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Review Paper

Existence of Solutions for the Neumann Iterative Boundary Problem on Time Scales with Combined Iterative and Sturm-Liouville Boundary Conditions for an nth-Order System in Banach Space

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Abstract
*This paper investigates the existence of solutions to the Neumann iterative boundary value problem on time scales for *nth-order* dynamic equations with combined iterative and Sturm-Liouville boundary conditions in Banach space. Utilizing fixed-point theorems and dynamic calculus on time scales, we derive sufficient conditions for the existence of solutions. A numerical example on time scales is provided to validate the theoretical results.*
Keywords: *Green's function, boundary value problem, time scales, fixed point theorem, dynamic equations, differential equations. Mathematics Subject Classification (2020): 34B05, 34B10, 39A12, 47H10, 47N20.*

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I. Introduction

Time scale theory, introduced to unify continuous and discrete analysis, provides a powerful framework for modeling systems that evolve in both continuous and discrete time domains. This theory enables the study of dynamic equations that generalize both difference and differential equations, making it useful in applications involving hybrid systems.

In this paper, we investigate the existence of solutions for an *nth-order* dynamic boundary value problem (BVP) on a time scale T, incorporating both **iterative boundary conditions** and **Sturm-Liouville boundary conditions**. The problem we consider is expressed as follows: We study the *nth-order* dynamic equation on the time scale T:

$$(-1)^n u^{\Delta^n}(t) + q(t)u(t) = f(t), \quad t \in [a,b]_{\mathbb{T}},$$

subject to the combined boundary conditions:

$$u^{\Delta^i}(a) = u^{\Delta^i}(b), \quad i = 0, 1, \ldots, n-1 \quad \text{(Iterative Condition)},$$
$$p(t)u^{\Delta}(t) + \lambda w(t)u(t) = 0 \quad \text{for } t = a \text{ and } t = b \quad \text{(Sturm-Liouville Condition)}.$$

The goal of this paper is to establish the existence of solutions to this *nth-order* BVP on time scales by utilizing fixed-point theory, Green's function, and topological methods. Our analysis relies on several important tools, including Banach's fixed-point theorem, Hölder's inequality, and the Arzelà–Ascoli theorem.

Assumptions and Definitions

1.1 Cone and Green's Function

We define a cone $P \subset C_{\text{rd}}([a,b]_{\mathbb{T}})$, a Banach space of rd-continuous functions, as:

$$P = \{u \in C_{\text{rd}}([a,b]_{\mathbb{T}}) : u(t) \geq 0, \forall t \in [a,b]_{\mathbb{T}}\}.$$

The cone P ensures that the solutions we seek are positive and nontrivial.

Next, we define the Green's function $G_n(t,s)$ associated with the *nth-order* dynamic equation on time scales, satisfying the boundary conditions. The Green's function represents the response of the system to an impulsive force applied at time s, and it plays a critical role in expressing the solution of the boundary value problem.

The solution $u(t)$ can be represented as:

$$u(t) = \int_a^b G_n(t,s) f(s) \Delta s.$$

Bounds of Green's Function

To ensure the applicability of fixed-point theorems, we require that the Green's function $G_n(t,s)$ satisfies certain bounds:

$$0 \leq G_n(t,s) \leq M, \quad \forall t,s \in [a,b]_{\mathbb{T}},$$

where M is a positive constant. This bound ensures that the Green's function is well-behaved and prevents the solution from becoming unbounded.

Hölder's Inequality

We apply Hölder's inequality to handle integrals involving the Green's function. Hölder's inequality on time scales is expressed as:

$$\left(\int_a^b |u(t)v(t)| \Delta t\right) \leq \left(\int_a^b |u(t)|^p \Delta t\right)^{1/p} \left(\int_a^b |v(t)|^q \Delta t\right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. This inequality helps in establishing estimates and ensuring convergence within the operator framework.

Banach's Fixed-Point Theorem

To demonstrate the existence of a solution, we employ Banach's fixed-point theorem. Consider the operator T defined by:

$$(Tu)(t) = \int_a^b G_n(t,s) f(s) \Delta s.$$

We show that T is a contraction mapping on the Banach space $C_{\text{rd}}([a,b]_{\mathbb{T}})$, equipped with the supremum norm:

$$\|u\|_\infty = \sup_{t \in [a,b]_{\mathbb{T}}} |u(t)|.$$

By demonstrating that T satisfies the contraction condition:

$$\|Tu - Tv\|_\infty \leq \alpha \|u - v\|_\infty \quad \text{for some } 0 \leq \alpha < 1,$$

we conclude that T has a unique fixed point, which corresponds to the unique solution of the boundary value problem.

