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**Review Paper**

# Transient Analysis of a Finite Capacity Queueing System with Catastrophes and State-Dependent Environmental Change Parameter

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***Abstract:*** *This paper presents a transient analysis of a finite capacity queueing system with catastrophes and state-dependent environmental change parameter. The tendency of the Poisson rate at which the system moves from environmental state F to E increases or decreases as the number of customers in the queue. Also, at some random times, the number of customers is immediately reset to zero whenever a catastrophe occurs at the system. Transient solution is obtained by using the technique of probability generating function. The Steady state solution of the model is obtained by using the property of Laplace transform. Furthermore, we derive and discuss several particular cases of the queueing model, both with and without catastrophes.*

***Keywords:*** *Transient analysis, Catastrophes, Environmental change, Finite capacity, Probability generating function.*

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## I. Introduction:

In queueing literature, analytical results for the transient behavior of queueing models are not as prevalent as those for steady-state conditions. Steady-state measures fail to provide insights into the system's transient behavior, which is crucial for understanding its dynamics over finite periods. While steady-state results are well-suited for assessing long-term performance measures, transient solutions are essential for analyzing how the system evolves over time. Among various methods available, the probability generating function technique is particularly effective for deriving transient solutions. However, even for a simple M/M/1/N queue, obtaining analytical expressions for transient behavior proves to be quite challenging. In this context, we have successfully derived the transient solution for a finite capacity queueing system characterized by a state-dependent environmental change parameter, taking into account the effects of catastrophes.

Recently, numerous authors have introduced a new class of queueing systems that incorporate the effects of catastrophes. In particular, birth and death models have been extended to include the assumption that the number of customers is instantly reset to zero at certain random times. These catastrophes occur at the service facility as a Poisson process with a specified rate $\xi$. When a catastrophe happens, all customers present are immediately removed, the server is temporarily inactivated, and it becomes available for service only when a new customer arrives in the system.

The queueing system with catastrophes was first examined by Krishna Kumar and Arivudainambi [9] in 2000, followed by a study in 2003 by Crescenzo et al. [5], who derived the transient probabilities for the M/M/1 queue model with catastrophes. Jain and Kanethia [8] explored the transient analysis of a queue influenced by environmental and catastrophic effects. Kumar, D. [11] further explores a queueing system that incorporates catastrophes, state-dependent input parameters, and environmental changes to assess their impact on the system behavior. Liu and Liu [16]

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investigated the transient probabilities of an M/PH/1 queue model with catastrophes, which serves as a generalization of the M/M/1 queue model under similar conditions. Additionally, a substantial number of research papers have addressed population processes affected by catastrophes. Notable contributions in this area include studies by Brockwell [1,2], Kyriakidis [12], and Swift [13], who have all discussed various birth and death models incorporating the effects of catastrophes.

In this paper, we introduce an additional factor of environmental change, whereby the changes in the environment influence the state of the queueing system. Specifically, the state of the queueing system is a function of these environmental change factors. When the environment shifts, the system transitions from environmental state E to state F at a certain rate $\beta$, while it moves from state F to state E at a rate of bn.

The primary objective of this paper is to develop a finite capacity queueing system that incorporates the effects of catastrophes and a state-dependent environmental change parameter. In Section 2, we outline the assumptions and definitions related to the model. Section 3 provides a detailed analysis of the queueing model, while specific cases are explored in Section 4. Finally, Sections 5 and 6 present the steady-state results and discuss the applications of the model.

## II. Assumptions and Definitions of the Model:

a) The customers arrive in the system one by one in accordance with a Poisson process at a single service station. The arrival pattern is non-homogeneous i.e., there may exists two arrival rates, namely $\lambda_1$ and 0 of which only one is operative at any instant.

b) The customers are served one by one at the single service channel. The service time is exponentially distributed. Further, corresponding to arrival rate $\lambda_1$ the Poisson service rate is $\mu_1$ and the service rate corresponding to the arrival rate 0 is $\mu_2$. The state of the system when operating with arrival rate $\lambda_1$ and service rate $\mu_1$ is designated as E whereas the other with arrival rate 0 and service rate $\mu_2$ is designated as F.

