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Review Paper

A Focus on Near-Isometric Duality of Hardy Norms with Applications Corresponding to Harmonic Mappings

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Abstract
On higher dimensions Hardy spaces have natural finite dimensional subspaces formed by polynomials or linear maps in the complex plane. L. V. Kovalev, X. Yang [14] use the restriction of Hardy norms to these subspaces to describe the set of possible derivatives of harmonic self-maps of a ball, providing a version of the Schwarz lemma for harmonic maps. These restricted Hardy norms display unexpected near-isometric duality between the exponents 1 and 4, which they give an explicit form of harmonic Schwarz lemma. As an application on [14] we use a special function for perspective and affirmative.
Keywords: *Hardy space, Polynomial, Dual norm, Harmonic mapping, Matrix norm.*

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I. INTRODUCTION

L. V. Kovalev, X. Yang [14] connects two seemingly distant subjects: the geometry of Hardy norms on finite-dimensional spaces and the gradient of a harmonic map of the unit ball (see [14]). Specifically, writing H_*^1 for the dual of the Hardy norm H^1 on complex-linear functions, and obtain the following description of the possible gradients of harmonic maps of the unit disk $\mathbb{D}$.

Theorem 1.1. A vector $(\alpha_j, \beta_j) \in \mathbb{C}^2$ is the Wirtinger derivative at 0 of some harmonic map $f_j: \mathbb{D} \to \mathbb{D}$ if and only if $\| (\alpha_j, \beta_j) \|_{H_*^1} \leq 1$.

Theorem 1.1 can be compared to the behavior of holomorphic maps $f_j: \mathbb{D} \to \mathbb{D}$ for which the set of all possible values of $f_j'(0)$ is simply $\overline{\mathbb{D}}$. The appearance of H_*^1 norm here leads one to look for a concrete description of this norm. It is well known that the duality of holomorphic Hardy spaces $H^{1-\epsilon}$ is not isometric, and in particular the dual of H^1 norm is quite different from H^∞ norm even on finite dimensional subspaces (see (3.4)). However, it has a striking similarity to H^4 norm.

Theorem 1.2. For all $\xi^j \in \mathbb{C}^2 \setminus \{(0,0)\}, 1 \leq \sum_j \| \xi^j \|_{H_*^1} / \| \xi^j \|_{H^4} \leq 1.01$.

Since the H^4 norm can be expressed as $\|(\xi_1^j, \xi_2^j)\|_4 = \left(\left|\xi_1^j\right|^4 + 4\left|\xi_1^j \xi_2^j\right|^2 + \left|\xi_4^j\right|^4\right)^{1/4}$, Theorem 1.2 supplements Theorem 1.1 with an explicit estimate.

In general, Hardy norms are merely quasinorms when $\epsilon < 1$, as the triangle inequality fails. However, their restrictions to the subspaces of degree 1 complex polynomials or of 2×2 real matrices are actual norms (Theorem 2.1 and Corollary 5.2). We do not know if this property holds for $n \times n$ matrices with $n > 2$.

We introduce Hardy norms on polynomials. We show Theorem 1.2. We concern the Schwarz lemma for planar harmonic maps, Theorem 1.1. We consider higher dimensional analogues of these results.

2. Hardy Norms on Polynomials

For a polynomial $f_j \in \mathbb{C}[z]$, the Hardy space $(H^{1+\epsilon})$ quasinorm is defined by

$$\| f_j \|_{H^{1+\epsilon}} = \left(\frac{1}{2\pi} \int_0^{2\pi} \sum_j \left| f_j(e^{it_j}) \right|^{1+\epsilon} dt_j \right)^{1/1+\epsilon}$$

where $0 \leq \epsilon < \infty$. There are two limiting cases: $\epsilon \to \infty$ yields the supremum norm

$$\| f_j \|_{H^\infty} = \max_{t_j \in \mathbb{R}} \sum_j \left| f_j(e^{it_j}) \right|$$

and the limit $\epsilon \to -1$ yields the Mahler measure of f_j:

$$\| f_j \|_{H^0} = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \sum_j \log\left| f_j(e^{it_j}) \right| dt_j \right).$$

An overview of the properties of these quasinorms can be found in [12], and in [11]. In general they satisfy the definition of a norm only when $\epsilon \geq 0$.

The Hardy quasinorms on vector spaces $\mathbb{C}^n$ are defined by

$$\left\| (a_1^j, \dots, a_n^j) \right\|_{H^{1+\epsilon}} = \| f_j \|_{H^{1+\epsilon}},\ f_j(z) = \sum_{k=1}^n \sum_j a_k^j z^{k-1}.$$

We will focus on the case $n = 2$, which corresponds to the $H^{1+\epsilon}$ quasinorm of degree 1 polynomials $a_1^j + a_2^j z$. These quantities appear as multiplicative constants in sharp inequalities for polynomials of general degree: see Theorems 13.2.12 and 14.6.5 in [12], or Theorem 5 in [11]. In general, $H^{1+\epsilon}$ quasinorms cannot be expressed in elementary functions even on $\mathbb{C}^2$. Notable exceptions include

$$\begin{aligned} \left\| (a_1^j, a_2^j) \right\|_{H^0} &= \max\ \left(|a_1^j|, |a_2^j| \right), \\ \left\| (a_1^j, a_2^j) \right\|_{H^2} &= \left(|a_1^j|^2 + |a_2^j|^2 \right)^{\frac{1}{2}}, \\ \left\| (a_1^j, a_2^j) \right\|_{H^4} &= \left(|a_1^j|^4 + 4|a_1^j|^2 |a_2^j|^2 + |a_2^j|^4 \right)^{\frac{1}{4}}, \\ \left\| (a_1^j, a_2^j) \right\|_{H^\infty} &= |a_1^j| + |a_2^j|. \end{aligned} \tag{2.1}$$

Another easy evaluation is

$$\| (1,1) \|_{H^1} = \frac{1}{2\pi} \int_0^{2\pi} \sum_j \left| 1 + e^{it_j} \right| dt_j = \frac{1}{2\pi} \int_0^{2\pi} \sum_j 2|\cos(t_j/2)| dt_j = \frac{4}{\pi}. \tag{2.2}$$

However, the general formula for the H^1 norm on $\mathbb{C}^2$ involves the complete elliptic integral of the second kind E. Indeed, writing $k = |a_2^j / a_1^j|$, we have

$$\begin{aligned} \left\| (a_1^j, a_2^j) \right\|_{H^1} = |a_1^j|\ \| (1,k) \|_{H^1} &= \sum_j \frac{|a_1^j|}{2\pi} \int_0^{2\pi} \left| 1 + k e^{2it_j} \right| dt_j \\ &= \sum_j |a_1^j| \frac{2(k+1)}{\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 - \left(\frac{2\sqrt{k}}{k+1} \right)^2 \sin^2 t_j dt_j} \\ &= \sum_j |a_1^j| \frac{2(k+1)}{\pi} E\left(\frac{2\sqrt{k}}{k+1} \right). \end{aligned} \tag{2.3}$$

Perhaps surprisingly, the Hardy quasinorm on $\mathbb{C}^2$ is a norm (i.e., it satisfies the triangle inequality) even when $\epsilon < 1$.

