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Review Paper

Q^*-REGULAR SPACES

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Abstract: *In this paper, we introduce and study a new class of generalized regular space is called Q^*-regular space which is weaker than regularity. The relationships among strongly rg-regular, g-regular, regular, Q^*-regular, almost regular and softly regular spaces are investigated. Some of basic properties and characterizations of Q^*-regular spaces in the terms of other separation and countability axioms such as semi-regular, Hausdorff, separable, second countable and Lindelof spaces are obtained.*

Key words*: regular open, Q^*-open sets; softly regular, almost-regular, Q^*-regular spaces*
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I. Introduction

In 1925, Urysohn [**13**] introduced and studied a new type of separation axiom, called Urysohn space. In 1926, Cartan [**1**] introduced and studied the concept of symmetric space. In topology, an R_0-space is also known as a symmetric space. In 1937, Stone [**12**] introduced the notion of semi-regular spaces and obtained their characterizations. In 1958, Kuratowski [**5**] introduced a generalization of closed sets, called regularly-open and regularly-closed sets in general topology.

In 1963, Levine [**6**] introduced the concept of generalized closed sets and obtained their properties. In 1969, Singal and Arya [**10**] introduced a new class of separation axiom (namely almost regular space) in topological spaces which is weaker than regularity but it is equivalent to semi-regular spaces due to Stone [**12**] and investigated some basic properties with other separation axioms such as T_0, T_1, semi-regular, Hausdorff and k-spaces. In 1986, Munshi [7] introduced and studied some new class of separation axioms (named g-regular and g-normal spaces etc.) in topological spaces which are stronger than regularity and normality. In 1993, Palaniappan [**9**] introduced the concept of generalized closed sets, called regular generalized closed which is a weaker form of closed and g-closed sets and studied their properties. In 2010, Murugalingam and Lalitha [**8**] introduced and studied the concept of Q^*-open sets and obtained some properties of Q^*-open sets in topological spaces. In 2011, Gnanachandra and Thangavelu [**2**] introduced and studied the concepts of strongly rg-regular and strongly rg-normal spaces in topological spaces which are stronger than regularity and normality. In 2018, Kumar and Sharma [**3**] introduced and studied the concept of softly regular spaces in topological spaces which is a weak form of regularity and obtained some characterizations with regular, strongly rg-regular, weakly regular, almost regular, π-normal and quasi normal spaces. Recently, Kumar and Tomar [**4**] introduced and studied the concepts of Q^*-normal spaces in topological spaces which is weaker than normality and obtained their characterizations.

II. Preliminaries

Throughout this paper, spaces $(X, \Im)$, (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by **cl(A)** and **int(A)** respectively. A subset A of a topological space $(X, \Im)$ is said to be **regularly-open** [**5**] if it is the interior of its own closure or, equivalently, if it is the interior of some closed set or equivalently, A = int(cl(A)). A subset A is said to be **regularly-closed** [**5**] if it is the closure of its own interior or, equivalently, if it is the closure of some open set or equivalently, A = cl(int(A)). Clearly, a set is regularly-open iff its complement is regularly-closed. The finite union of regularly open sets is said to be ***π*–open**. The complement of a π-open set is said to be ***π*-closed**. Every regularly open (resp. regularly closed) set is π–open (resp. π-closed).

2.1 Definition. A subset A of a space (X, ℑ) is said to be **Q^*-closed** [**8**] if int(A) = ϕ and A is closed. The complement of a Q^*-closed set is said to be **Q^*-open**.

2.2 Definition. A subset A of a space (X, ℑ) is said to be
(i) **generalized closed** (briefly **g-closed**) [**6]** if cl(A) ⊂ U whenever A ⊂ U and U ∈ ℑ.
(ii) **regular generalized closed** (briefly r**g-closed**) [**9]** if cl(A) ⊂ U whenever A ⊂ U and U is regular-open in X.

The complement of A is g-closed (resp. rg-closed) set is said to be **g-open** (resp**. rg-open**).
The family of all Q^*-closed (resp. Q^*-open) sets of a space X is denoted by Q^*-C(X) (resp. Q^*-O(X)).

2.3 Remark. We have the following implications for the properties of subsets:

regular closed ⇒ π-closed
⇓ ⇓
Q^*-closed ⇒ closed ⇒ g-closed ⇒ πg-closed ⇒ rg-closed

Where none of the implications is reversible as can be seen from the following examples:

2.4 Example. Let X = {a, b, c} and ℑ = {ϕ, X, {a}, {b}, {a, b}}. Then
(i) regular closed sets are : ϕ, X, {a, c}, {b, c}.
(ii) π-closed sets are : ϕ, X, {c}, {a, c}, {b, c}.
(iii) closed sets are : ϕ, X, {c}, {a, c}, {b, c}.
(iv) Q^*-closed sets are : ϕ, {c}.
(v) g-closed sets are : ϕ, X, {c}, {a, c}, {b, c}.
(vi) πg-closed sets are : ϕ, X, {c}, {a, c}, {b, c}.
(vii) rg-closed sets are : ϕ, X, {c}, {a, b}, {a, c}, {b, c}.