c) The Poisson rate at which the system goes from environmental state E to F is denoted by $\beta$. Also the Poisson rate $b_n$, at which the system moves from environmental state F to E tends to increase or decrease according as the numbers in the queue (say n) increase or decrease from some fixed number (say N). We therefore define,

$$b_n = \alpha\left[1+\varepsilon(n-N)\right] \text{ with } n \geq N-\frac{1}{\varepsilon}$$

$$\text{and } 0 \leq N-\frac{1}{\varepsilon} \leq n \leq M$$

Where M denotes the size of the waiting space and $\varepsilon$ is a positive number such that $\varepsilon \geq \frac{1}{N}$. This restriction on M is necessary to avoid a negative value of bn. When n=N or $\varepsilon$ =0, bn gives the normal rate as $\alpha$.

d) When the system is not empty, catastrophes occur according to a Poisson process with rate $\xi$. The effect of each catastrophe is to make the queue instantly empty. Simultaneously, the system becomes ready to accept the new customers.

e) The queue discipline is first-come-first-served.

f) The capacity of the queueing system is restricted to M. i.e., if at any instant there are M units in the queue then the units arriving at that instant will not be permitted to join the queue and will be considered lost for the system.

Define,

$P_n(t)$ = Joint probability that at time t the system is in state E and n units are in the queue, including the one in service.

$Q_n(t)$ = Joint probability that at time t the system is in state F and n units are in the queue, including the one in service.

$R_n(t)$= The probability that at time t there are n units in the queue, including the one in service.

Obviously,

$$R_n(t) = P_n(t) + Q_n(t)$$

Let us measure time t from an instant when there are zero customers in the queue and the system is in the environmental state E so that the initial conditions associated with $P_n(t)$ and $Q_n(t)$ becomes,

$$P_n(0) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$Q_n(0) = 0$ ; for all n.

## III. Formulation of the Queueing Model and Transient Analysis:

The differential-difference equations governing the system are.

$$\frac{d}{dt}P_0(t) = -(\lambda_1 + \beta + \xi)P_0(t) + \mu_1 P_1(t) + b_0 Q_0(t) + \xi\sum_{n=0}^{M} P_n(t); \quad n = 0 \quad (1)$$

$$\frac{d}{dt}P_n(t) = -(\lambda_1 + \mu_1 + \beta + \xi)P_n(t) + \mu_1 P_{n+1}(t) + \lambda_1 P_{n-1}(t) + b_n Q_n(t); \quad 0 < n < M \quad (2)$$

$$\frac{d}{dt}P_M(t) = -(\mu_1 + \beta + \xi)P_M(t) + \lambda_1 P_{M-1}(t) + b_M Q_M(t); \quad n = M \quad (3)$$

$$\frac{d}{dt}Q_0(t) = -(b_0 + \xi)Q_0(t) + \mu_2 Q_1(t) + \beta P_0(t) + \xi\sum_{n=0}^{M} Q_n(t); \quad n = 0 \quad (4)$$

$$\frac{d}{dt}Q_n(t) = -(\mu_2 + b_n + \xi)Q_n(t) + \mu_2 Q_{n+1}(t) + \beta P_n(t); \quad 0 < n < M \quad (5)$$

$$\frac{d}{dt}Q_M(t) = -(\mu_2 + b_M + \xi)Q_M(t) + \beta P_M(t); \quad n = M \quad (6)$$

Define, the Laplace Transform as

$$\text{L.T. } [f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt = \bar{f}(s) \quad (7)$$

Now, taking the Laplace transforms of equations (1)–(6) and using the initial conditions, we get

$$(s + \lambda_1 + \beta + \xi)\bar{P}_0(s) - 1 = \mu_1 \bar{P}_1(s) + b_0 \bar{Q}_0(s) + \xi\sum_{n=0}^{M}\bar{P}_n(s) \quad (8)$$

$$(s + \lambda_1 + \mu_1 + \beta + \xi)\bar{P}_n(s) = \mu_1 \bar{P}_{n+1}(s) + \lambda_1 \bar{P}_{n-1}(s) + b_n \bar{Q}_n(t) \quad (9)$$

$$(s + \mu_1 + \beta + \xi)\bar{P}_M(s) = \lambda_1 \bar{P}_{M-1}(s) + b_M \bar{Q}_M(s) \quad (10)$$