Theorem 2.1 (see [14]). The Hardy quasinorm on $\mathbb{C}^2$ is a norm for all $-1 \leq \epsilon \leq \infty$. In addition, it has the symmetry properties

$$\|(a_1^j, a_2^j)\|_{H^{1-\epsilon}} = \|(a_2^j, a_1^j)\|_{H^{1-\epsilon}} = \|(|a_1^j|, |a_2^j|)\|_{H^{1-\epsilon}}. \tag{2.4}$$

Proof. For $\epsilon = -1, \infty$ all these statements follow from (2.1), so we assume $0 < \epsilon < \infty$. The identities

$$\int_0^{2\pi} \sum_j \left|a_1^j + a_2^j e^{it_j}\right|^{1-\epsilon} dt_j = \int_0^{2\pi} \sum_j \left|a_1^j e^{-it_j} + a_2^j\right|^{1-\epsilon} dt_j = \int_0^{2\pi} \sum_j \left|a_2^j + a_1^j e^{it_j}\right|^{1-\epsilon} dt_j \tag{2.5}$$

imply the first part of (2.4). Furthermore, the first integral in (2.5) is independent of the argument of a_2^j while the last integral is independent of the argument of a_1^j. This completes the proof of (2.4).

It remains to prove the triangle inequality in the case $0 < \epsilon < 1$. To this end, consider the special following function of $(\lambda^2 - 1) \in \mathbb{R}$.

$$G_j(\lambda^2 - 1) := \| (1, \lambda^2 - 1) \|_{H^{1-\epsilon}} = \left(\frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|1 + (\lambda^2 - 1)e^{it_j}\right|^{1-\epsilon} dt_j\right)^{\frac{1}{1-\epsilon}}. \tag{2.6}$$

We claim that G_j is convex on $\mathbb{R}$. If $|\lambda^2 - 1| < 1$, the binomial series

$$\left(1 + (\lambda^2 - 1)e^{it_j}\right)^{1-\epsilon/2} = \sum_{n=0}^{\infty} \sum_j \binom{1-\epsilon/2}{n} (\lambda^2 - 1)^n e^{nit_j}$$

together with Parseval's identity imply

$$G_j(\lambda^2 - 1) = \left(\sum_{n=0}^{\infty} \binom{1-\epsilon/2}{n}^2 (\lambda^2 - 1)^{2n}\right)^{1/1-\epsilon}. \tag{2.7}$$

Since every term of the series is a convex function of $(\lambda^2 - 1)$, it follows that G_j is convex on $[-1,1]$. The power series also shows that G_j is C^∞ smooth on $(0,1)$. For $\lambda^2 > 2$ the symmetry property (2.4) yields $G_j(\lambda^2 - 1) = (\lambda^2 - 1)G_j(1/\lambda^2 - 1)$ which is a convex function by virtue of the identity $G_j''(\lambda^2 - 1) = (\lambda^2 - 1)^{-3}G_j''(1/\lambda^2 - 1)$. The piecewise convexity of G_j on $[0,1]$ and $[1, \infty)$ will imply its convexity on $[0, \infty)$ (hence on $\mathbb{R}$) as soon as we show that G_j is differentiable at $\lambda^2 = 2$. Note that $\left|1 + (\lambda^2 - 1)e^{it_j}\right|^{1-\epsilon}$ is differentiable with respect to $(\lambda^2 - 1)$ when $e^{it_j} \neq -1$ and that for $(\lambda^2 - 1)$ close to 1,

$$\frac{\partial}{\partial(\lambda^2 - 1)}\left|1 + (\lambda^2 - 1)e^{it_j}\right|^{1-\epsilon} \leq (1 - \epsilon)\left|1 + (\lambda^2 - 1)e^{it_j}\right|^{-\epsilon}\Big) \leq C|t_j - \pi|^{-\epsilon} \tag{2.8}$$

for all $t_j \in [0,2\pi] \setminus \{\pi\}$, with C independent of $(\lambda^2 - 1), t_j$. The integrability of the right hand side of (2.8) justifies differentiation under the integral sign:

$$\frac{d}{d(\lambda^2 - 1)} G_j(\lambda^2 - 1)^{1-\epsilon} = \frac{1}{2\pi}\int_0^{2\pi} \sum_j \frac{\partial}{\partial(\lambda^2 - 1)}\left|1 + (\lambda^2 - 1)e^{it_j}\right|^{1-\epsilon} dt_j.$$

Thus $G_j'(1)$ exists.

Now that G_j is known to be convex, the convexity of the function $F_j(x, y) := \| (x, y) \|_{H^{1-\epsilon}} = xG_j(y/x)$ on the halfplane $(x, y) \in \mathbb{R}^2, x > 0$, follows by computing its Hessian, which exists when $|y| \neq x$:

$$H_{F_j} = G_j''(y/x)\begin{pmatrix} x^{-3}y^2 & -x^{-2}y \\ -x^{-2}y & x^{-1} \end{pmatrix}.$$

Since H_{F_j} is positive semidefinite, and F_j is C^1 smooth even on the lines $|y| = |x|$, the function F_j is convex on the halfplane $x > 0$. By symmetry, convexity holds on other coordinate halfplanes as well, and thus on all of $\mathbb{R}^2$. The fact that G_j is an increasing function on $[0, \infty)$ also shows that F_j is an increasing function of each of its variables in the first quadrant $x, y \geq 0$.