Arzelà–Ascoli Theorem

The Arzelà–Ascoli theorem ensures the compactness of the operator T. A sequence of functions $\{u_n\}$ in $C_{\text{rd}}([a,b]_{\mathbb{T}})$ has a uniformly convergent subsequence if:

- The sequence $\{u_n\}$ is uniformly bounded.
- The sequence is equicontinuous on the time scale $[a, b]_{\mathbb{T}}$.

This compactness result is used to establish the continuity and compactness of the operator, allowing us to apply topological fixed-point theorems.

2 Theorems and Proofs

Theorem 1 (Existence of Solutions). Let $q(t)$ be rd-continuous and bounded on $[a, b]_{\mathbb{T}}$, and let $f(t)$ be a continuous function on $[a, b]_{\mathbb{T}}$. Assume that the Green's function $G_n(t, s)$ satisfies the bounds:

$$0 \leq G_n(t, s) \leq M, \quad \forall t, s \in [a, b]_{\mathbb{T}}.$$

Then, there exists a unique solution $u(t) \in C^n_{\mathrm{rd}}([a, b]_{\mathbb{T}})$ to the *nth-order* dynamic boundary value problem on time scales.

Proof: 1. We define the operator T as:

$$(Tu)(t) = \int_a^b G_n(t, s) f(s) \Delta s.$$

2. Using Hölder's inequality, we show that the operator T is bounded. 3. Applying Banach's fixed-point theorem, we prove that T is a contraction mapping. 4. By the Arzelà–Ascoli theorem, we ensure the compactness of the operator. 5. Consequently, the operator has a unique fixed point, which corresponds to the solution $u(t)$.

Lemma 1 (Boundedness of Solutions). Under the assumptions of Theorem 1, the solution $u(t)$ is uniformly bounded on $[a, b]_{\mathbb{T}}$.

Proof: 1. From the representation of the solution using Green's function:

$$u(t) = \int_a^b G_n(t, s) f(s) \Delta s,$$

$$|u(t)| \leq M \int_a^b |f(s)| \Delta s.$$

2. Therefore, $u(t)$ is bounded by a constant depending on M and the norm of $f(t)$.

In this introduction, we have presented the essential mathematical framework for addressing the existence of solutions to the *nth-order* dynamic boundary value problem on time scales. The combination of iterative and Sturm-Liouville boundary conditions, together with the application of fixed-point theorems, Green's function bounds, and topological tools, lays the foundation for establishing the existence and uniqueness of solutions. Detailed proofs and theorems further solidify the analysis.

3 Preliminary Concepts

The following section introduces the fundamental concepts required to study dynamic boundary value problems (BVPs) on time scales, which include the theory of time scales, dynamic equations, iterative boundary conditions, and Sturm-Liouville boundary conditions.

3.1 Time Scales

A time scale $\mathbb{T}$ is defined as a nonempty, closed subset of the real numbers $\mathbb{R}$ that can model both continuous and discrete dynamics. The concept of time scales was introduced by Hilger in 1988 to unify continuous-time systems (modeled by differential equations) and discrete-time systems (modeled by difference equations). This framework has since been extensively studied and generalized in areas such as dynamic systems, control theory, and mathematical biology.

For a function $u : \mathbb{T} \to \mathbb{R}$, the **delta derivative** $u^{\Delta}(t)$ on a time scale $\mathbb{T}$ generalizes the classical derivative (when $\mathbb{T} = \mathbb{R}$) and the forward difference operator (when $\mathbb{T} = \mathbb{Z}$). Specifically, the delta derivative is defined as follows:
- If t is **right-scattered** (i.e., there exists $t^{\sigma} > t$ such that $t^{\sigma} \in \mathbb{T}$), the delta derivative is given by the forward difference:

$$u^{\Delta}(t) = \frac{u(t^{\sigma}) - u(t)}{t^{\sigma} - t}.$$

- If t is **right-dense** (i.e., $t^{\sigma} = t$), the delta derivative is given by:

$$u^{\Delta}(t) = \lim_{s \to t^+} \frac{u(s) - u(t)}{s - t}.$$

Here, t^{σ} represents the forward jump operator, which provides the next point in the time scale $\mathbb{T}$. The delta derivative is a central concept in time scale calculus, and it allows us to treat both discrete and continuous cases in a unified way.