$$(s + b_0 + \xi)\bar{Q}_0(s) = \mu_2 \bar{Q}_1(s) + \beta \bar{P}_0(s) + \xi\sum_{n=0}^{M}\bar{Q}_n(s) \quad (11)$$

$$(s + \mu_2 + b_n + \xi)\bar{Q}_n(s) = \mu_2 \bar{Q}_{n+1}(s) + \beta \bar{P}_n(s) \quad (12)$$

$$(s + \mu_2 + b_M + \xi)\bar{Q}_M(s) = \beta \bar{P}_M(s) \quad (13)$$

Define, the probability generating functions

$$P(z,s) = \sum_{n=0}^{M}\bar{P}_n(s) z^n \quad (14)$$

$$Q(z,s) = \sum_{n=0}^{M}\bar{Q}_n(s) z^n \quad (15)$$

$$R(z,s) = \sum_{n=0}^{M}\bar{R}_n(s) z^n \quad (16)$$

where

$$\bar{R}_n(s) = \bar{P}_n(s) + \bar{Q}_n(s)$$

Multiplying equations (8)–(10) by the suitable powers of z, summing over all n and using equations (14)–(16), we have.

$$\alpha\varepsilon z^2 Q'(z,s)+\alpha(1-\varepsilon N)zQ(z,s)+\left[\lambda_1 z^2 - z\{s+\lambda_1+\mu_1+\xi+\beta\}+\mu_1\right]P(z,s)$$

$$=\lambda_1 z^{M+1}(z-1)\overline{P}_M(s)+\mu_1(1-z)\overline{P}_0(s)-z-\xi z\sum_{n=0}^{M}\overline{P}_n(s) \qquad (17)$$

Similarly, from equations (11)–(13) and using (14)–(16), we have

$$\alpha\varepsilon z^2 Q'(z,s)+\left[z\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}-\mu_2\right]Q(z,s)-\beta zP(z,s)$$

$$=\mu_2(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s) \qquad (18)$$

Eliminating P(z,s) from equations (17) and (18), we have

$$Q'(z,s)+\frac{\eta_1(z)}{\eta_2(z)}Q(z,s)=\frac{1}{\eta_2(z)}\left[z_1+z_2\overline{Q}_0(s)+z_3\overline{P}_0(s)+z_4\overline{P}_M(s)+z_5\sum_{n=0}^{M}\overline{P}_n(s)+z_6\sum_{n=0}^{M}\overline{Q}_n(s)\right] \qquad (19)$$

where

$$\eta_1(z)=\left[a_2\lambda_1 z^3+\{\alpha\beta(1-\varepsilon N)-a_2a_3\}z^2+(a_2\mu_1+a_3\mu_2)z-\mu_1\mu_2\right]$$

$$\eta_2(z)=\alpha\varepsilon z^2\left[\lambda_1 z^2+\mu_1-a_1 z\right]$$

$$z_1=-\beta z^2$$

$$z_2=\mu_2(z-1)\left[\lambda_1 z^2-a_3 z+\mu_1\right]$$

$$z_3=\beta\mu_1 z(1-z)$$

$$z_4=\beta\lambda_1 z^{M+2}(z-1)$$

$$z_5=-\beta\xi z^2$$

$$z_6=\xi z\left[\lambda_1 z^2-a_3 z+\mu_1\right]$$

$$a_1=[s+\mu_1+\lambda_1+\xi]$$

$$a_2=[s+\mu_2+\xi+\alpha(1-\varepsilon N)]$$

$$a_3=[s+\lambda_1+\mu_1+\beta+\xi]$$

In equation (19), the co-efficient of Q(z,s) can be re-written as

$$\frac{\eta_1(z)}{\eta_2(z)}=\frac{A_1}{z}+\frac{B_1}{z^2}+\frac{C}{2\lambda_1}\left[\frac{2\lambda_1 z-a_1}{\lambda_1 z^2-a_1 z+\mu_1}\right]+\frac{D_1}{X^2(z)-a^2} \qquad (20)$$

where

$$A_1=\frac{a_2}{\alpha\varepsilon}+A$$

$$A=\frac{\mu_1 a_4-a_1 a_5}{\alpha\varepsilon\mu_1^{\,2}}$$

$$B_1=\left[\alpha\beta(1-\varepsilon N)-a_2a_3+a_1a_2\right]\frac{1}{\lambda_1\alpha\varepsilon}+\frac{B}{\alpha\varepsilon}$$

