{r-1}^T t_{r-1}} \leq \frac{1}{c} = \epsilon_2$ and $\frac{\|s_{r-1}\|^2}{s_{r-1}^T t_{r-1}} \geq \frac{1}{(L+c)} = \epsilon_1$, and

$$\epsilon_1 \leq \theta_r \leq \epsilon_2. \tag{21}$$

III. Convergence analysis

In this section we consider some lemmas in order to establish the global convergence of the newly proposed approach.

Lemma 3.1 *The search direction given by (8)-(10) satisfies sufficient decent property, i.e* $J_r^T d_r \leq -\lambda\|J_r\|^2$, $\forall r \geq 0$.

Proof . From eqs. (8)-(10),(19), (21) we can obtain

$$\begin{aligned}
J_r^T d_r &= -\theta_r\|J_r\|^2 + J_r^T \bar{\beta}_r s_{r-1} - J_r^T \Phi_r J_{r-1} \\
&= -\theta_r\|J_r\|^2 + J_r^T \left[\frac{J_r^T J_{r-1}}{J_{r-1}^T J_{r-1}}\right] s_{r-1} - J_r^T \left[\frac{J_r^T s_{r-1}}{J_{r-1}^T J_{r-1}}\right] J_{r-1} \\
&= -\theta_r\|J_r\|^2 + \left[\frac{(J_r^T s_{r-1})(J_r^T J_{r-1}) - (J_r^T J_{r-1})(J_{r-1}^T s_{r-1})}{J_{r-1}^T J_{r-1}}\right] \\
&= -\theta_r\|J_r\|^2 \\
&\leq -\epsilon_2\|J_r\|^2.
\end{aligned}$$

By letting $\lambda = \epsilon_2$. Thus, we get

$$J_r^T d_r \leq -\lambda\|J_r\|^2. \tag{22}$$

Lemma 3.2 *Suppose the assumption (S_1)-(S_3) are true, sequence $\{a_r\}$ and $\{z_r\}$ be generated by New Algorithm, then a positive integer D exist such that, $\{a_r\}$ and $\{z_r\}$ are bounded. Moreover,*

$$\|d_r\| \le D, \tag{23}$$

$$\lim_{r\to\infty} \|a_r - z_r\| = 0, \tag{24}$$

and

$$\lim_{r\to\infty} \|a_{r+1} - a_r\| = 0. \tag{25}$$

Proof To show that the sequences $\{a_r\}$ and $\{z_r\}$ are bounded, let $\bar{x} \in \bar{A}$. By the monotonicity attribute of J we have

$$\zeta(z_k)^T(v_k - \bar{x}) \ge \sigma\alpha_k^2\|d_k\|^2 > 0. \tag{26}$$

Now, By (14) and (16), we have

$$\begin{aligned}
\|a_{r+1} - \bar{a}\|^2 &= \|P_Q[a_r - \mu_r J(z_r)] - \bar{a}\|^2 \\
&\le \|a_r - \mu_r J(z_r) - \bar{x}\|^2 \\
&= \|a_r - \bar{a}\|^2 - 2\mu_r\langle J(z_r), a_r - \bar{a}\rangle + \mu_r^2\|J(z_r)\|^2 \\
&\le \|a_r - \bar{a}\|^2 - 2\mu_r\langle J(z_r), a_r - z_r\rangle + \mu_r^2\|J(z_r)\|^2 \\
&= \|a_r - \bar{a}\|^2 - \frac{\langle J(z_r), a_r - z_r\rangle^2}{\|J(z_r)\|^2} \\
&\le \|a_r - \bar{a}\|^2,
\end{aligned} \tag{27}$$

which means

$$\|a_{r+1} - \bar{a}\| \le \|a_r - \bar{a}\| \ \forall r \ge 0. \tag{28}$$

This implies that $\{\|a_r - \bar{a}\|\}$ is decreasing. Hence $\{a_r\}$ is bounded. Furthermore, combining Lipschitz continuity of J with $\bar{a} \in \bar{A}$, we have

$$\begin{aligned}
\|J(a)\| &= \|J(a) - J\bar{a}\| \\
&\le L\|a_r - \bar{a}\| \\
&\le L\|a_0 - \bar{a}\|.
\end{aligned}$$

By letting $L\|a_0 - \bar{x}\| \le \delta$, hence, $\{J_r\}$ is bounded. i.e,

$$\|J(a)\| \le \delta. \tag{29}$$

From the definition of d_r, (21) and Cauchy-Schwatz inequality, we have

$$\begin{aligned}
\|d_r\| &= \| - \theta_r J_r + \bar{\beta}_r s_{r-1} - \Phi_r J_{r-1}\| \\
&= \theta_r\|J_r\| + \left|\frac{J_r^T J_{r-1}}{J_{r-1}^T J_{r-1}}\right| \|s_{r-1}\| - \left|\frac{J_r^T s_{r-1}}{J_{r-1}^T J_{r-1}}\right| \|J_{r-1}\| \\
&\le \theta_r\|J_r\| + \frac{\|J_r^T\|\|J_{r-1}\|\|s_{r-1}\|}{\|J_{r-1}^T\|\|J_{r-1}\|} + \frac{\|J_r^T\|\|s_{r-1}\|\|J_{r-1}\|}{\|J_{r-1}^T\|\|J_{r-1}\|} \\
&\le \theta_r\|J_r\| + \frac{\|J_r\|\|J_{r-1}\|\|s_{r-1}\| + \|J_r\|\|s_{r-1}\|\|J_{r-1}\|}{\|J_{r-1}^T\|\|J_{r-1}\|} \\
&\le \theta_r\|J_r\| + \|J_r\|(\|J_{r-1}\|\|s_{r-1}\| + \|s_{r-1}\|\|J_{r-1}\|) \\
&\le \|J_r\|(\theta_r + 2) \\
&\le \delta(\epsilon_2 + 2).
\end{aligned}$$

Let $D = \delta(\epsilon_2 + 2)$, we have

$$\|d_r\| \leq D. \tag{30}$$

Thus, search direction d_r is bounded.