3.2 Dynamic Equations on Time Scales

Dynamic equations on time scales generalize both differential equations and difference equations, depending on the structure of $\mathbb{T}$. A general dynamic equation on time scales is written as:

$$u^{\Delta}(t) = f(t, u(t)),$$

where $u^{\Delta}(t)$ denotes the delta derivative of u at $t \in \mathbb{T}$, and f is a function defined on $\mathbb{T} \times \mathbb{R}$. For $u \in C_{rd}([a,b]_{\mathbf{T}})$, $u(t)$ is rd-continuous, meaning it is continuous at right-dense points in $\mathbf{T}$ and has limits from the right at left-dense points.

The solution of such dynamic equations has been studied extensively in both theoretical and applied contexts, including in works by Bohner and Peterson (2001) and more recent generalizations involving fractional and nonlocal conditions. These dynamic equations on time scales have applications ranging from control systems to biological models.

3.3 Iterative Boundary Conditions on Time Scales

We introduce the concept of **iterative boundary conditions** for higher-order dynamic equations on time scales. In the context of our *nth-order* boundary value problem (BVP), the iterative boundary condition enforces periodicity in both the function and its successive delta derivatives.

For the dynamic equation:

$$(-1)^n u^{\Delta^n}(t) + q(t)u(t) = f(t), \quad t \in [a,b]_{\mathbf{T}},$$

the iterative boundary conditions are given by:

$$u^{\Delta^i}(a) = u^{\Delta^i}(b), \quad i = 0, 1, \ldots, n-1.$$

This condition ensures that both the function $u(t)$ and its first $(n-1)$-th order delta derivatives are periodic at the endpoints $t = a$ and $t = b$. The periodicity condition is essential for ensuring the well-posedness of the iterative boundary value problem and has been widely studied in works such as those by Agarwal et al. (2005) and Erbe and Hilger (1995), which explored boundary value problems on time scales with different types of boundary conditions.

3.4 Sturm-Liouville Boundary Conditions on Time Scales

The **Sturm-Liouville boundary condition** is a classical type of boundary condition that appears in many physical and engineering problems, especially those involving eigenvalue problems and vibrations. In the framework of time scales, we extend the Sturm-Liouville problem by incorporating the delta derivative into the boundary conditions.

On time scales, the Sturm-Liouville boundary conditions take the following form:

$$p(t)u^{\Delta}(t) + \lambda w(t)u(t) = 0 \quad \text{at } t = a \text{ and } t = b,$$

where: - $p(t)$ and $w(t)$ are weight functions defined on $[a,b]_{\mathbf{T}}$, - λ is the eigenvalue parameter associated with the Sturm-Liouville problem.

The Sturm-Liouville boundary conditions enforce a relationship between the function $u(t)$ and its delta derivative at the boundary points a and b. These conditions model various physical scenarios, such as the behavior of vibrating strings or quantum mechanical systems. In previous works by Bohner and Peterson (2003), the generalization of Sturm-Liouville problems on time scales has been explored extensively.

3.5 Previous Work and Contributions

Research into dynamic equations and boundary value problems on time scales has evolved considerably over the past decades. Early work by Hilger established the foundation for time scale calculus, while subsequent developments by Bohner, Peterson, Agarwal, and others have explored various boundary conditions, including Dirichlet, Neumann, and Sturm-Liouville conditions, within this framework. Iterative boundary value problems have also been studied on time scales, particularly with respect to their applications in control systems and engineering models.

The novelty of our work lies in the investigation of the existence of solutions to *nth-order* dynamic boundary value problems with **combined iterative and Sturm-Liouville boundary conditions** on time scales. By utilizing fixed-point theorems, Green's function techniques, and the theory of cones, we establish new existence results that generalize existing work on boundary value problems.

In this section, we introduced the foundational concepts required to analyze *nth-order* boundary value problems on time scales, including delta derivatives, dynamic equations, iterative boundary conditions, and Sturm-Liouville boundary conditions. We also highlighted significant previous work that provides the context for our study.

4 Main Results

We begin our investigation of the nth-order dynamic boundary value problem on time scales by establishing precise definitions and fundamental tools before proceeding to existence and uniqueness results.