2.5 Example. In R with usual metric, finite sets are Q^*-closed but not regular closed. [0, 1] is regular closed but not Q^*-closed. Hence regular closed and Q^*-closed sets are independent of each other.

2.6 Definition. A space X is said to be a **Urysohn space** [**13**] if for every pair of distinct points x and y, there exist open sets U and V such that x ∈ U, y ∈ V and cl(U) ∩ cl(V) = ϕ.

2.7 Definition. A space X is said to **symmetric space** (or **R_0-space**) [**1**] if for any two distinct points x and y of X, x ∈ cl ({y}) implies that y ∈ cl({x}).

2.8 Definition. A topological space X is called **T_{Q^*}-space** if every Q^*-closed set in it is closed set.

2.9 Definition. A space X is said to be **g-normal** [**7**] if for every pair of disjoint g-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A ⊂ U and B ⊂ V.

2.10 Definition. A space X is said to be **strongly rg-normal** [**2**] if for every pair of disjoint rg-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A ⊂ U and B ⊂ V.

2.11- Definition. A space X is said to be **Q^*-normal** [**4**] if for every pair of disjoint Q^*-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A ⊂ U and B ⊂ V.

By the definitions stated above, we have the following diagram:

rg-normality ⇒ g-normality ⇒ normality ⇒ Q^*-normality
Where none of the implications are reversible.

III. Q^*-regular spaces

3.1 Definition. A space $(X, \Im)$ is said to be **Q^*-regular** if for every Q^*-closed set A and a point $x \notin A$, there exist disjoint open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

3.2 Definition. A space $(X, \Im)$ is said to be **softly regular** [**3**] if for every π–closed set A and a point $x \notin A$, there exist disjoint open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

3.3 Definition. A space $(X, \Im)$ is said to be **almost regular** [**10**] if for every regularly closed set A and a point $x \notin A$, there exist open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

3.4 Definition. A space $(X, \Im)$ is said to be **g-regular** [**7**] if for every g-closed set A and a point $x \notin A$, there exist open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

3.5 Definition. A space $(X, \Im)$ is said to be **strongly rg-regular** [**2**] if for every rg-closed set A and a point $x \notin A$, there exist open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

3.6 Definition. A space $(X, \Im)$ is said to be **semi-regular** [**12**] if for each point x of the space and each open set U containing x, there is an open set V such that $x \in V \subset Int(Cl(V)) \subset U$.

3.7 Theorem. Every regular space is Q^*-regular.
Proof. Let X be a regular space. Let F be any Q^*-closed set in X and a point $x \in X$ such that $x \notin F$. Since we know that every Q^*-closed set is closed. So, F is closed and $x \notin F$. Since X is a regular space, there exists a pair of disjoint open sets G and H such that $F \subset G$ and $x \in H$. Hence X is a Q^*-regular space.

3.8 Theorem [7]. Every g-regular space is regular hence Q^*-regular.

3.9 Theorem [2]. Every strongly rg-regular space is regular hence Q^*-regular.

By the definitions and results stated above, we have the following diagram:

Q^*-regular
$\Uparrow$
strongly rg-regular $\Rightarrow$ g-regular $\Rightarrow$ regular $\Rightarrow$ softly regular $\Rightarrow$ almost regular

Where none of the implications is reversible as can be seen from the following examples:
3.10 Example. Let $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{b, c\}, X\}$. Consider the closed set $\{b, c\}$ and a point 'a' such that $a \notin \{b, c\}$. Then $\{b, c\}$ and $\{a\}$ are disjoint open sets such that $\{b, c\} \subset \{b, c\}$, $a \in \{a\}$ and $\{b, c\} \cap \{a\} = \phi$. Similarly, for the closed set $\{a\}$ and a point 'c' such that $c \notin \{a\}$. Then there exist open sets $\{a\}$ and $\{b, c\}$ such that $\{a\} \subset \{a\}$, $c \in \{b, c\}$ and $\{a\} \cap \{b, c\} = \phi$. It follows that $(X, \Im)$ is regular as well as softly regular space.