$$B=-\frac{a_5}{\mu_1}$$

$$C=\frac{\lambda_1(a_1a_5-\mu_1a_4)}{\alpha\varepsilon\mu_1^{\,2}}$$

$$D_1=D+\frac{Ca_1}{2\lambda_1}$$

$$D = a_1\left(\mu_1 a_4 - a_1 a_5\right)\frac{1}{\mu_1^{\,2}\alpha\varepsilon} + \frac{\lambda_1 a_5}{\mu_1\alpha\varepsilon}$$

$$X(z) = \left[z\lambda_1^{1/2} - \frac{a_1}{2\lambda_1^{1/2}}\right]$$

$$a = \left[\frac{a_1^{\,2}}{4\lambda_1} - \mu_1\right]^{1/2}$$

$$a_4 = a_3\mu_2 + \frac{a_1}{\lambda_1}\left[\alpha\beta\left(1-\varepsilon N\right) - a_2 a_3 + a_1 a_2\right]$$

$$a_5 = \frac{\mu_1}{\lambda_1}\left[\alpha\beta\left(1-\varepsilon N\right) - a_2 a_3 + a_1 a_2\right] + \mu_1\mu_2$$

Using equation (20) in equation (19) and integrating (19) w. r. t. z, we have

$$Q(z,s) = \frac{L_1(z) + L_2(z)\overline{Q}_0(s) + L_3(z)\overline{P}_0(s) + L_4(z)\overline{P}_M(s) + L_5(z)\sum_{n=0}^{M}\overline{P}_n(s) + L_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{L(z)} \qquad (21)$$

where

$$L(z) = z^{A_1}\left(\lambda_1 z^2 - a_1 z + \mu_1\right)^{C/2\lambda_1}\left[\frac{X(z)-a}{X(z)+a}\right]^{D_1/2a}\cdot\exp\left(-\frac{B_1}{z}\right)$$

$$L_j(z) = \int_0^z \frac{z_j}{\eta_2(z)} L(z)\,dz\,; \qquad j = 1, 2, 3, 4, 5, 6.$$

Again eliminating $Q'(z,s)$ from equations (17) and (18), and using Q(z,s) from equation (21), we have

$$P(z,s) = \frac{L_7(z) + \overline{Q}_0(s)L_8(z) + \overline{P}_0(s)L_9(z) + \overline{P}_M(s)L_{10}(z) + \sum_{n=0}^{M}\overline{P}_n(s)L_{11}(z) + \sum_{n=0}^{M}\overline{Q}_n(s)L_{12}(z)}{B(z)L(z)} \qquad (22)$$

where

$$L_7(z) = L_1(z)\,g(z) - z\,L(z)$$
$$L_8(z) = L_2(z)\,g(z) - \mu_2(z-1)\,L(z)$$
$$L_9(z) = L_3(z)\,g(z) - \mu_1(z-1)\,L(z)$$
$$L_{10}(z) = L_4(z)\,g(z) + \lambda_1 z^{M+1}(z-1)\,L(z)$$
$$L_{11}(z) = L_5(z)\,g(z) - \xi z\,L(z)$$
$$L_{12}(z) = L_6(z)\,g(z) - \xi z\,L(z)$$
$$g(z) = (s+\xi)z + \mu_2(z-1)$$
$$B(z) = \lambda_1 z^2 + \mu_1 - a_1 z$$

Adding equations (21) and (22), we have

$$R(z,s) = \frac{C_1(z) + C_2(z)\overline{Q}_0(s) + C_3(z)\overline{P}_0(s) + C_4(z)\overline{P}_M(s) + C_5(z)\sum_{n=0}^{M}\overline{P}_n(s) + C_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B(z)\,L(z)} \qquad (23)$$

where

$C_i(z) = B(z)\,L_i(z) + L_{i+6}(z)$ ; i=1, 2, 3, 4, 5, 6.