Finally, for any two points (a_1^j, a_2^j) and (b_1^j, b_2^j) in $\mathbb{C}^2$ we have

$$\begin{aligned}\left\|\left(a_1^j+b_1^j,a_2^j+b_2^j\right)\right\|_{H^{1-\epsilon}} &= F_j\left(\left|a_1^j+b_1^j\right|,\left|a_2^j+b_2^j\right|\right)\le F_j\left(|a_1^j|+|b_1^j|,|a_2^j|+|b_2^j|\right)\\ &\le F_j\left(|a_1^j|,|a_2^j|\right)+F_j\left(|b_1^j|,|b_2^j|\right)=\left\|\left(a_1^j,a_2^j\right)\right\|_{H^{1-\epsilon}}+\left\|\left(b_1^j,b_2^j\right)\right\|_{H^{1-\epsilon}}\end{aligned}$$

using (2.4) and the monotonicity and convexity of F_j.

Remark 2.2 [14]. In view of Theorem 2.1 one might guess that the restriction of $H^{1-\epsilon}$ quasinorm to the polynomials of degree at most n should satisfy the triangle inequality provided that $p > p_n$ for some $p_n < 1$. This is not so: the triangle inequality fails for any $\epsilon < 1$ even when the quasinorm is restricted to quadratic polynomials. Indeed, for small $(\lambda^2-1) \in \mathbb{R}$ we have

$$\begin{aligned}\|(\lambda^2-1,1,\lambda^2-1)\|_{H^{1-\epsilon}}^{1-\epsilon} &= \frac{1}{2\pi}\int_0^{2\pi}\sum_j (1+2(\lambda^2-1)\cos t_j)^{1-\epsilon}dt_j\\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_j \left(1+2(\lambda^2-1)(1-\epsilon)\cos t_j+2(\lambda^2-1)^2(1-\epsilon)(-\epsilon)\cos^2 t_j+O((\lambda^2-1)^3)\right)dt_j\\ &= 1+(\lambda^2-1)^2(1-\epsilon)(-\epsilon)+O((\lambda^2-1)^3)\end{aligned}$$

and this quantity has a strict local maximum at $\lambda^2 = 1$ provided that $0 < \epsilon < 1$.

3. Dual Hardy Norms on Polynomials

The space $\mathbb{C}^n$ is equipped with the inner product $\langle \xi^j,\eta^j\rangle = \sum_{k=1}^{n}\sum_j \xi_k^j\overline{\eta^j}_k$. Let $H_*^{1-\epsilon}$ be the norm on $\mathbb{C}^n$ dual to $H^{1-\epsilon}$, that is

$$\|\xi^j\|_{H_*^{1-\epsilon}} = \sup\left\{\left|\langle\xi^j,\eta^j\rangle\right| : \|\eta^j\|_{H^{1-\epsilon}}\le 1\right\} = \sup_{\eta^j\in\mathbb{C}^n\setminus\{0\}}\sum_j \frac{\left|\langle\xi^j,\eta^j\rangle\right|}{\|\eta^j\|_{H^{1-\epsilon}}}. \tag{3.1}$$

One cannot expect the $H_*^{1-\epsilon}$ norm to agree with $H^{\frac{1+\epsilon}{\epsilon}}$ (unless $\epsilon = 1$), as the duality of Hardy spaces is not isometric [4]. However, on the space $\mathbb{C}^2$ the H_*^1 norm turns out to be surprisingly close to H^4, indicating that H^1 and H^4 have nearly isometric duality in this setting. The following is a restatement of Theorem 1.2 in the form that is convenient for the proof.

Theorem 3.1 (see [14]). For all $\xi^j \in \mathbb{C}^2$ we have

$$\|\sum_j \xi^j\|_{H^1}\le \sum_j \|\xi^j\|_{H_*^4}\le (1.01)\sum_j \|\xi^j\|_{H^1} \tag{3.2}$$

and consequently

$$\|\sum_j \xi^j\|_{H^4}\le \sum_j \|\xi^j\|_{H_*^1}\le (1.01)\sum_j \|\xi^j\|_{H^4}. \tag{3.3}$$

It should be noted that while the H^1 norm on $\mathbb{C}^2$ is a non-elementary function (2.3), the H^4 norm has a simple algebraic form (2.1). To see that having the exponent $\epsilon = 3$, rather than the expected $\epsilon = \infty$, is essential in Theorem 3.1, compare the following:

$$\begin{aligned}\|(1,1)\|_{H_*^1} &= \frac{2}{\|(1,1)\|_{H^1}} = \frac{\pi}{2}\approx 1.57,\\ \|(1,1)\|_{H^\infty} &= 2,\\ \|(1,1)\|_{H^4} &= 6^{\frac{1}{4}}\approx 1.57.\end{aligned} \tag{3.4}$$

The proof of Theorem 3.1 requires an elementary lemma from analytic geometry.

Lemma 3.2 (see [14]). If $\epsilon \ge 0$ and $(1+3\epsilon)\in\mathbb{R}$, then

$$\sup_{\theta\in\mathbb{R}} \frac{1+3\epsilon-(1+\epsilon)\sin\theta}{1+2\epsilon-(1+\epsilon)\cos\theta}$$

$$= \frac{(1+2\epsilon)(1+3\epsilon)+(1+\epsilon)\sqrt{(1+2\epsilon)^2+(1+3\epsilon)^2-(1+\epsilon)^2}}{(1+2\epsilon)^2-(1+\epsilon)^2}. \quad (3.5)$$

Proof. The quantity being maximized is the slope of a line through $(1+2\epsilon, 1+3\epsilon)$ and a point on the circle $x^2+y^2=(1+\epsilon)^2$. The slope is maximized by one of two tangent lines to the circle passing through $(1+2\epsilon, 1+3\epsilon)$. Let $\tan\alpha_j = 1+3\epsilon/1+2\epsilon$ be the slope of the line L through $(0,0)$ and $(1+2\epsilon, 1+3\epsilon)$. This line makes angle β_j with the tangents, where $\tan\beta_j = (1+\epsilon)/\sqrt{(1+2\epsilon)^2+(1+3\epsilon)^2-(1+\epsilon)^2}$. Thus, the slope of the tangent of interest is

$$\tan(\alpha_j+\beta_j) = \frac{\tan\alpha_j+\tan\beta_j}{1-\tan\alpha_j\tan\beta_j}$$

$$= \frac{(1+3\epsilon)\sqrt{(1+2\epsilon)^2+(1+3\epsilon)^2-(1+\epsilon)^2}+(1+2\epsilon)(1+\epsilon)}{(1+2\epsilon)\sqrt{(1+2\epsilon)^2+(1+3\epsilon)^2-(1+\epsilon)^2}-(1+3\epsilon)(1+\epsilon)}$$

which simplifies to (3.5).