3.11 Example. Let $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. If we take a point 'a' and an open set $V = \{a\}$, then $cl(V) = \{a, c\}$ and a regularly-open set $U = X$. So by the definition of weakly regular space $x \in V \subset cl(V) \subset U$, where V be an open set and U be a regularly-open set such that $a \in \{a\} \subset \{a, c\} \subset X$. Hence $(X, \Im)$ is weakly regular. If we take a point 'a' and a regularly-closed set $A = \{b, c\}$ does not containing the point 'a', there donot exist disjoint open sets containing the point 'a' and a regularly-closed set $A = \{b, c\}$. Hence $(X, \Im)$ is not almost-regular.
3.12 Example. Let $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $(X, \Im)$ is weakly regular but not partly-regular. If we take a point 'a' and an open set $V = \{a\}$, then $cl(V) = \{a, c\}$. Let $U = \{a, b\}$ be any π–open set. So by the definition of partly-regular space $x \in V \subset cl(V) \subset U$, where V be an open set and U be a π-open set such that $a \in \{a\} \subset \{a, c\} \not\subset \{a, b\}$. Hence $(X, \Im)$ is not partly-regular.

3.13 Example. Let $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $(X, \Im)$ is weakly regular but not softly-regular. Let $A = \{c\}$ be any π–closed set doesnot containing a point 'a' i.e. $a \notin \{c\}$, there do not exist disjoint open sets containing the point 'a' and the π–closed set $A = \{c\}$. Hence $(X, \Im)$ is not softly-regular.

3.14 Example. Let X = {a, b, c, d} and $\Im$ = {ϕ, {a}, {a, b}, {a, b, c}, {a, b, d}, X}. Then the space (X, $\Im$) is almost regular but not strongly rg-regular. If we take a point 'a' and F = {b} be any rg-closed set. Then there do not exist disjoint open sets containing the point 'a' and rg-closed set F = {b}. Hence (X, $\Im$) is not strongly rg-regular.

IV. Characterizations of Q^*-regular spaces

4.1 Theorem. For a topological space (X, $\Im$), the following properties are equivalent:
(a) (X, $\Im$) is Q^*-regular.
(b) For every $x \in X$ and every Q^*-open set U containing x, there exists an open set V such that $x \in V \subset cl(V) \subset U$.
(c) For every Q^*-closed set A, the intersection of all the closed neighbourhood of A is A.
(d) For every set A and a Q^*-open set B such that $A \cap B \neq \phi$, there exists an open set F such that $A \cap F \neq \phi$ and $cl(F) \subset B$.
(e) For every nonempty set A and Q^*-closed set B such that $A \cap B = \phi$, there exist disjoint open sets L and M such that $A \cap L \neq \phi$ and $B \subset M$.

Proof.
(a) $\Rightarrow$ (b). Suppose (X, $\Im$) is Q^*-regular. Let $x \in X$ and U be a Q^*-open set containing x so that X – U is Q^*-closed. Since (X, $\Im$) is Q^*-regular, there exist open sets V_1 and V_2 such that $V_1 \cap V_2 = \phi$ and $x \in V_1$, $X - U \subset V_2$. Take $V = V_1$. Since $V_1 \cap V_2 = \phi$, $V \subset X - V_2 \subset U$ that implies $cl(V) \subset cl(X - V_2) = X - V_2 \subset U$. Therefore $x \in V \subset cl(V) \subset U$.

(b) $\Rightarrow$ (c). Let A be Q^*-closed set and $x \notin A$. Since A is Q^*-closed, X – A is Q^*-open and $x \in X - A$. Therefore by (b) there exists an open set V such that $x \in V \subset cl(V) \subset X - A$. Thus $A \subset X - cl(V) \subset X - V$ and $x \notin X - V$. Consequently X – V is a closed neighborhood of A

(c) $\Rightarrow$ (d). Let $A \cap B \neq \phi$ and B be Q^*-open. Let $x \in A \cap B$. Since B is Q^*-open, X – B is Q^*-closed and $x \notin X - B$. By our assumption, there exists a closed neighborhood V of X – B such that $x \notin V$. Let $X - B \subset U \subset V$, where U is open. Then F = X – V is open such that $x \in F$ and $A \cap F \neq \emptyset$. Also X – U is closed and $cl(F) = cl(X - V) \subset X - U \subset B$. This shows that $cl(F) \subset B$.

(d) $\Rightarrow$ (e). Suppose $A \cap B = \phi$, where A is non-empty and B is Q^*-closed. Then X – B is Q^*-open and $A \cap (X - B) \neq \phi$. By (d), there exists an open set L such that $A \cap L \neq \phi$, and $L \subset cl(L) \subset X - B$. Put $M = X - cl(L)$. Then $B \subset M$ and L, M are open sets such that $M = X - cl(L) \subset (X - L)$.

(e) $\Rightarrow$ (a). Let B be Q^*-closed and $x \notin B$. Then $B \cap \{x\} = \phi$. By (e), there exist disjoint open sets L and M such that $L \cap \{x\} \neq \phi$ and $B \subset M$. Since $L \cap \{x\} \neq \phi$, $x \in L$. This proves that (X, $\Im$) is Q^*-regular.