Since, $\sum_{n=0}^{M} R_n(t) = 1$

Therefore,

$$R(1,s)=\sum_{n=0}^{M}\overline{R}_n(s)=\frac{1}{s}$$

Thus equation (23) for z=1, gives

$$R(1,s)=\frac{1}{s}=\lim_{z\to 1}R(z,s) \qquad (24)$$

Also

$$P(0,s)=\overline{P}_0(s)=\lim_{z\to 0}P(z,s) \qquad (25)$$

and $$Q(0,s)=\overline{Q}_0(s)=\lim_{z\to 0}Q(z,s) \qquad (26)$$

The equations (24), (25) and (26) on solution gives the values of $\overline{P}_0(s)$, $\overline{Q}_0(s)$, $\overline{P}_M(s)$, $\sum_{n=0}^{M}\overline{P}_n(s)$ and $\sum_{n=0}^{M}\overline{Q}_n(s)$. Let them be $P_0$, $Q_o$, $P_M$, $\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}Q_n$ respectively then from relation (24), we have

$$R(z,s)=\frac{C_1(z)+C_2(z)Q_0+C_3(z)P_0+C_4(z)P_M+C_5(z)\sum_{n=0}^{M}P_n+C_6(z)\sum_{n=0}^{M}Q_n}{B(z)\,L(z)} \qquad (27)$$

The Laplace transform of various state probabilities for the number of customers in the queue, including the one in service can be obtained as the coefficients of different powers of z in the binomial expansion of equation (27).

Again, since

P(1,s)=Laplace transform of the probability that the system is in environmental state E.

and

Q(1,s)=Laplace transform of the probability that the system is in environmental state F.

We have from equations (21) and (22) on setting z=1 and $\overline{P}_0(s)=P_0$, $\overline{Q}_0(s)=Q_0$, $\overline{P}_M(s)=P_M$, $\sum_{n=0}^{M}\overline{P}_n(s)=\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}\overline{Q}_n(s)=\sum_{n=0}^{M}Q_n$

$$P(1,s)=\frac{L_1(1)+L_2(1)Q_0+L_3(1)P_0+L_4(1)P_M+L_5(1)\sum_{n=0}^{M}P_n+L_6(1)\sum_{n=0}^{M}Q_n}{L(1)} \qquad (28)$$

$$Q(1,s)=\frac{L_7(1)+L_8(1)Q_0+L_9(1)P_0+L_{10}(1)P_M+L_{11}(1)\sum_{n=0}^{M}P_n+L_{12}(1)\sum_{n=0}^{M}Q_n}{B(1)\,L(1)} \qquad (29)$$

These on inversions give the respective probabilities for the system to be in the environmental states E and F.

## IV. Particular Case:

Setting $\varepsilon=0$ or n=N in equations (17) and (18), (i.e., when the rate of change of environmental state F to E is constant), we have

$$X_1(z)P(z,s)+X_2(z)Q(z,s)+X_3(z)=0 \qquad (30)$$

$$X_4(z)P(z,s)+X_5(z)Q(z,s)+X_6(z)=0 \qquad (31)$$

where

$$X_1(z)=-\left[\lambda_1 z^2-z\,(s+\lambda_1+\mu_1+\beta+\xi)+\mu_1\right]$$

$$X_2(z)=-\alpha z$$

$$X_3(z)=-\left[\mu_1(z-1)\,\overline{P}_0(s)+\lambda_1 z^{M+1}(1-z)\overline{P}_M(s)+z+\xi z\sum_{n=0}^{M}\overline{P}_n(s)\right]$$

$X_4(z)=\beta z$

$X_5(z)=[\mu_2 - z(s+\mu_2+\alpha+\xi)]$

$X_6(z)=\left[\mu_2(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)\right]$

From equations (30) and (31), we have

$$P(z,s)=\frac{X_2(z)X_6(z)-X_3(z)X_5(z)}{X_1(z)X_5(z)-X_2(z)X_4(z)} \quad (32)$$

$$Q(z,s)=\frac{X_4(z)X_3(z)-X_1(z)X_6(z)}{X_1(z)X_5(z)-X_2(z)X_4(z)} \quad (33)$$