From (27), we obtain

$$\left((a_r - z_r)^T J(z_r)\right)^2 \leq \|J(z_r)\|^2 \left(\|a_r - \bar{a}\|^2 - \|a_{r+1} - \bar{a}\|^2\right). \tag{31}$$

Also by (11), we get

$$\sigma^2 \alpha_r^4 \|d_r\|^4 \leq \alpha_r^2 \left(J(z_r)^T d_r\right)^2. \tag{32}$$

Combining (27) and (33), we have

$$\sigma^2 \alpha_r^4 \|d_r\|^4 \leq \|J(z_r)\|^2 \left(\|a_r - \bar{a}\|^2 - \|a_{r+1} - \bar{a}\|^2\right). \tag{33}$$

Thus, $\{\|a_r - \bar{a}\|\}$ is convergent and $\{J(z_r)\}$ is bounded. Introducing limit to (33) implies

$$\sigma^2 \lim_{r\to\infty} \alpha_r^4 \|d_r\|^4 \leq 0, \tag{34}$$

additionally, we obtain

$$\lim_{r\to\infty} \alpha_r \|d_r\| = 0. \tag{35}$$

By (14), (15) and (13), we have

$$\begin{aligned}
\|a_{r+1} - \bar{a}\| &= \|P_Q[a_r - \eta_r J(z_r)] - a_r\| \\
&= \|a_r - \eta_r J(z_r) - a_r\| \\
&= \|\eta_r J(z_r)\| \\
&\leq \|a_r - z_r\|, \ \forall r \geq 0.
\end{aligned}$$

The proof. ■

Lemma 3.3 *Suppose the assumptions (S_1)-(S_3) are true, the stepsize α_r satisfying the line search in (11) $\forall r \geq 0$.*

Proof . Let assume there exist $r_0 \geq 0$ such that, the line search given by (11) not true, meaning that, for any $i = 0, 1, 2, ..,.$ we have

$$-J(a_{r_0} + \rho^i d_r)^T d_{r_0} < \rho^i \|d_{r_0}\|^2. \tag{36}$$

From the Lipschitz continuity property of J, allowing $i \to \infty$ we obtain

$$-J_{r_0}^T d_{r_0} \leq 0. \tag{37}$$

Also by (22), we get

$$-J_{r_0}^T d_{r_0} \geq \lambda \|J_{r_0}\|^2. \tag{38}$$

This contradict (36). Hence, the line search (11) is well defined. ■

Theorem 3.4 *Suppose the sequence $\{a_r\}$ is generated by New Algorithm and (S_1)-(S_3) hold, then*

$$\liminf_{r\to\infty} \|J(a)\| = 0. \tag{39}$$

Proof . Suppose by contradiction that (38) does not hold, i.e, a positive constant τ exists such that

$$\|J_r\| \geq \tau \ \forall r \geq 0. \tag{40}$$

Furthermore, from (3.2) and definition z_r, we can deduct

$$\lim_{r\to\infty} \alpha_r \|d_r\| = 0. \tag{41}$$

If $\alpha_r \neq \eta$, $\alpha_r = \rho^{-1}$ does not satisfy line search in (11). This implies

$$-J(a_r + \rho^{-1}\alpha_r d_r)^T < \sigma \rho^{-1} \alpha_r \|d_r\|^2,$$

If $\alpha_r \neq \eta$, $\alpha_r = \rho^{-1}$ does not satisfy line search in (11). This implies

$$-J(a_r + \rho^{-1}\alpha_r d_r)^T < \sigma\rho^{-1}\alpha_r \|d_r\|^2,$$

by combining it with(16), (22) and Cauchy-Swatch inequality, we obtain

$$\begin{aligned}\lambda\|J_r\|^2 &\leq -J_r^T d_r \\ &= \left(J\left(a_r + \rho^{-1}\alpha_r d_r\right) - J_r\right)^T d_r - J\left(a_r + \rho^{-1}\alpha_r d_r\right)^T d_r \\ &\leq L\rho^{-1}\alpha_r\|d_r\|^2 + \sigma\rho^{-1}\alpha_r\|d_r\|^2.\end{aligned}$$

This resulted in

$$\alpha_r \geq \lambda \frac{\rho\|J_r\|^2}{(L+\sigma)\,\|d_r\|^2}.$$

Also, by (40) and (30), we get

$$\begin{aligned}\alpha_r\|d_r\| &\geq \frac{\lambda\rho}{(L+\sigma)}\frac{\|J_r\|^2}{\|d_r\|} \\ &\geq \frac{\lambda\rho}{(L+\sigma)}\frac{\tau^2}{D}.\end{aligned}$$