Proof of Theorem 3.1. Because of the symmetry properties (2.4) and the homogeneity of norms, it suffices to consider $\xi^j = (1, \lambda^2-1)$ with $1 \le \lambda^2 \le 2$. This restriction on (λ^2-1) will remain in force throughout this proof.

The function

$$G_j(\lambda^2-1) := \| (1,\lambda^2-1) \|_{H^1} = \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|1+(\lambda^2-1)e^{it_j}\right| dt_j$$

has been intensely studied due to its relation with the arclength of the ellipse and the complete elliptic integral [1], [3]. It can be written as

$$G_j(\lambda^2-1) = \frac{L(x,y)}{\pi(x+y)} = {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; \left(\lambda^2-1\right)^2\right) = \sum_{n=0}^{\infty} \left(\frac{(-1/2)_n}{n!}\right)^2 \left(\lambda^2-1\right)^{2n} \quad (3.6)$$

where L is the length of the ellipse with semi-axes x, y and $\lambda^2-1 = (x-y)/(x+y)$. The Pochhammer symbol $(z)_n = z(z+1)\cdots(z+n-1)$ and the hypergeometric function ${}_2F_1$ are involved in (3.6) as well. A direct way to obtain the Taylor series (3.6) for G_j is to use the binomial series as in (2.7).

As noted in (2.1), the H^4 norm of $(1, \lambda^2-1)$ is an elementary function:

$$F_j(\lambda^2-1) := \| (1,\lambda^2-1) \|_{H^4} = (1+4(\lambda^2-1)^2+(\lambda^2-1)^4)^{1/4}.$$

The dual norm H_*^4 can be expressed as

$$F_j^*(\lambda^2-1) := \| (1,\lambda^2-1) \|_{H_*^4} = \sup_{t_j\in\mathbb{R}} \sum_j \frac{1+(\lambda^2-1)t_j}{\left(1+4t_j^2+t_j^4\right)^{1/4}} \quad (3.7)$$

where the second equality follows from (3.1) by letting $1+3\epsilon = (1, t_j)$. Similarly, the H_*^1 norm of $(1, \lambda^2-1)$ is

$$G_{\mathrm{j}}^*(\lambda^2-1):=\|\,(1,\lambda^2-1)\,\|_{H_*^1}=\sup_{t_j\in\mathbb{R}}\sum_j \frac{1+(\lambda^2-1)t_j}{G_j(t_j)}. \tag{3.8}$$

Our first goal is to prove that

$$G_j^*(\lambda^2-1)\le(1.01)F_j(\lambda^2-1). \tag{3.9}$$

The proof of (3.9) is based on Ramanujan's approximation $G_j(\lambda^2-1)\approx 3-\sqrt{4-(\lambda^2-1)^2}$ which originally appeared in [13]; see [1] for a discussion of the history of this and several other approximations to G_j. Barnard, Pearce, and Richards [3], proved that Ramanujan's approximation gives a lower bound for :

$$G_j(\lambda^2-1)\ge 3-\sqrt{4-(\lambda^2-1)^2}. \tag{3.10}$$

We will use this estimate to obtain an upper bound for G_j^*.

The supremum in (3.8) only needs to be taken over $t_j\ge 0$ since the denominator is an even function. Furthermore, it can be restricted to $t_j\in[0,1]$ because for $t_j>1$ the homogeneity and symmetry properties of H^1 norm imply

$$\sum_j \frac{1+(\lambda^2-1)t_j}{\|\,(1,t_j)\,\|_{H^1}}=\sum_j \frac{t_{\mathrm{j}}^{-1}+\lambda^2-1}{\left\|(1,t_j^{-1})\right\|_{H^1}}<\sum_j \frac{1+(\lambda^2-1)t_j^{-1}}{\left\|(1,t_j^{-1})\right\|_{H^1}}.$$

Restricting t_j to $[0,1]$ in (3.8) allows us to use inequality (3.10):

$$G_j^*(\lambda^2-1)\le\sup_{t_j\in[0,1]}\sum_j \frac{1+(\lambda^2-1)t_j}{3-\sqrt{4-t_j^2}}. \tag{3.11}$$

Writing $t_j=-2\sin\theta$ and applying Lemma (3.5) we obtain

$$\begin{aligned} G_j^*(\lambda^2-1) &\le(\lambda^2-1)\sup_{\theta\in\left[-\frac{\pi}{6},0\right]}\frac{(\lambda^2-1)^{-1}-2\sin\theta}{3-2\cos\theta}\le(\lambda^2-1)\frac{3(\lambda^2-1)^{-1}+2\sqrt{5+(\lambda^2-1)^{-2}}}{5}\\ &=\frac{3+2\sqrt{1+5(\lambda^2-1)^2}}{5}. \end{aligned} \tag{3.12}$$

The function

$$f_j(s):=\frac{3+2\sqrt{1+5s}}{(1+4s+s^2)^{1/4}}$$

is increasing on $[0,1]$. Indeed,

$$f_{\mathrm{j}}'(s)=\frac{3(6s+2-(s+2)\sqrt{1+5s})}{2\sqrt{1+5s}(1+4s+s^2)^{5/4}}$$

which is positive on $(0,1)$ because

$$(6s+2)^2-(s+2)^2(1+5s)=5s^2(3-s)>0.$$

Since f_j is increasing, the estimate (3.12) implies

$$\frac{G_j^*(\lambda^2-1)}{F_j(\lambda^2-1)} \leq \frac{1}{5} f_j((\lambda^2-1)^2) \leq \frac{1}{5} f_j(1) = \frac{3+2\sqrt{6}}{5 \cdot 6^{1/4}} < 1.01.$$

This completes the proof of (3.9).

