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Review Paper

About the common of the same heights root vectors of two completely continuous operators in Hilbert spaces.

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In [1] ,[2] it is proved the necessary and sufficient conditions for existence of common eigenvalues oftwo bundles

$$A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + ... + \lambda^n A_n \qquad (1)$$

$$B(\lambda) = B_0 + \lambda B_1 + \lambda^2 B_2 + ... + \lambda^m B_m \qquad (2)$$

These results was generalized in [3] for the case of several polynomial bundles in Hilbert spaces. In the]4] ,[5] it is given tue sufficient conditions(in the special cases the necessary andsufficient conditions) for existence of common eigenvalues and eigenvectors of two and more completely continuous operators [7], [8], and also two and more polynomial bundles [9] in Hilbert space.Now we give the sufficient conditions for existence of the common root vectors of the identical heights of two completely continuous operators. We introduce some necessary definitions and notions for understanding of the presented material.

Definitions [4],[5],[6].

1. λ is an eigenvalue of operator A ifthere is nonzero element x such, that $Ax - \lambda x = 0$

2.Element x is called a root vector of height k if the following equalities

$(A - \lambda E)^k x = 0$ and $(A - \lambda E)^{k-1} x \neq 0$ are satisfied.

Let $A(\lambda) = A_0 + \lambda A_1 - ... + \lambda^{n-1} A_{n-1} + \lambda^n A_n$ be a polynomial bundle where A_i are bounded operators, acting in Hilbert space H .

3.If for some nonzerovector y_0 we have $A(c) y_0 = y_0$,then y_0 is called an eigenvector of operator $A(\lambda)$,corresponding to eigenvalue c .

4.The vector y_k is called a $k-th$ -associated vector to theeigenvector y_0 if the following equalities

$$y = A(c)y$$

$$y_1 = A(c)y_1 + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y$$

$$\ldots\ldots\ldots \qquad (3)$$

$$y_k = A(c)y_k + \frac{1}{1!}\frac{\partial A(c)}{\partial c}y_{k-1} + \ldots + \frac{1}{k!}\frac{\partial^k A(c)}{\partial c^k}y$$

are fulfilled.

The system of linear independent eigen and associated vectors is called a chain of eigen and associated (e.a) vectors of operator $A(\lambda)$,corresponding to eigenvalue c. The number of e.a. vectors in the chain of e.a. vectors is called a length of eigenvector y_0. Totality, all independent eigensnd associated vectors ,corresponding to all eigen vectors with eigenvalue c is called the multiplicity of eigenvalue c.

5.M.V.Keldysh built the derivative systems with the help of the formulas

$$\left[\frac{d^k}{dt^k}e^{\lambda t}\left(x_k + \frac{1}{1!}x_{k-1} + \ldots + \frac{1}{k!}x_0\right)\right](t=0) \quad k = 1, 2, \ldots, s \qquad (4)$$

Now our purpose to give the sufficient conditions for the existence of common root vectors of the same heights of two completely continuous operators A and B.

For each m we need to study the spectral properties of operators

$$A(m,\lambda) = (A - \lambda E)^m \quad (5)$$

and $B(m,\lambda) = (B - \lambda E)^m$ (6)

For definiteness and without loss of generality, we work with the operator $B(m,\lambda)$.

There is the connection between the eigenvalues of operator bundle $B(m,\lambda)$ and the eigenvalues of the operator

$$B(m) = \begin{pmatrix} 0 & -B & 0 & \ldots & 0 \\ 0 & 0 & -B & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ B_0 & B_1 & B_2 & \ldots & B_{m-1} \end{pmatrix} \quad (7)$$

$B_i = C_m^i B$

where $(i = 0,1,...,m-1)$

Operator (7)is a completely continuous and it acts in direct sum of m copies of Hilbert space H. Further, we install that the first component of eigenvector of operator $B(m)$ coincides with the eigenvector of operator $B(m,\lambda)$. At first, let be $(x_{0,0}, x_{0,1},..., x_{0,m-1})$ a root vector of an operator $B(m)$ of height 1.

It is easy to see that the first component of eigenvector of operator $B(m)$

coincides with the eigenvector of operator $B(m,\lambda) = (B - \lambda E)^{m-}$ corresponding to its eigenvalue, and the second, third, ,…,- n - th components of this eigenvector match the elements of derivative systems, built on corresponding the eigen- vector of operator B by formulas (4).Really, let $(x_{00}, x_{01},..., x_{0,m-1})$ be eigenvector of operator

$B(m)$,then

$$-\begin{pmatrix} 0 & -B & 0 & ... & 0 \\ 0 & 0 & -B & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ B_0 & B_1 & B_2 & ... & B_{m-1} \end{pmatrix} \begin{pmatrix} x_{00} \\ x_{01} \\ \vdots \\ x_{0,n-1} \end{pmatrix} = \lambda \begin{pmatrix} x_{00} \\ x_{01} \\ \vdots \\ x_{0,n-1} \end{pmatrix}$$

or

$$x_{0,1} = -\lambda B^{-1} x_{0,0}$$

$$x_{0,2} = -\lambda B^{-1} x_{0,1} \quad (8)$$

……………..

$$x_{0,m-1} = -\lambda B^{-1} x_{0,m-2}$$

$$B_0 x_{0,0} + ... + B_{n-1} x_{0,n-1,m-1} = x$$

Substituting the expression for $x_{0,1}$ from the first equation of (8) into the second equation ,and then, substituting the expression, obtained for $x_{0,2}$, into the third equation and continuing this process, we have

$$x_{0,k} = (-1)^k \lambda B^{-k} \ x_{0,0} \quad (9)$$

Using the expressions (9) in last equation of (8)

with taking account $B_i = C_m^i B$ $(i = 0,1,...,m-1)$

and acting on both parts of last equation of (8) by operator B^{m-1} we have that $B(m,\lambda)x_{0,0} = (B-\lambda E)^{m-} x_{0,0} = 0$. A similar one can be proven for the operator A and $A(m,\lambda) = (A-\lambda E)^m = 0$.