Thus, we have

$$R(z,s)=\frac{\begin{array}{l}\mu_2(z-1)[X_2(z)-X_1(z)]\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)[X_2(z)-X_1(z)]+\mu_1(1-z)\\ [X_4(z)-X_5(z)]\overline{P}_0(s)+\lambda_1 z^{M+1}[X_5(z)-X_4(z)](1-z)\overline{P}_M(s)+\\ z[X_5(z)-X_4(z)]+\xi z\sum_{n=0}^{M}\overline{P}_n(s)[X_5(z)-X_4(z)]\end{array}}{-z^2 s^2+sX_7(z)+(1-z)X_8(z)-z^2\xi(\alpha+\beta+\xi)} \quad (34)$$

where

$X_7(z)=\lambda_1 z^3 - z^2(\lambda_1+\mu_1+\mu_2+\alpha+\beta+2\xi)+z(\mu_1+\mu_2)$

$X_8(z)=-z^2\lambda_1(\alpha+\mu_2+\xi)+z[\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)]-\mu_1\mu_2$.

$P(1,s)=\sum_{n=0}^{M}\overline{P}_n(s)=\frac{s+\alpha+\xi}{s(s+\alpha+\beta+\xi)}$

$Q(1,s)=\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{\beta}{s(s+\alpha+\beta+\xi)}$

and

$$\sum_{n=0}^{M}\overline{R}_n(s)=\sum_{n=0}^{M}\overline{P}_n(s)+\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{1}{s}$$

Relation (34) is a polynomial in z and holds for all values of z, including the three zeros of the denominator. Hence $\overline{P}_0(s)$, $\overline{Q}_0(s)$ and $\overline{P}_M(s)$ are obtained by setting the numerator equal to zero and substituting the three zeros, $\alpha_1$, $\alpha_2$ and $\alpha_3$ (say) of the denominator (at each of which the numerator must vanish).

Now, letting $\alpha\to\infty$, $\beta\to 0$ and setting $\mu_1=\mu_2=\mu$ (say) in relation (34), we have

$$r(z,s)=\frac{(1-z)\mu\overline{R}_0(s)-(1-z)\lambda_1 z^{M+1}\overline{P}_M(s)-z-\xi z/s}{\lambda_1 z^2 - z(s+\lambda_1+\mu+\xi)+\mu} \quad (35)$$

where

$\overline{R}_0(s)=\overline{P}_0(s)+\overline{Q}_0(s)$

$r(z,s)=\lim_{\beta\to 0}\left[\lim_{\alpha\to\infty}R(z,s)\right]$

Relation (35) is a polynomial in z and exists for all values of z, including the two zeros of the denominator. Hence, $\overline{R}_0(s)$ and $\overline{P}_M(s)$ can be evaluated as before.

## V. Steady State Results:

This can be obtained by the well-known property of the Laplace transform given below:

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,\bar{f}(s)$, If the limit on the left hand side exists.

Thus if $R(z)=\sum_{n=0}^{M} R_n z^n$

where,

$$R_n = \lim_{s\to 0} s\bar{R}_n(s)$$

Then $R(z)=\lim_{s\to 0} s\, R(z,s)$

By employing this property, from equation (23) we have

$$R(z)=\frac{Q_0 N_1(z)+N_2(z)P_0+N_3(z)P_M+N_4(z)\sum_{n=0}^{M}P_n+N_5(z)\sum_{n=0}^{M}Q_n+N}{B_1(z)\,N(z)} \qquad (36)$$

where,

$$N_i(z)=B_1(z)L'_{i+1}(z)+L'_{i+7}(z)\Big|_{s=0} \quad ; \ i=1, 2, 3, 4, 5.$$

$$B_1(z)=B(z)\big|_{s=0}$$

$$L'_j(z)=\int\left[\frac{z_j}{\eta_2(z)}L(z)\right]_{\epsilon=0} dz \quad ; \ j=2, 3, 4, 5, 6.$$

$$L'_k(z)=L_k(z)\big|_{s=0} \quad ; \ k=8, 9, 10, 11, 12.$$

$$N(z)=L(z)\big|_{s=0}$$

N' = the constant of integration.

The unknown quantities $Q_0$, $P_0$, $P_M$, $\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}Q_n$ can be evaluated as before.