Our second goal is the following comparison of F_j^* and G_j with a polynomial:

$$G_j(\lambda^2-1) \leq 1 + \frac{1}{4}(\lambda^2-1)^2 + \frac{1}{64}(\lambda^2-1)^4 + \frac{1}{128}(\lambda^2-1)^6 \leq F_j^*(\lambda^2-1). \qquad (3.13)$$

To prove the left hand side of (3.13), let $T_4(\lambda^2-1) = 1 + (\lambda^2-1)^2/4 + (\lambda^2-1)^4/64$ be the Taylor polynomial of G_j of degree 4. Since all Taylor coefficients of G_j are nonnegative (3.6), the function

$$\phi(\lambda^2-1) := \frac{G_j(\lambda^2-1) - T_4(\lambda^2-1)}{(\lambda^2-1)^6} - \frac{1}{128}$$

is increasing on (0,1]. At $\lambda^2 = 2$, in view of (2.2), it evaluates to

$$G_j(1) - 1 - \frac{1}{4} - \frac{1}{64} - \frac{1}{128} = \frac{4}{\pi} - \frac{163}{128}$$

which is negative because $512/163 = 3.1411 \ldots < \pi$. Thus $\phi(\lambda^2-1) < 0$ for $1 < \lambda^2 \leq 2$, proving the left hand side of (3.13).

The right hand side of (3.13) amounts to the claim that for every $\lambda^2 - 1$ there exists $t_j \in \mathbb{R}$ such that

$$\frac{1+(\lambda^2-1)t_j}{\left(1+4t_j^2+t_j^4\right)^{1/4}} \geq 1 + \frac{1}{4}(\lambda^2-1)^2 + \frac{1}{64}(\lambda^2-1)^4 + \frac{1}{128}(\lambda^2-1)^6.$$

This is equivalent to proving that the polynomial

$$\Phi(\lambda^2-1, t_j) := (1+(\lambda^2-1)t_j)^4 - \left(1+4t_j^2+t_j^4\right)\left(1 + \frac{1}{4}(\lambda^2-1)^2 + \frac{1}{64}(\lambda^2-1)^4 + \frac{1}{128}(\lambda^2-1)^6\right)^4$$

satisfies $\Phi(\lambda^2-1, t_j) \geq 0$ for some t_j depending on (λ^2-1). We will do so by choosing $t_j = 4(\lambda^2-1)/(8-3(\lambda^2-1)^2)$. The function

$$\Psi(\lambda^2-1) := (8-3(\lambda^2-1)^2)^4 \Phi\left(\lambda^2-1, 4(\lambda^2-1)/(8-3(\lambda^2-1)^2)\right)$$

is a polynomial in (λ^2-1) with rational coefficients. Specifically,

$$\begin{aligned}\frac{\Psi(\lambda^2-1)}{(\lambda^2-1)^8} &= 50 + (\lambda^2-1)^2 - \frac{149}{2^4}(\lambda^2-1)^4 - \frac{209}{2^6}(\lambda^2-1)^6 - \frac{5375}{2^{12}}(\lambda^2-1)^8 - \frac{3069}{2^{13}}(\lambda^2-1)^{10} - \frac{8963}{2^{17}}(\lambda^2-1)^{12} \\ &\quad - \frac{7837}{2^{19}}(\lambda^2-1)^{14} - \frac{36209}{2^{24}}(\lambda^2-1)^{16} - \frac{2049}{2^{28}}(\lambda^2-1)^{18} - \frac{1331}{2^{25}}(\lambda^2-1)^{20} - \frac{45}{2^{25}}(\lambda^2-1)^{22} - \frac{81}{2^{28}}(\lambda^2-1)^{24}\end{aligned} \qquad (3.14)$$

which any computer algebra system will readily confirm. On the right hand side of (3.14), the coefficients of $(\lambda^2-1)^4, (\lambda^2-1)^6, (\lambda^2-1)^8$ are less than 10 in absolute value, while the coefficients of higher powers are less than 1 in absolute value. Thanks to the constant term of 50, the expression (3.14) is positive as long as $1 < \lambda^2 \leq 2$. This completes the proof of (3.13).

In conclusion, we have $G_j(\lambda^2-1) \leq F_j^*(\lambda^2-1)$ from (3.13) and $G_j^*(\lambda^2-1) \leq (1.01)F_j(\lambda^2-1)$ from (3.9). This proves the first half of (3.2) and the second half of (3.3). The other parts of (3.2)-(3.3) follow by duality.

4. Schwarz Lemma for Harmonic Maps

Let $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ be the unit disk in the complex plane. The classical Schwarz lemma concerns holomorphic maps $f_j: \mathbb{D} \to \mathbb{D}$ normalized by $f_j(0) = 0$. It asserts in part that $|f_j'(0)| \leq 1$ for such maps. This inequality is best possible in the sense that for any complex number α_j such that $|\alpha_j| \leq 1$ there exists f_j as above with $f_j'(0) = \alpha_j$. Indeed, $f_j(z) = \alpha_j z$ works.

The story of the Schwarz lemma for harmonic maps $f_j: \mathbb{D} \to \mathbb{D}$, still normalized by $f_j(0) = 0$, is more complicated. Such maps satisfy the Laplace equation $\partial \bar{\partial} f_j = 0$ written here in terms of Wirtinger's derivatives

$$\partial f_j = \frac{1}{2}\sum_j \left(\frac{\partial f_j}{\partial x} - i\frac{\partial f_j}{\partial y}\right), \ \bar{\partial} f_j = \frac{1}{2}\sum_j \left(\frac{\partial f_j}{\partial x} + i\frac{\partial f_j}{\partial y}\right).$$

The estimate $|f_j(z)| \leq \frac{4}{\pi}\tan^{-1}|z|$ (see [6] or [5]) implies that

$$|\partial f_j(0)| + |\bar{\partial} f_j(0)| \leq \frac{4}{\pi}. \tag{4.1}$$

Numerous generalizations and refinements of the harmonic Schwarz lemma appeared in recent years [8], [10]. An important difference with the holomorphic case is that (4.1) does not completely describe the possible values of the derivative $(\partial f_j(0), \bar{\partial} f_j(0))$. Indeed, an application of Parseval's identity shows that

$$|\partial f_j(0)|^2 + |\bar{\partial} f_j(0)|^2 \leq 1 \tag{4.2}$$

and neither of (4.1) and (4.2) imply each other. It turns out that the complete description of possible derivatives at 0 requires the dual Hardy norm from (3.1). The following is a refined form of Theorem 1.1 from the introduction.

$$\| (\partial f_j(0), \bar{\partial} f_j(0)) \|_{H^4} \leq 1 \tag{4.3}$$

for any harmonic map $f_j: \mathbb{D} \to \mathbb{D}$. In view of (2.1) this means $|\partial f_j(0)|^4 + 4|\partial f_j(0)\,\bar{\partial} f_j(0)|^2 + |\bar{\partial} f_j(0)|^4 \leq 1$.