This contradicts (40).Hence (39) holds. The proof completed. ■

IV. Experimental Results

Using the similar line search in (11), in this section, we report the numerical experiments conducted on three competitive algorithms. However we set the following parameters in our **Algorithm 1**; $c = 0.1$, $\rho = 0.9$ and $\sigma = 0.001$. Two well-known existing algorithms, **PCG** [17] and **ETCG** [12] were employed as a points of comparison. All the parameters used in **PCG** and **ETCG** are the same as in [17] and [12]. The idea behind the experiment was that, an approach with lower number of iterations (**IT**), function evaluations (**FE**) and CPU time (**TM**) is considered to be the best solver. Five different dimensions of 1000, 5000, 10000, 50000, 100000, and six different initial points $t_1 = (1,\cdots,1)^T$, $t_2 = \left(1-\frac{1}{n}, 1-\frac{2}{n}, \cdots, 0\right)^T$, $t_3 = (2,\cdots,2)^T$, $t_4 = \left(\frac{-1}{4}, \frac{-1}{8}, \cdots, \frac{-1}{4n}\right)^T$, $t_5 = (0.6,\cdots,0.6)^T$,and $t_6 = (0.1,\cdots,0.1)^T$ were used. All the codes have been written and run in Matlab 8.3.0 (R2014a) on a personal computer with an Intel (R) processor, 8GB of RAM, and CPU N4200 @ 1.10GHz in the experiment. The terminating criterion is when $\|J_r\| \leq 10^{-6}$ or **NI** exceeds 1000. The following 5 problems were selected for the experiment, $J = (J_1, J_2, J_3, \ldots J_n)$.

Problem 1: [1]

$$\begin{aligned}J_1 &= e^{(a_1)} - 1; \\ J_i &= e^{(a_i)} + a_i - 1, \quad for\ i = 1,2,...,n.\end{aligned}$$

Problem 2: [20]

$$Ja_i = log\frac{|(a_i)+1|}{n} - a_i, \quad for\ i = 2,...,n.$$

Problem 3: [6]

$$J_i = 2a_i - \sin(|a_i|), \quad for\ i = 1,2,...,n.$$

Problem 4: [16]

$$J_i = cos(a_i) + (a_i) - 1, \quad for\ i = 1,2,...,n.$$

Problem 5: [28]

$$J_i = e^{a_i} - 1, \quad for\ i = 1,2,...,n.$$

Problem 6: [6]

$$\begin{aligned}J_1 &= 2a_1 - a_2 + e^{a_1} - 1 \\ J_i &= -a_{i-1} + 2a_i - a_{i+1} + e^{a_i - 1} \\ J_n &= -J_{n-1} + 2a_n + e^{a_n} - 1, \quad for\ i = 2,3,...,n-1.\end{aligned}$$

Table 1: Numerical results for Algorithm 1, PCG and ETCG on Example 1

		Algorithm 1				PCG				ETCG			
DIMENSION	IP	IT	FE	TM	$\|J\|$	IT	FE	TM	$\|J\|$	IT	FE	TM	$\|J\|$
1000	a1	9	21	0.060869	4.72E-11	28	30	0.049456	4.03E-07	17	36	0.033368	8.73E-11
	a2	8	18	0.012803	8.51E-11	29	31	0.05113	4.03E-07	15	33	0.016004	7.31E-11
	a3	8	18	0.013223	1.43E-11	29	31	0.047804	5.66E-07	15	33	0.017473	3.66E-11
	a4	9	21	0.012417	8.7E-12	21	23	0.033304	4.51E-07	15	33	0.019694	8.74E-11
	a5	10	19	0.012685	4.79E-12	20	22	0.031945	7.41E-07	15	33	0.016938	8.05E-11
	a6	2	25	0.0086	0	20	22	0.028815	4.9E-07	17	41	0.018784	6.58E-11
5000	a1	10	21	0.042341	8.2E-11	26	28	0.124678	5.64E-07	45	99	0.025778	6.64E-11
	a2	10	19	0.041195	8.85E-12	28	30	0.12515	5.01E-07	46	104	0.027612	6.97E-11
	a3	9	17	0.035784	1.64E-11	27	29	0.121363	7.92E-07	17	38	0.06779	3.28E-11
	a4	9	19	0.043	2.05E-11	22	24	0.102931	4.5E-07	16	35	0.068802	3.23E-11
	a5	10	19	0.043339	1E-11	21	23	0.109244	7.25E-07	15	33	0.061072	8.14E-11
	a6	2	25	0.024855	0	20	22	0.100434	7.42E-07	16	35	0.063566	3.64E-11
10000	a1	11	22	0.068549	5.33E-12	25	27	0.18327	8.87E-07	16	35	0.062799	3.54E-11
	a2	9	19	0.064932	1.25E-11	26	28	0.179835	8.88E-07	19	43	0.073904	2.79E-11
	a3	8	17	0.055527	2.22E-11	27	29	0.184644	5.62E-07	45	99	0.027082	6.64E-11
	a4	9	19	0.060792	2.21E-11	22	24	0.16003	6.36E-07	46	104	0.02323	6.97E-11
	a5	10	19	0.060109	1.41E-11	22	24	0.161363	4.58E-07	17	38	0.102644	4.53E-11
	a6	2	25	0.040974	0	20	22	0.147226	9.84E-07	16	35	0.097024	4.56E-11
50000	a1	10	22	0.25084	1.16E-11	25	27	0.647123	4.97E-07	16	35	0.096786	2.3E-11
	a2	10	19	0.236975	2.78E-11	26	28	0.704322	4.98E-07	16	35	0.09739	5.09E-11
	a3	9	17	0.200535	3.75E-11	26	28	0.660825	6.98E-07	16	35	0.098673	5E-11
	a4	10	19	0.250531	2.04E-11	23	25	0.59537	6.36E-07	18	43	0.120582	3.91E-11
	a5	9	19	0.225576	3.13E-11	23	25	0.608633	4.57E-07	45	99	0.023511	6.64E-11
	a6	2	25	0.141121	0	21	23	0.678327	9.31E-07	46	104	0.024926	6.97E-11
100000	a1	11	22	0.460992	1.63E-11	23	25	1.324672	8.82E-07	17	38	0.398892	9.92E-11
	a2	9	19	0.418471	3.93E-11	25	27	1.4427	7.83E-07	17	37	0.379142	2.04E-11
	a3	8	17	0.370166	4.99E-11	26	28	1.508439	4.95E-07	16	35	0.359132	5.14E-11
	a4	10	19	0.410437	2.23E-11	23	25	1.359832	9E-07	17	37	0.380325	2.25E-11
	a5	9	19	0.414397	4.42E-11	23	25	1.342447	6.46E-07	17	37	0.380744	2.23E-11
	a6	2	25	0.291097	0	22	24	1.364657	5.85E-07	18	43	0.416464	8.7E-11