**Particular case:**

Relation (34), on applying the theory of Laplace transforms gives

$$R(z)=\frac{\begin{array}{l}\mu_2(1-z)\left\{\alpha z+z(\lambda_1+\mu_1+\beta+\xi)-\lambda_1 z^2-\mu_1\right\}Q_0+\\ \mu_1(1-z)[\beta z-\{\mu_2-z\,(\mu_2+\alpha+\xi)\}\,]P_0+\lambda_1 z^{M+1}(1-z)\{\mu_2-z\,(\mu_2+\alpha+\xi)-\beta z\}\,P_M+\\ \xi z/(\alpha+\beta+\xi)\left[\beta\left\{\lambda_1 z^2-z(\lambda_1+\mu_1+\alpha+\beta+\xi)+\mu_1\right\}+(\alpha+\xi)\{\mu_2-z(\mu_2+\alpha+\beta+\xi)\}\right]\end{array}}{\begin{array}{l}z^3\lambda_1(\mu_2+\alpha+\xi)-z^2[\lambda_1(\mu_2+\alpha+\xi)+\{\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)\}+\xi(\alpha+\beta+\xi)]\\ +z[\{\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)\}+\mu_1\mu_2]-\mu_1\mu_2\end{array}} \qquad (37)$$

or, we can write

$$R(z)=\frac{T(z)Q_0+N(z)P_0+L(z)P_M+M(z)}{K(z)} \qquad (38)$$

Where T(z), N(z) and L(z) are the co-efficient of $Q_0$, $P_0$ and $P_M$ respectively in the numerator of equation (37) and K(z) is the denominator of equation (37).

Equation (38) is a polynomial in z and holds for all values of z, including three zeros of the denominator. Hence $Q_0$, $P_0$ and $P_M$ can be obtained by setting the numerator equal to zero. Substituting the three zeros $b_1$, $b_2$ and $b_3$ (say) of the denominator (at each of which the numerator must vanish).

Three equations determining the constants $Q_0$, $P_0$ and $P_M$ are

$$T(b_1)Q_0 + N(b_1)P_0 + L(b_1)P_M = -M(b_1) \quad (39)$$

$$T(b_2)Q_0 + N(b_2)P_0 + L(b_2)P_M = -M(b_2) \quad (40)$$

$$T(b_3)Q_0 + N(b_3)P_0 + L(b_3)P_M = -M(b_3) \quad (41)$$

After solving these equations, we have

$$Q_0 = \frac{-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}}{A}$$

$$P_0 = \frac{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}}{A}$$

$$P_M = \frac{-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}}{A}$$

where

$$A = \begin{vmatrix} T(b_1) & N(b_1) & L(b_1) \\ T(b_2) & N(b_2) & L(b_2) \\ T(b_3) & N(b_3) & L(b_3) \end{vmatrix}$$

$A_{ij}$ is the co-factor of the (i, j)$^{th}$ element of A.
By putting the values of $Q_0$, $P_0$ and $P_M$ in equation (38), we have

$$R(z) = \frac{T(z)[-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}] + N(z)[M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}] + L(z)[-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}] + A\cdot M(z)}{A\cdot K(z)} \quad (42)$$

**Mean Queue Length:**
Define,
$L_q$= Expected number of customers in the queue including the one in service.

Then $\quad L_q = R'(z)\big|_{z=1}$

Therefore, from equation (42), we have

$$L_q = \frac{\begin{aligned} &K(1)[T'(1)\{-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}\} + N'(1)\{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\} \\ &+ L'(1)\{-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}\} + A\cdot M'(1)] - [T(1)\{-M(b_1)A_{11} + M(b_2)A_{21} \\ &- M(b_3)A_{31}\} + N(1)\{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\} + L(1)\{-M(b_1)A_{13} + M(b_2)A_{23} \\ &- M(b_3)A_{33}\} + A\cdot M(1)]\,K'(1)\end{aligned}}{A\cdot[K(1)]^2} \quad (43)$$

where dashes denotes the first derivative with respect to z.

Now, Relation (35) on applying the theory of Laplace transforms gives

$$r(z) = \frac{(1-z)\mu R_0 - (1-z)\lambda_1 z^{M+1} P_M - \xi z}{\lambda_1 z^2 - z(\lambda_1 + \mu + \xi) + \mu} \quad (44)$$

where

$$r(z) = \lim_{s\to 0} s\, r(z,s)$$

Equation (44) is a polynomial in z and holds for all values of z, including the two zeros of the denominator. Therefore, $R_0$ and $P_M$ can be found by setting the numerator equal to zero. Substituting the two zeros $a_1$ and $a_2$ (say) of the denominator (at each of which the numerator must also vanish).