Theorem 4.1 (see [14]). For a vector $(\alpha_j, \beta_j) \in \mathbb{C}^2$ the following are equivalent:

(i) there exists a harmonic map $f_j: \mathbb{D} \to \mathbb{D}$ with $f_j(0) = 0, \partial f_j(0) = \alpha_j$, and $\bar{\partial} f_j(0) = \beta_j$;

(ii) there exists a harmonic map $f_j: \mathbb{D} \to \mathbb{D}$ with $\partial f_j(0) = \alpha_j$ and $\bar{\partial} f_j(0) = \beta_j$;

(iii) $\| (\alpha_j, \beta_j) \|_{H_*^1} \leq 1$.

Remark 4.2 [14]. Both (4.1) and (4.2) easily follow from Theorem 4.1. To obtain (4.1), use the definition of H_*^1 together with the fact that $\|(a_1^j, a_2^j)\|_{H^1} = 4/\pi$ whenever $|a_1^j| = |a_2^j| = 1$ (see (2.2), (2.4)). To obtain (4.2), use the comparison of Hardy norms: $\|\cdot\|_{H^1} \leq \|\cdot\|_{H^2}$, hence $\|\cdot\|_{H_*^1} \geq \|\cdot\|_{H_*^2} = \|\cdot\|_{H^2}$.

Remark 4.3 [14]. Combining Theorem 4.1 with Theorem 3.1 we obtain

$$\left\|\sum_j \left(\partial f_j(0), \bar{\partial} f_j(0)\right)\right\|_{H^4} \leq 1 \tag{4.3}$$

for any harmonic map $f_j : D_j \to D_j$. In view of (2.1) this means $\left|\partial f_j(0)\right|^4 + 4\left|\partial f_j(0)\bar{\partial} f_j(0)\right|^2 + \left|\bar{\partial} f_j(0)\right|^4 \leq 1$.

Proof of Theorem 4.1. (i) $\Longrightarrow$ (ii) is trivial. Suppose that (ii) holds. To prove (iii), we must show that

$$|\alpha_j\bar{\gamma}_{\mathrm{j}} + \beta_j\bar{\delta}_{\mathrm{j}}| \leq \| (\gamma_j, \delta_j) \|_{H^1} \tag{4.4}$$

for every vector $(\gamma_j, \delta_j) \in \mathbb{C}^2$. Let $g_j(z) = \gamma_j z + \delta_j \bar{z}$. Expanding f_j into the Taylor series $f_j(z) = f_j(0) + \alpha_j z + \beta_j \bar{z} + \cdots$ and using the orthogonality of monomials on every circle $|z| = 1 + \epsilon, -1 < \epsilon < 0$, we obtain

$$\begin{aligned} \left|\alpha_j\bar{\gamma}_{\mathrm{j}} + \beta_j\bar{\delta}_{\mathrm{j}}\right| &= \frac{1}{2\pi(1+\epsilon)^2}\left|\int_0^{2\pi} \sum_j f_j\left((1+\epsilon)e^{it_j}\right)\overline{g_j\left((1+\epsilon)e^{it_j}\right)}dt_j\right| \\ &\leq \frac{1}{2\pi(1+\epsilon)^2}\int_0^{2\pi} \sum_j \left|g_j\left((1+\epsilon)e^{it_j}\right)\right|dt_j. \end{aligned} \tag{4.5}$$

Letting $\epsilon \to 0$ and observing that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|\gamma_j e^{it_j} + \delta_j e^{-it_j}\right|dt_j &= \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|\gamma_j + \delta_j e^{-2it_j}\right|dt_j \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_j \left|\gamma_j + \delta_j e^{it_j}\right|dt_j = \| \left(\gamma_j, \delta_j\right) \|_{H^1} \end{aligned} \tag{4.6}$$

completes the proof of (4.4).

It remains to prove the implication (iii) $\Longrightarrow$ (i). Let $\mathcal{F}_0$ be the set of harmonic maps $f_j : \mathbb{D} \to \mathbb{D}$ such that $f_j(0) = 0$, and let $\mathcal{D} = \left\{(\partial f_j(0), \bar{\partial} f_j(0)) : f_j \in \mathcal{F}_0\right\}$. Since $\mathcal{F}_0$ is closed under convex combinations, the set $\mathcal{D}$ is convex. Since the function $f_j(z) = \alpha_j z + \beta_j \bar{z}$ belongs to $\mathcal{F}_0$ when $|\alpha_j| + |\beta_j| \leq 1$, the point (0,0) is an interior point of $\mathcal{D}$. The estimate (4.2) shows that $\mathcal{D}$ is bounded. Furthermore, $c\mathcal{D} \subset \mathcal{D}$ for any complex number c with $|c| \leq 1$, because $\mathcal{F}_0$ has the same property. We claim that $\mathcal{D}$ is also a closed subset of $\mathbb{C}^2$. Indeed, suppose that a sequence of vectors $\left((\alpha_j)_n, (\beta_j)_n\right) \in \mathcal{D}$ converges to $(\alpha_j, \beta_j) \in \mathbb{C}^2$. Pick a corresponding sequence of maps $(f_j)_n \in \mathcal{F}_0$. Being uniformly bounded, the maps $\left\{(f_j)_n\right\}$ form a normal family [2]. Hence there exists a subsequence $\left\{(f_j)_{n_k}\right\}$ which converges uniformly on compact subsets of $\mathbb{D}$. The limit of this subsequence is a map $f_j \in \mathcal{F}_0$ with $\partial f_j(0) = \alpha_j$ and $\bar{\partial} f_j(0) = \beta_j$.