Table 2: Numerical results for Algorithm 1, PCG and ETCG on Example 2

		Algorithm 1				PCG				ETCG			
DIMENSION	IP	IT	FE	TM	$\|J\|$	IT	FE	TM	$\|J\|$	IT	FE	TM	$\|J\|$
1000	a1	10	15	0.04763	3.43E-11	11	10	0.017873	4.41E-07	5	7	0.012967	3.6E-11
	a2	10	13	0.017355	3.01E-11	12	10	0.015517	4.98E-07	4	6	0.010377	1.46E-11
	a3	9	11	0.016866	3.52E-11	9	13	0.017134	3.03E-07	4	6	0.008872	5.12E-13
	a4	10	13	0.018387	1.44E-11	10	15	0.01721	5.35E-07	5	7	0.012007	4.35E-13
	a5	10	13	0.017593	3.31E-11	10	14	0.019349	6.17E-07	4	6	0.009455	3.2E-11
	a6	11	16	0.019542	2.79E-11	12	15	0.014384	8.76E-07	6	8	0.011247	3.6E-11
500	a1	10	15	0.054948	7.74E-11	13	20	0.04568	9.33E-07	1001	1002	0.403196	0.306853
	a2	10	13	0.05034	6.66E-11	11	15	0.048635	9.39E-08	1001	1002	0.398438	0.306853
	a3	9	11	0.044925	7.71E-11	10	11	0.06161	6.34E-07	5	7	0.027898	1.26E-12
	a4	10	13	0.048992	3.21E-11	10	14	0.052829	9.93E-08	4	6	0.023262	3.77E-13
	a5	10	13	0.056027	7.32E-11	11	16	0.056135	5.63E-07	3	5	0.020441	3.49E-11
	a6	11	16	0.059817	6.12E-11	12	16	0.062272	1.16E-07	4	6	0.024287	2.83E-11
10000	a1	11	16	0.088058	9.95E-12	9	18	0.077485	5.72E-07	4	6	0.024477	1.08E-12
	a2	10	13	0.077468	9.4E-11	12	15	0.085291	1.31E-07	6	8	0.032159	1.26E-12
	a3	10	12	0.077019	9.88E-12	10	14	0.077849	8.89E-07	1001	1002	0.397414	0.306853
	a4	10	13	0.088657	4.53E-11	10	15	0.081483	1.38E-07	1001	1002	0.390675	0.306853
	a5	11	14	0.083742	9.41E-12	13	16	0.090358	7.89E-07	5	7	0.042592	3.77E-13
	a6	11	16	0.086318	8.63E-11	12	18	0.095795	1.63E-07	4	6	0.041046	1.11E-13
50000	a1	11	16	0.321398	2.22E-11	11	15	0.325659	1.18E-07	3	5	0.029246	1.21E-11
	a2	11	14	0.311366	1.91E-11	13	15	0.274709	2.9E-07	5	8	0.044938	5.84E-12
	a3	10	12	0.279162	2.2E-11	11	13	0.304094	8.61E-07	5	8	0.043875	2E-13
	a4	11	14	0.328419	9.23E-12	11	17	0.298894	3.05E-07	7	10	0.05665	2.22E-13
	a5	11	14	0.308316	2.1E-11	12	15	0.351294	1.63E-07	1001	1002	0.46609	0.306853
	a6	12	17	0.354863	1.75E-11	12	20	0.364556	3.62E-07	1001	1002	0.771554	0.306853
10000	a1	11	16	0.607099	3.15E-11	13	22	0.593268	1.67E-07	28	54	1.792144	4.97E-14
	a2	11	14	0.580303	2.7E-11	12	15	0.602717	4.09E-07	26	52	1.31723	7.43E-11
	a3	10	12	0.52254	3.12E-11	12	13	0.702541	1.13E-07	35	71	1.111911	7.09E-11
	a4	11	14	0.571712	1.31E-11	11	16	0.581445	4.31E-07	33	65	1.122	4.97E-14
	a5	11	14	0.577911	2.97E-11	12	14	0.717245	2.3E-07	28	56	0.894512	9.66E-11
	a6	12	17	0.665448	2.48E-11	12	18	0.786817	5.11E-07	31	60	0.955379	6.8E-11