**If $\xi = 0$ (i.e., Catastrophes are not allowed in the system):**
From equation (35), on applying the theory of Laplace transforms gives

$$r(z) = \frac{\mu R_0 - \lambda_1 z^{M+1} P_M}{\mu - \lambda_1 z} \quad (45)$$

The condition, $\lim_{z\to 1} r(z) = 1$ gives

$$\mu R_0 - \lambda_1 P_M = \mu - \lambda_1 \quad (46)$$

As r(z) is analytic, the numerator and denominator of equation (45) must vanish simultaneously for z= $\mu/\lambda_1$, which is a zero of its denominator. Equating the numerator of equation (45) to zero for z= $\mu/\lambda_1$ we have

$$R_0 = \rho^{-M} P_M, \qquad \rho = \lambda_1/\mu < 1 \quad (47)$$

Relation (46) and (47) gives

$$R_0 = \frac{1-\rho}{1-\rho^{M+1}}, \quad P_M = \frac{(1-\rho)\rho^M}{1-\rho^{M+1}}$$

Now, from equation (45), we have

$$r(z) = \frac{1-\rho}{1-\rho^{M+1}} \cdot \left[\frac{1-(\rho z)^{M+1}}{1-\rho z}\right] \quad (48)$$

which is a well-known result of the M/M/1 queue with finite waiting space M.
When there is an infinite waiting space, the corresponding expression for r(z) is obtained by letting M tends to infinity in equation (48), If (ρ,|z|) <1.

$$r(z) = \frac{1-\rho}{1-\rho z} \quad (49)$$

which is again a well-known result of M/M/1 queue with infinite waiting space.

**Key Aspects of the M/M/1 Queue with Catastrophes:**

Kumar and Arivudainambi [9] explores an M/M/1 queueing model that incorporates the effects of catastrophes, which are sudden events causing the queue to drop to zero i.e., all customers in the system are removed and reset to zero at once. This approach is useful for modeling systems that may experience random, complete resets, such as in certain network or service systems where failures can clear out all ongoing tasks.

**Steady-state probabilities:**

In a queue with catastrophes, the steady-state distribution can still exist if the arrival rate λ and service rate μ are balanced in such a way that the system does not grow unbounded. The steady-state probabilities are altered by the rate of catastrophes with rate ξ.
When a catastrophe occurs at the service facility with rate ξ, the steady-state distribution $\{p_n;\ n \geq 0\}$ of the M/M/1 queue with catastrophes corresponds to

$$p_0 = (1-\rho)\ ;\ n = 0 \quad (50)$$

$$p_n = (1-\rho)\rho^n\ ;\ n = 1, 2, 3, \ldots \quad (51)$$

where

$$\rho = \frac{(\lambda+\mu+\xi) - \sqrt{\lambda^2+\mu^2+\xi^2+2\lambda\xi+2\mu\xi-2\lambda\mu}}{2\mu} \quad (52)$$

Thus equations (50)-(52) provide the steady- state distribution for the queueing system. Obviously, the steady state distribution exists if and only if $\rho < 1$. It is also observed that the results of equations (50)-(52) agree with the model discussed above and with Chao, X [3].

## VI. Application of the model:

Queueing models with catastrophes and environmental change can be applied suitably to many practical situations in biological and agricultural sciences, etc. In agricultural scenario, if a crop has some sort of infection through one type of insects caused due to change in temperature i.e., environment; then for such type of infection some sort of chemical agent or compounds can be applied. The number of bacteria that destroy the crop, in the large part, depends on the effectiveness and quantity of chemical reagents used. It means that the application of these chemical reagents can lead to the elimination of all the insects or a segment of it. The impact these chemical reagents have on bacteria that makes them become zero instantaneously can be considered as the happening of a catastrophe.

## VII. Conclusion:

In this paper, we established a queueing model and obtained the transient solution of a limited capacity queueing system with catastrophes and state-dependent environmental change parameter. Further, we have derived the steady state result and the mean queue length of the model. We have also obtained some particular cases both with and without catastrophes.

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