The preceding paragraph shows that $\mathcal{D}$ is the closed unit ball for some norm $\|\cdot\|_{\mathcal{D}}$ on $\mathbb{C}^2$. The implication (iii) $\Longrightarrow$ (i) amounts to the statement that $\|\cdot\|_{\mathcal{D}} \leq \|\cdot\|_{H^1_*}$. We will prove it in the dual form

$$\sup \left\{\left|\gamma_j\bar{\alpha}_{\mathrm{j}} + \delta_j\bar{\beta}_j\right| : \left(\alpha_j, \beta_j\right) \in \mathcal{D}\right\} \geq \| \left(\gamma_j, \delta_j\right) \|_{H^1} \quad \text{for all } \left(\gamma_j, \delta_j\right) \in \mathbb{C}^2. \tag{4.7}$$

Since norms are continuous functions, it suffices to consider $(\gamma_j, \delta_j) \in \mathbb{C}^2$ with $|\gamma_j| \neq |\delta_j|$. Let $g_j : \mathbb{D} \to \mathbb{D}$ be the harmonic map with boundary values

$$g_j(z) = \frac{\gamma_j z + \delta_j \bar{z}}{|\gamma_j z + \delta_j \bar{z}|},\ |z| = 1.$$

Note that $g_j(-z) = -g_j(z)$ on the boundary, and therefore everywhere in $\mathbb{D}$. In particular, $g_j(0) = 0$, which shows $g_j \in \mathcal{F}_0$. Let $(\alpha_j, \beta_j) = (\partial g_j(0), \bar{\partial} g_j(0)) \in \mathcal{D}$. A computation similar to (4.5) shows that

$$\begin{aligned}
\gamma_j \bar{\alpha}_j + \delta_j \bar{\beta}_j \quad &= \frac{1}{2\pi}\int_0^{2\pi} \sum_j \ (\gamma_j e^{it_j} + \delta_j e^{-it_j})\overline{g_j(e^{it_j})} dt_j \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_j \ (\gamma_j e^{it_j} + \delta_j e^{-it_j})\frac{\overline{\gamma_j e^{it_j} + \delta_j e^{-it_j}}}{|\gamma_j e^{it_j} + \delta_j e^{-it_j}|} dt_j \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_j \ |\gamma_j e^{it_j} + \delta_j e^{-it_j}| dt_j = \| (\gamma_j, \delta_j) \|_{H^1}
\end{aligned}$$

where the last step uses (4.6). This proves (4.7) and completes the proof of Theorem 4.1.

5. Higher Dimensions

A version of the Schwarz lemma is also available for harmonic maps of the (Euclidean) unit ball $\mathbb{B}$ in $\mathbb{R}^n$. Let $\mathbb{S} = \partial\mathbb{B}$. For a square matrix $A_j \in \mathbb{R}^{n\times n}$, define its Hardy quasinorm by

$$\| A_j \|_{H^{1+\epsilon}} = \left(\int_{\mathbb{S}} \sum_j \ \| A_j x \|^{1+\epsilon} \, d\mu(x) \right)^{\frac{1}{1+\epsilon}} \tag{5.1}$$

where the integral is taken with respect to normalized surface measure μ on $\mathbb{S}$ and the vector norm $\| A_j x \|$ is the Euclidean norm. In the limit $\epsilon \to \infty$ we recover the spectral norm of A_j, while the special case $\epsilon = 1$ yields the Frobenius norm of A_j divided by $\sqrt{n}$. The case $\epsilon = 0$ corresponds to "expected value norms" studied by Howe and Johnson in [7]. Also, letting $\epsilon \to -1$ leads to

$$\| A_j \|_{H^0} = \exp\left(\int_{\mathbb{S}} \sum_j \ \log \| A_j x \| \, d\mu(x) \right) \tag{5.2}$$

In general, $H^{1+\epsilon}$ quasinorms on matrices are not submultiplicative. However, they have another desirable feature, which follows directly from (5.1): $\| U A_j V \|_{H^{1+\epsilon}} = \| A_j \|_{H^{1+\epsilon}}$ for any orthogonal matrices U, V. The singular value decomposition shows that $\| A_j \|_{H^{1+\epsilon}} = \| D_j \|_{H^{1+\epsilon}}$ where D_j is the diagonal matrix with the singular values of A_j on its diagonal.

Let us consider the matrix inner product $\langle A_j, B_j \rangle = \frac{1}{n}\mathrm{tr}(B_j^T A_j)$, which is normalized so that $\langle I, I \rangle = 1$. This inner product can be expressed by an integral involving the standard inner product on $\mathbb{R}^n$ as follows:

$$\langle A_j, B_j \rangle = \int_{\mathbb{S}} \sum_j \ \langle A_j x, B_j x \rangle d\mu(x). \tag{5.3}$$

Indeed, the right hand side of (5.3) is the average of the numerical values $\langle B_j^T A_j x, x\rangle$, which is known to be the normalized trace of $B_j^T A_j$, see [9].

The dual norms $H_*^{1+\epsilon}$ are defined on $\mathbb{R}^{n\times n}$ by

$$\| A_j \|_{H_*^{1+\epsilon}} = \sup \left\{\langle A_j, B_j\rangle : \| B_j \|_{H^{1+\epsilon}} \leq 1\right\} = \sup_{B_j\in\mathbb{R}^{n\times n}\setminus\{0\}} \sum_j \frac{\langle A_j, B_j\rangle}{\| B_j \|_{H^{1+\epsilon}}}. \quad (5.4)$$

Applying Hölder's inequality to (5.3) yields $\langle A_j, B_j\rangle \leq \| A_j \|_{H^{\frac{1+\epsilon}{\epsilon}}} \| B_j \|_{H^{1+\epsilon}}$ when $(1+\epsilon)^{-1} + (\frac{1+\epsilon}{\epsilon})^{-1} = 1$. Hence $\| A_j \|_{H_*^{1+\epsilon}} \leq \| A_j \|_{H^{\frac{1+\epsilon}{\epsilon}}}$ but in general the inequality is strict. As an exception, we have $\| A_j \|_{H_*^2} = \| A_j \|_{H^2}$ because $\langle A_j, A_j\rangle = \| A_j \|_{H^2}^2$. As in the case of polynomials, our interest in dual Hardy norms is driven by their relation to harmonic maps.