Table 3: Numerical results for Algorithm 1, PCG and ETCG on Example 3

		Algorithm 1				PCG				ETCG			
DIMENSION	IP	IT	FE	TM	||J||	IT	FE	TM	||J||	IT	FE	TM	||J||
1000	a1	11	15	0.032525	1.6E-11	13	19	0.014753	3.51E-07	51	105	0.048035	7.84E-11
	a2	10	13	0.013588	3.05E-11	19	14	0.015291	8.39E-07	49	101	0.043965	6.32E-11
	a3	10	13	0.013323	1.22E-11	10	16	0.014205	5.37E-07	47	97	0.03848	7.09E-11
	a4	10	13	0.013486	5.31E-11	10	16	0.012864	6.84E-07	50	103	0.048492	7.35E-11
	a5	10	13	0.013424	3.63E-11	14	17	0.014255	9.87E-07	49	101	0.040638	7.55E-11
	a6	12	22	0.015891	2.2E-11	14	17	0.013406	5.8E-07	51	105	0.043168	8.87E-11
5000	a1	11	15	0.045346	3.58E-11	14	17	0.03923	7.85E-07	46	95	0.026844	6.95E-11
	a2	10	13	0.042382	6.81E-11	14	15	0.04277	8.18E-07	46	95	0.024888	7.21E-11
	a3	10	13	0.042889	2.73E-11	14	21	0.045119	1.11E-07	53	109	0.185852	6.31E-11
	a4	11	14	0.041704	1.08E-11	15	21	0.045808	1.42E-07	50	103	0.177298	8.48E-11
	a5	10	13	0.040145	8.12E-11	17	15	0.042985	9.63E-07	48	99	0.183426	9.51E-11
	a6	12	22	0.048431	4.93E-11	17	15	0.047574	5.66E-07	51	105	0.179481	9.86E-11
10000	a1	11	15	0.063469	5.06E-11	12	19	0.061433	4.84E-07	51	105	0.179642	6.08E-11
	a2	10	13	0.056798	9.64E-11	12	19	0.066456	1.07E-07	53	109	0.186898	7.14E-11
	a3	10	13	0.055798	3.86E-11	13	19	0.066087	1.57E-07	46	95	0.024067	6.95E-11
	a4	11	14	0.068203	1.53E-11	13	19	0.066795	2.01E-07	46	95	0.033067	7.21E-11
	a5	11	14	0.06086	1.04E-11	14	19	0.072546	1.26E-07	53	109	0.290443	8.92E-11
	a6	12	22	0.07378	6.97E-11	14	18	0.063571	8E-07	51	105	0.268567	7.2E-11
50000	a1	12	16	0.252089	1.03E-11	13	21	0.297596	1.01E-07	49	101	0.258852	8.07E-11
	a2	11	14	0.231592	1.96E-11	11	21	0.249944	2.4E-07	52	107	0.275238	8.37E-11
	a3	10	13	0.21576	8.63E-11	12	23	0.250218	3.52E-07	51	105	0.270958	8.6E-11
	a4	11	14	0.240989	3.42E-11	13	23	0.251049	4.49E-07	54	111	0.282943	6.06E-11
	a5	11	14	0.229717	2.33E-11	13	23	0.256866	2.82E-07	46	95	0.0252	6.95E-11
	a6	13	23	0.292736	1.42E-11	15	23	0.268788	1.66E-07	46	95	0.024324	7.21E-11
100000	a1	12	16	0.453772	1.45E-11	14	23	0.461282	1.42E-07	55	113	1.229654	7.18E-11
	a2	11	14	0.41231	2.77E-11	15	23	0.454661	3.39E-07	52	107	1.092488	9.66E-11
	a3	11	14	0.410322	1.11E-11	13	21	0.469631	4.98E-07	51	105	1.223603	6.5E-11
	a4	11	14	0.415152	4.83E-11	12	23	0.478729	6.35E-07	54	111	1.251676	6.73E-11
	a5	11	14	0.411057	3.3E-11	12	23	0.477595	3.99E-07	53	109	1.238306	6.92E-11
	a6	13	23	0.573398	2E-11	15	23	0.476256	2.35E-07	55	113	1.304278	8.13E-11