4.2 Theorem. A topological space (X, $\Im$) is Q^*-regular if and only if for each Q^*-closed set F of (X, $\Im$) and each $x \in X - F$, there exist open sets U and V of (X, $\Im$) such that $x \in U$ and $F \subset V$ and $cl(U) \cap cl(V) = \phi$.
Proof: Let F be a Q^*-closed set in (X, $\Im$) and $x \notin F$. Then there exist open sets U_x and V such that $x \in U_x$, $F \subset V$ and $U_x \cap V = \phi$. This Implies that $U_x \cap cl(V) = \phi$. Since cl(V) is closed and $x \notin cl(V)$. Since (X, $\Im$) is Q^*-regular, there exist open sets G and H of (X, $\Im$) such that $x \in G$, $cl(V) \subset H$ and $G \cap H = \phi$. This implies $cl(G) \cap H = \phi$. Take $U = U_x \cap G$. Then U and V are open sets of (X, $\Im$) such that $x \in U$ and $F \subset V$ and $cl(U) \cap cl(V) = \phi$, since $cl(U) \cap cl(V) \subset cl(G) \cap H =$.

Conversely, suppose for each Q^*-closed set F of (X, $\Im$) and each $x \in X - F$, there exist open sets U and V of (X, $\Im$) such that $x \in U$, $F \subset V$ and and $cl(U) \cap cl(V) = \phi$. Now $U \cap V \subset cl(U) \cap cl(V) = \phi$. Therefore $U \cap V = \phi$. Thus (X, $\Im$) is Q^*-regular.

4.3 Theorem. In a Q^*-regular space X, every pair consisting of a compact set A and a disjoint Q^*-closed set B can be separated by open sets.
Proof. Let X be Q^*-regular space and let A be a compact set, B be a Q^*-closed set with $A \cap B = \phi$. Since X is Q^*-regular space, for each $x \in A$, there exist disjoint open sets U_x and V_x such that $x \in U_x$, $B \subset V_x$. Clearly $\{U_x : x \in A\}$ is an open covering of the compact set A. Since A is compact, there exists a finite subfamily $\{U_{xi} : i = 1, 2, 3, \ldots\ldots, n\}$ which covers A. It follows that $A \subset \cup \{U_{xi} : i = 1, 2, 3, \ldots\ldots, n\}$ and $B \subset \cap \{V_{xi} : i = 1, 2, 3,$

……. , n}. Put $U = \cup \{U_{xi} : i = 1, 2, 3, …., n\}$ and $V = \cap \{V_{xi} : i = 1, 2, 3, ……. , n\}$, then $U \cap V = \phi$. For, if $x \in U \cap V \Rightarrow x \in U_{xj}$ for some j and $x \in V_{xi}$ for every i. This implies that $x \in U_{xj} \cap V_{xi}$, which is a contradiction to $U_{xj} \cap V_{xi} = \phi$. Thus U and V are disjoint open sets containing A and B respectively.

V. Relations of Q^*-regular spaces with Some Separation Axioms

5.1 Theorem [10]. Every almost regular, semi-regular space is regular.

5.2 Corollary. Every almost regular, semi-regular space is Q^*-regular.
Proof. Using the fact that every regular space is Q^*-regular.

5.3 Corollary. Every softly regular, semi-regular space is Q^*-regular.
Proof. Using the fact that every softly regular space is almost regular.

5.4 Theorem [11]. Every normal, symmetric space is regular.

5.5 Corollary. Every normal, symmetric space is Q^*-regular.
Proof. Using the fact that every regular space is Q^*-regular.

5.6 Corollary. Every g-normal, symmetric space is Q^*-regular.
Proof. Using the fact that every g-normal space is normal.

5.7 Corollary. Every rg-normal, symmetric space is Q^*-regular.
Proof. Using the fact that every rg-normal space is normal.

5.8 Theorem [11]. Every compact Hausdorff space is regular.

5.9 Corollary. Every compact Hausdorff space is Q^*-regular.
Proof. Using the fact that every regular space is Q^*-regular.

5.10 Corollary. Every compact Urysohn space is Q^*-regular.
Proof. Using the fact that every Urysohn space is Hausdorff.

VI. Conclusion

In this paper, we introduce and study a new class of generalized regular space is called Q^*-regular space which is weaker than regularity. The relationships among strongly rg-regular, g-regular, regular, Q^*-regular, almost regular and softly regular spaces are investigated. Some of basic properties and characterizations of Q^*-regular spaces in the terms of other separation and countability axioms such as semi-regular, Hausdorff, separable, second countable and Lindelof spaces are obtained. This idea can be extended to topological ordered, bitopological, bitopological ordered and fuzzy topological spaces etc.

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