We proved that the spectrum of each operator $A(m) = \begin{pmatrix} 0 & -A & ... & 0 \\ 0 & 0 & -A & ... \\ ... & ... & ... & ... \\ A_0 & A_1 & ... & A_{n-1} \end{pmatrix}$ where $A_i = C_m^i A$ $(i = 0,1,...,m-1)$

and $B(m) = \begin{pmatrix} 0 & -B & ... & 0 \\ 0 & 0 & B & ... \\ ... & ... & ... & ... \\ B_0 & B_1 & ... & B_{n-1} \end{pmatrix} -$

where $B_i = C_m^i B$ $(i = 0,1,...,m-1)$ coincides with the spectrum of each completely continuous operators A and B , correspondingly.

Because operators $A(m)$ and $B(m)$ are completely continuous then they have discrete spectra.

Now we give the sufficient condition when the root vectors of the same height of the operators A and B coincide.

We consider the operators $A(m)=Tm=+iS(m)$ where

$$Tm=\frac{A(m)+[Am)]^{*}}{2}$$

$$Sm=\frac{A(m)-[Am)]^{*}}{2i}$$

and

$$B(m)=G(m)+iH(m)$$

where

$$Gm=\frac{B(m)+[B(m)]^{*}}{2}$$

$$Hm=\frac{B(m)-B(m)]^{*}}{2i}$$

Let $E_{m,t}$ and $F_{m,t}$ be the expansion of unity of operators - T_m and S_m ,corresponding [5].

Further, the following equalities are true [5].

1. $E_{m,a}=0$, $E_{m,b}=1$

$F_{m,c}=0$, $F_{m,,d}=1$

2. $E_{m,l}E_{m,n}=E_{m,k}$

$k=\min(l,n)$

$$F_{,mr}F_{m,s} = F_{m,q}$$

$$q = \min(r, s)$$

3. $E_{m,t} - E_{m,t-0} = P_{m,t}$

$$F_{sm,m,} - F_{sm,-0} = R_{m,s}$$

where $P_{tm,t}$ is a projective operator, that projects onto eigen subspace

T_m, corresponding to its eigenvalue t

and $R_{m,s}$ - is a two parameter projective operator, that projects onto the eigen subspace of operator $R(m)$ corresponding to eigenvalue s.

We denote $K_{p,t}$ and $L_{p,k}$ the expansions of unity of operators $G(m)$ A and $H(m)$ correspondingly. The following equalities are true

$$Kp, c = 0, \; Kp, d = 1$$

$$Lp, c = 0, \; Lp, d = 1$$

$$KmK_{\backslash\backslash p,p,n} = K_{p,k} \;\; K_{p,m}K_{,p,n} = K_{p,k}$$

$$k = \min(m, n)$$

$$Lp, sLp, q = Lp, r$$

$$r = \min(q, s)$$

3. $K_{p,t} - K_{p,t-0} = S_{p,t}$

$L_{p,t} - L_{p,t-} = N_{p,t}$

where Spm is two parameter projective operator, that projects onto the eigen subspace of operator $G(m)$ corresponding to its eigenvalue t . t

and $N_{m,t}$ is a projective operator, that projects on eigen subspace of operator, $H(m)$ corresponding to eigenvalue s .Considering that the operators $A(m)$ and $B(m)$ have the discrete spectra. We can formulate the

Theorem. If for some real numbers m, a, b, c, d and some element x from H^m

,and the following conditions

1. $P_{tm,a} R_{m,b} S,_{tm,c} H_{m,d} x \neq 0$

2. $P_{tm,-1,a}, R_{m-1,,b} x = 0$

$S_{tm,-1,c} H_{m,-1,d} x = 0$ are satisfiedthem the first component x_0 of element x is a common root vector of height m of both operators A and B .

Proof.

From $P_{tm,t} S_{tm,s} R_{p,u} H_{p,v} \neq 0$ (10)

it follows that its projection enters the projections of projective operators

$P_{tm,t} S_{tm\,p} \neq 0$ (11)

And $R_{p,u} H_{p,v} \neq 0$ (12)

Besides, the $P_{tm,t} S_{tm,s} R_{p,u} H_{p,v}$ projects on intersection of eigen space of operator $A(m)$,corresponding to its eigenvalue $a + ib$ and eigen subspace ofoperator $B(m)$,corresponding to its eigenvalue $c + id$ [7]. Further, from[7] condition (11) is the sufficient condition for existence of eigenvalue $a+ib$ of operator $A(m)$ and the (12) is the sufficient condition for the existence the eigenvalue $c+id$ of operator $B(m)$ .The condition (10)

of the Theorem means, that the sufficient conditions for existence of common eigen vector of two completely continuous operators $A(m)$ and $B(m)$ from]8,] and [9] are satisfied.

It is not difficult to see that the first component of common eigenvector of operators $A(m)$ and $B(m)$ is the common root vector of both operators A and B .The condition 2 of this Theorem means that this common eigenvector of operators $A(m)$ and $B(m)$ is not eigenvector of operators $A(m-1)$ and $B(m-1)$ with corresponding eigenvalues $a+ib$ and $c+id$

Thus the fires component of common eigenvector of operators $A(m)$ and $B(m)$ is a common m heighs root vector of operators A and B .

Theorem is proven.

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