Table 4: Numerical results for Algorithm 1, PCG and ETCG on Example 4

		Algorithm 1				PCG				ETCG			
DIMENSION	IP	IT	FE	TM	||J||	IT	FE	TM	||J||	IT	FE	TM	||J||
1000	a1	1	3	0.021703	0	1	2	0.007783	0	1	3	0.007552	0
	a2	1	2	0.007835	0	1	2	0.006638	0	1	2	0.006937	0
	a3	1	2	0.007448	0	1	2	0.006559	0	1	3	0.006529	0
	a4	1	2	0.006132	0	1	3	0.00854	0	1	2	0.005744	0
	a5	1	2	0.006795	0	1	2	0.006776	0	1	3	0.005578	0
	a6	1	2	0.006796	0	1	3	0.006373	0	1	3	0.006431	0
5000	a1	1	3	0.011662	0	1	3	0.010729	0	1	3	0.005993	0
	a2	1	2	0.010139	0	1	3	0.012322	0	1	3	0.004501	0
	a3	1	2	0.010241	0	1	3	0.010382	0	1	3	0.01261	0
	a4	1	2	0.010022	0	1	2	0.010502	0	1	3	0.012128	0
	a5	1	2	0.010976	0	1	2	0.011026	0	1	2	0.010972	0
	a6	1	3	0.011528	0	1	3	0.011378	0	1	3	0.010848	0
10000	a1	1	3	0.014923	0	1	2	0.014452	0	1	3	0.015469	0
	a2	1	2	0.013954	0	1	3	0.013895	0	1	3	0.012354	0
	a3	1	2	0.014586	0	1	3	0.014544	0	1	3	0.009897	0
	a4	1	2	0.013535	0	1	3	0.016536	0	1	3	0.004958	0
	a5	1	2	0.013824	0	1	3	0.015683	0	1	3	0.016392	0
	a6	1	3	0.0156	0	1	3	0.018182	0	1	2	0.013694	0
50000	a1	1	3	0.05146	0	1	2	0.047422	0	1	3	0.014481	0
	a2	1	2	0.04653	0	1	3	0.046049	0	1	3	0.013137	0
	a3	1	2	0.053044	0	1	3	0.046638	0	1	3	0.014775	0
	a4	1	2	0.045921	0	1	2	0.047077	0	1	3	0.016271	0
	a5	1	2	0.05135	0	1	3	0.05273	0	1	3	0.004474	0
	a6	1	3	0.068548	0	1	3	0.053207	0	1	3	0.004751	0
100000	a1	1	3	0.090481	0	1	3	0.081076	0	1	3	0.054675	0
	a2	1	2	0.083921	0	1	3	0.084041	0	1	2	0.049253	0
	a3	1	2	0.08209	0	1	3	0.082398	0	1	2	0.053993	0
	a4	1	2	0.092453	0	1	2	0.083571	0	1	2	0.048236	0
	a5	1	2	0.082792	0	1	3	0.096207	0	1	2	0.045259	0
	a6	1	3	0.096956	0	1	3	0.098183	0	1	3	0.059349	0

Table 5: Numerical results for Algorithm 1, PCG and ECTG 5 on Example 1

		Algorithm 1				PCG				ETCG			
DIMENSION	IP	IT	FE	TM	$\|J\|$	IT	FE	TM	$\|J\|$	IT	FE	TM	$\|J\|$
1000	a1	11	21	0.034372	2.54E-11	8	10	0.011308	4.39E-07	34	67	0.036465	6.08E-11
	a2	10	13	0.012075	2.73E-11	13	10	0.014611	7.26E-07	23	46	0.029076	8.1E-11
	a3	9	11	0.011891	3.44E-11	11	11	0.013783	4.99E-07	31	63	0.032035	8.72E-11
	a4	10	14	0.012485	2.91E-11	12	12	0.013363	3.45E-07	34	68	0.034854	7.05E-11
	a5	10	13	0.012114	2.48E-11	11	13	0.014066	1.53E-07	26	52	0.02542	7.45E-11
	a6	11	17	0.014431	9.6E-12	11	13	0.013818	8.06E-07	25	48	0.024128	7.07E-11
5000	a1	11	21	0.046392	5.67E-11	13	10	0.038143	9.82E-07	17	33	0.01284	8.07E-11
	a2	10	13	0.038718	6.1E-11	13	11	0.04239	1.51E-07	32	62	0.018379	6.03E-11
	a3	9	11	0.035545	7.69E-11	12	12	0.043566	4.87E-07	35	69	0.123905	8.16E-11
	a4	10	14	0.039532	6.52E-11	11	12	0.043014	7.71E-07	25	50	0.100317	6.52E-11
	a5	10	13	0.038907	5.55E-11	11	13	0.04469	3.42E-07	33	67	0.122247	7.02E-11
	a6	11	17	0.043903	2.15E-11	12	14	0.049688	7.86E-07	35	70	0.135148	9.46E-11
10000	a1	11	21	0.068849	8.02E-11	13	11	0.061118	6.06E-07	27	54	0.09846	1E-10
	a2	10	13	0.058424	8.63E-11	13	11	0.056827	2.13E-07	26	50	0.093498	9.49E-11
	a3	10	12	0.05647	9.88E-12	12	12	0.062053	6.89E-07	17	33	0.012711	8.07E-11
	a4	10	14	0.058735	9.21E-11	11	13	0.067199	4.76E-07	32	62	0.026767	6.03E-11
	a5	10	13	0.063247	7.85E-11	11	13	0.066526	4.84E-07	36	71	0.232319	6.92E-11
	a6	11	17	0.071292	3.04E-11	13	15	0.08075	1.03E-07	25	50	0.142691	9.22E-11
50000	a1	12	22	0.27671	1.63E-11	10	12	0.229279	1.26E-07	33	67	0.190399	9.93E-11
	a2	11	14	0.232332	1.75E-11	13	11	0.207531	4.76E-07	36	72	0.204536	8.02E-11
	a3	10	12	0.238967	2.21E-11	11	13	0.246735	1.43E-07	28	56	0.156732	8.48E-11
	a4	11	15	0.242508	1.88E-11	12	14	0.265459	9.87E-08	27	52	0.16356	8.05E-11
	a5	11	14	0.236267	1.59E-11	12	14	0.269366	4.72E-07	17	33	0.012097	8.07E-11
	a6	11	17	0.247444	6.79E-11	13	15	0.2905	2.31E-07	32	62	0.018109	6.03E-11
100000	a1	12	22	0.540089	2.3E-11	13	12	0.452903	1.78E-07	37	73	0.85963	9.28E-11
	a2	11	14	0.453622	2.49E-11	10	11	0.390873	6.74E-07	27	54	0.63136	7.42E-11
	a3	10	12	0.405906	3.12E-11	11	13	0.511976	2.02E-07	35	71	0.836694	7.99E-11
	a4	11	15	0.451781	2.65E-11	12	14	0.503105	1.4E-07	38	76	0.906562	6.46E-11
	a5	11	14	0.420963	2.25E-11	12	14	0.500701	6.68E-07	30	60	0.693342	6.83E-11
	a6	11	17	0.485583	9.6E-11	13	15	0.550563	3.26E-07	29	56	0.690443	6.48E-11