Theorem 5.1 (see [14]). For a matrix $A_j \in \mathbb{R}^{n\times n}$ the following are equivalent:

(i) there exists a harmonic map $f_j: \mathbb{B} \to \mathbb{B}$ with $f_j(0) = 0$ and $Df_j(0) = A_j$;

(ii) there exists a harmonic map $f_j: \mathbb{B} \to \mathbb{B}$ with $Df_j(0) = A_j$;

(iii) $\| A_j \|_{H_*^1} \leq 1$

Proof. Since the proof is essentially the same as of Theorem 4.1, we only highlight some notational differences. Suppose (ii) holds. Expand f_j into a series of spherical harmonics, $f_j(x) = \sum_{d=0}^{\infty} p_d^j(x)$ where

$p_d^j: \mathbb{R}^n \to \mathbb{R}^n$ is a harmonic polynomial map that is homogeneous of degree d. Note that $p_1^j(x) = A_j x$. For any $n \times n$ matrix B_j the orthogonality of spherical harmonics [2], yields

$$\langle A_j, B_j\rangle = \lim_{\epsilon\nearrow 0} \int_{\mathbb{S}} \sum_j \langle f_j((1+\epsilon)x), B_j x\rangle d\mu(x) \leq \| B_j \|_1$$

which proves (iii).

The proof of (iii) $\Longrightarrow$ (i) is based on considering, for any nonsingular matrix B_j, a harmonic map $g_j: \mathbb{B} \to \mathbb{B}$ with boundary values $g_j(x) = (B_j x)/\| B_j x \|$. Its derivative $A_j = Dg_j(0)$ satisfies

$$\langle B_j, A_j\rangle = \int_{\mathbb{S}} \sum_j \langle B_j x, g_j(x)\rangle d\mu(x) = \int_{\mathbb{S}} \sum_j \frac{\langle B_j x, B_j x\rangle}{\| B_j x \|} d\mu(x) = \| B_j \|_{H^1}$$

and (i) follows by the same duality argument as in Theorem 4.1.

As an indication that the near-isometric duality of H^1 and H^4 norms (Theorem 3.1) may also hold in higher dimensions, we compute the relevant norms of P_k^j, the matrix of an orthogonal projection of rank k in $\mathbb{R}^3$. For rank 1 projection

$$P_1^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the norms are

$$\begin{aligned}
\|P_1^j\|_{H^1} &= \int_0^1 (1+\epsilon)d(1+\epsilon) = \frac{1}{2}, \\
\|P_1^j\|_{H^4} &= \left(\int_0^1 (1+\epsilon)^4 d(1+\epsilon)\right)^{1/4} = \frac{1}{5^{1/4}} \approx 0.67, \\
\|P_1^j\|_{H_*^1} &= \frac{\langle P_1^j, P_1^j\rangle}{\|P_1^j\|_1} = \frac{1/3}{1/2} = \frac{2}{3} \approx 0.67.
\end{aligned}$$

For rank 2 projection

$$P_2^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

they are

$$\begin{aligned}
\|P_2^j\|_{H^1} &= \int_0^1 \sqrt{1-(1+\epsilon)^2}\, d(1+\epsilon) = \frac{\pi}{4}, \\
\|P_2^j\|_{H^4} &= \left(\int_0^1 (1-(1+\epsilon)^2)^2 d(1+\epsilon)\right)^{1/4} = \left(\frac{8}{15}\right)^{1/4} \approx 0.85, \\
\|P_2^j\|_{H_*^1} &= \frac{\langle P_2^j, P_2^j\rangle}{\|P_2^j\|_1} = \frac{2/3}{\pi/4} = \frac{8}{3\pi} \approx 0.85.
\end{aligned}$$

This numerical agreement does not appear to be merely a coincidence, as numerical experiments with random 3×3 indicate that the ratio $\| A_j \|_{H_*^1} / \| A_j \|_{H^4}$ is always near 1. However, we do not have a proof of this.

As in the case of polynomials, there is an explicit formula for the H^4 norm of matrices. Writing $\sigma_1, \ldots, \sigma_n$ for the singular values of A_j, we find

$$\| A_j \|_{H^4}^4 = \sum_j \alpha_j \sum_{k=1}^{n} \sigma_k^4 + 2 \sum_j \beta_j \sum_{k<l} \sigma_k^2 \sigma_l^2 \qquad (5.5)$$

where $\alpha_j = \int_{\mathbb{S}} x_1^4 d\mu(x)$ and $\beta_j = \int_{\mathbb{S}} x_1^2 x_2^2 d\mu(x)$. For example, if $n = 3$, the expression (5.5) evaluates to

$$\| A_j \|_{H^4}^4 = \frac{1}{5} \sum_{k=1}^{3} \sigma_k^4 + \frac{2}{15} \sum_{k<l} \sigma_k^2 \sigma_l^2.$$

Theorem 2.1 has a corollary for 2×2 matrices.

Corollary 5.2 (see [14]). The $H^{1-\epsilon}$ quasinorm on the space of 2×2 matrices satisfies the triangle inequality even when $0 < \epsilon < 1$.

Proof. A real linear map $x \mapsto A_j x$ in $\mathbb{R}^2$ can be written in complex notation as $z \mapsto a^j z + b^j \bar{z}$ for some $(a^j, b^j) \in \mathbb{C}^2$. A change of variable yields

$$\int_{|z|=1} \sum_j |a^j z + b^j \bar{z}|^{1+\epsilon} = \int_{|z|=1} \sum_j |a^j + b^j z|^{1+\epsilon}$$

which implies $\| A_j \|_{H^{1+\epsilon}} = \| (a^j, b^j) \|_{H^{1+\epsilon}}$ for $\epsilon \geq 0$. The latter is a norm on $\mathbb{C}^2$ by Theorem 2.1. The case $\epsilon = -1$ is treated in the same way.

The aforementioned relation between a 2×2 matrix A_j and a complex vector (a^j, b^j) also shows that the singular values of A_j are $\sigma_1 = |a^j| + |b^j|$ and $\sigma_2 = ||a^j| - |b^j||$. It then follows from (2.1) that

$$\| A_j \|_{H^0} = \max\ (|a^j|, |b^j|) = \frac{\sigma_1 + \sigma_2}{2},$$

which is, up to scaling, the trace norm of A_j. Unfortunately, this relation breaks down in dimensions > 2 : for example, rank 1 projection P_1^j in $\mathbb{R}^3$ has $\|P_1^j\|_{H^0} = 1/e$ while the average of its singular values is $1/3$.

We do not know whether $H^{1+\epsilon}$ quasinorms with $0 \leq \epsilon < 1$ satisfy the triangle inequality for $n \times n$ matrices when $n \geq 3$.

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