Table 6: Numerical results for Algorithm 1, PCG and ETCG on Example 6

		Algorithm 1				PCG				ETCG			
DIMENSION	IP	IT	FE	TM	$\|J\|$	IT	FE	TM	$\|J\|$	IT	FE	TM	$\|J\|$
1000	a1	1	8	0.022653	0	9	11	0.013086	2.28E-07	1	3	0.00622	0
	a2	1	4	0.00633	0	9	11	0.013947	2E-07	1	3	0.00499	0
	a3	1	3	0.005529	0	10	12	0.014094	2.62E-07	1	3	0.00533	0
	a4	1	5	0.005513	0	10	12	0.013017	2.53E-07	1	3	0.00618	0
	a5	1	4	0.005877	0	11	13	0.013723	2.64E-07	1	3	0.0059	0
	a6	1	13	0.007454	0	11	13	0.014462	2.65E-07	1	6	0.00576	0
5000	a1	1	8	0.012884	0	9	11	0.03582	5.1E-07	1	8	0.00649	0
	a2	1	4	0.009827	0	9	11	0.034015	4.46E-07	1	9	0.00523	0
	a3	1	3	0.00953	0	10	12	0.043407	5.85E-07	1	3	0.00883	0
	a4	1	5	0.010453	0	10	12	0.038015	5.66E-07	1	3	0.0087	0
	a5	1	4	0.009639	0	11	13	0.041242	5.9E-07	1	3	0.00844	0
	a6	1	13	0.015151	0	11	13	0.041011	5.94E-07	1	3	0.00857	0
10000	a1	1	8	0.015995	0	9	11	0.049641	7.21E-07	1	3	0.00829	0
	a2	1	4	0.011215	0	9	11	0.06224	6.31E-07	1	6	0.01051	0
	a3	1	3	0.010341	0	10	12	0.055498	8.27E-07	1	8	0.00515	0
	a4	1	5	0.011703	0	10	12	0.055655	8.01E-07	1	9	0.00534	0
	a5	1	4	0.012718	0	11	13	0.061168	8.34E-07	1	3	0.0101	0
	a6	1	13	0.021376	0	11	13	0.061395	8.39E-07	1	3	0.0103	0
50000	a1	1	8	0.046036	0	10	12	0.19571	7.04E-07	1	3	0.01004	0
	a2	1	4	0.029402	0	10	12	0.191261	6.16E-07	1	3	0.01024	0
	a3	1	3	0.027236	0	11	13	0.23373	8.07E-07	1	3	0.01261	0
	a4	1	5	0.037285	0	11	13	0.207349	7.81E-07	1	6	0.01285	0
	a5	1	4	0.029996	0	12	14	0.236744	8.14E-07	1	8	0.00529	0
	a6	1	13	0.062889	0	12	14	0.235091	8.19E-07	1	9	0.00624	0
100000	a1	1	8	0.079274	0	10	12	0.412189	9.95E-07	1	3	0.0291	0
	a2	1	4	0.050967	0	10	12	0.345391	8.71E-07	1	3	0.02532	0
	a3	1	3	0.043807	0	12	14	0.422036	1.06E-07	1	3	0.02425	0
	a4	1	5	0.056675	0	12	14	0.416034	1.02E-07	1	3	0.02509	0
	a5	1	4	0.051322	0	13	15	0.441327	1.07E-07	1	3	0.02481	0
	a6	1	13	0.131088	0	13	15	0.446176	1.07E-07	1	6	0.04185	0

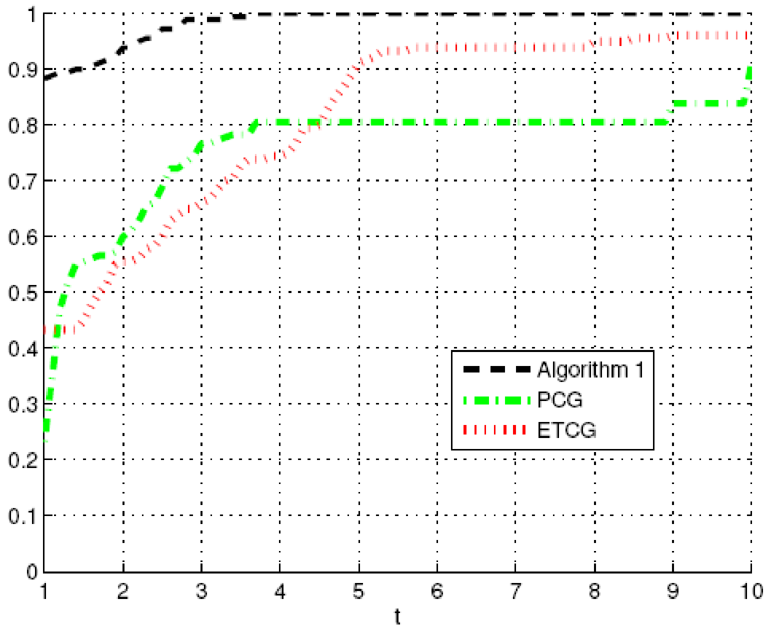

Figure 1: NI performance profile

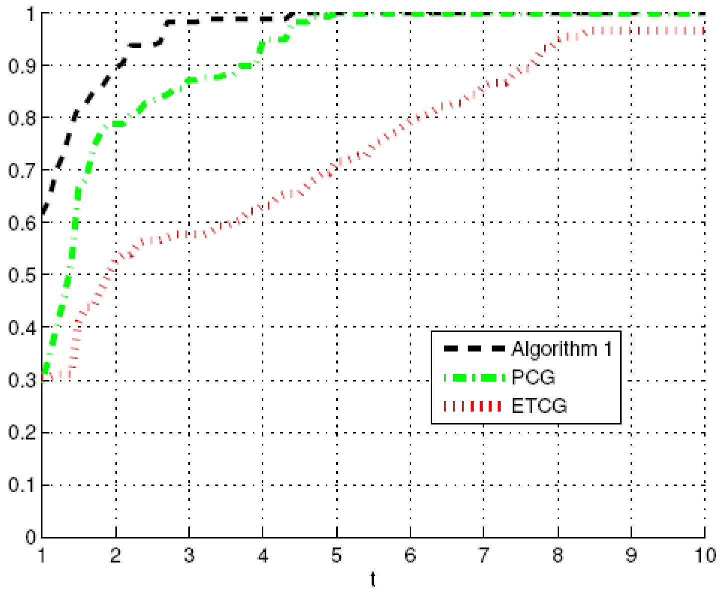

Figure 2: FE performance profile

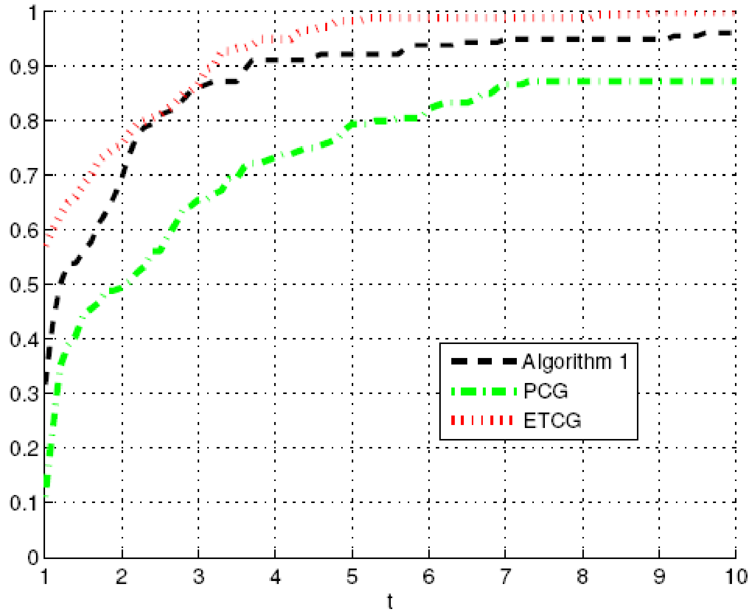

Figure 3: ET performance profile

Results for the Numerical experiments conducted are reported in table 1 − 6. To make it more clearer, data from these tables are represented in figure 1 − 3. In this analysis, an algorithm with higher percentage is considered most effective. In summary, all the competing algorithms were able to solve the employed problems with regards to fixed metrics (Number of iterations, Function evaluations and CPU time) 100% Successfully. In figure 1, number of iterations were presented in which our new algorithm 1 achieved 57% success rate compared to PCG and ETCG with 15% and 28% success rate rates. Figure 2 expresses that, Algorithm 1 has 51% success rate of the functions evaluation, whereas PCG and ETCG had gotten 24% and 25% respectively. CPU time performance was presented in figure 3, where the proposed Algorithm 1 won average score of 32% in comparison with PCG algorithm with 28% and ETCG algorithm with 40% winning rate. Generally, figure 1−3 suggests that, our proposed Algorithm 1 is more efficient and reliable when it comes to solving a monotone nonlinear equation(1).

V. Conclusion Statement

In this paper, we proposed a novel three-term derivative-free method for solving nonlinear monotone equations with convex constraints, inspired by the projection technique developed by Solodov and Svaiter. The key innovation lies in the combination of a spectral gradient parameter with a newly introduced PRP-like conjugate gradient CG coefficient, which integrate the search direction while maintaining a derivative-free framework. The global convergence of the proposed algorithm has been rigorously established under standard assumptions. Our method is not only theoretically sound but also demonstrated exceptional numerical performance on benchmark problems which validate the superior efficiency and reliability of our algorithm in comparison to popular conjugate gradient methods. In particular, the approach has proven to be highly effective in handling large-scale nonlinear monotone equations with convex constraints.

VI. Competing Interest Declaration

The authors declare that they have no competing financial interests or personal relationships that could have influenced the work presented in this paper.

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