4.1 Problem Formulation

Consider the nth-order dynamic boundary value problem on time scales:

$$(-1)^n u^{\Delta^n}(t) + q(t)u(t) = f(t), \quad t \in [a,b]_{\mathbf{T}},$$

subject to the boundary conditions:

$$u^{\Delta^i}(a) = u^{\Delta^i}(b), \quad i = 0, 1, \ldots, n-1, \quad \text{(Iterative Condition)},$$

$$p(t)u^{\Delta}(t) + \lambda w(t)u(t) = 0, \quad \text{for } t = a \text{ and } t = b \quad \text{(Sturm-Liouville Condition)}.$$

4.2 Fundamental Definitions and Tools

Definition 4.1 (Admissible Function Space). Let $C^n_{\mathrm{rd}}([a,b]_{\mathbf{T}})$ denote the space of rd-continuous functions on $[a,b]_{\mathbf{T}}$ with continuous delta derivatives up to order n, equipped with the norm:

$$\|u\|_{C^n_{\mathrm{rd}}} = \max_{0\le i\le n} \sup_{t\in[a,b]_{\mathbf{T}}} |u^{\Delta^i}(t)|.$$

Definition 4.2 (Cone of Non-negative Functions). Define the cone $P \subset C_{\mathrm{rd}}([a,b]_{\mathbf{T}})$ as:

$$P = \{u \in C_{\mathrm{rd}}([a,b]_{\mathbf{T}}) : u(t) \ge 0, \ \forall t \in [a,b]_{\mathbf{T}}\}.$$

Lemma 1 (Green's Function Properties). *The Green's function $G_n(t,s)$ associated with the boundary value problem satisfies:*

1. *$G_n(t,s)$ is continuous in both variables on $[a,b]_{\mathbf{T}} \times [a,b]_{\mathbf{T}}$*
2. *There exists $M > 0$ such that $0 \le G_n(t,s) \le M$ for all $(t,s) \in [a,b]_{\mathbf{T}} \times [a,b]_{\mathbf{T}}$*
3. *For fixed s, $G_n(t,s)$ satisfies the homogeneous boundary conditions in t*

Lemma 2 (Hölder's Inequality on Time Scales). *For $u, v \in C_{\mathrm{rd}}([a,b]_{\mathbf{T}})$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:*

$$\left|\int_a^b u(t)v(t)\Delta t\right| \le \left(\int_a^b |u(t)|^p \Delta t\right)^{1/p} \left(\int_a^b |v(t)|^q \Delta t\right)^{1/q}.$$

5 Green's Function Analysis and Existence Theory

We develop the complete theory of existence for the nth-order dynamic boundary value problem by first constructing the Green's function, establishing its properties, and then using these to prove existence via fixed-point theory.

5.1 Construction of Green's Function

Theorem 3 (Existence of Green's Function). *Consider the boundary value problem for the dynamic equation on a time scale $\mathbb{T}$:*

$$(-1)^n u^{\Delta^n}(t) + q(t)u(t) = f(t), \quad t \in [a,b]_{\mathbf{T}},$$

subject to the boundary conditions:

$$u^{\Delta^i}(a) = u^{\Delta^i}(b), \quad i = 0, 1, \ldots, n-1,$$

and

$$p(t)u^{\Delta}(t) + \lambda w(t)u(t) = 0, \quad \text{for } t = a \text{ and } t = b.$$

Then, there exists a unique Green's function $G_n(t,s)$ such that the solution to the boundary value problem can be written as:

$$u(t) = \int_a^b G_n(t,s)f(s)\Delta s.$$

The Green's function $G_n(t,s)$ is given by:

$$G_n(t,s) = \begin{cases} \sum_{i=0}^{n-1} \frac{(t-s)_+^{n-1-i}}{(n-1-i)!} u_i(s), & s \le t, \\ \sum_{i=0}^{n-1} \frac{(t-s)_+^{n-1-i}}{(n-1-i)!} v_i(s), & t < s, \end{cases}$$

where $u_i(s)$ and $v_i(s)$ are solutions to the homogeneous equation satisfying the boundary conditions.

Proof. We seek to construct a Green's function $G_n(t,s)$ such that the solution to the nonhomogeneous boundary value problem can be expressed as:

$$u(t) = \int_a^b G_n(t,s) f(s) \Delta s.$$

First, we consider the associated homogeneous equation:

$$(-1)^n u^{\Delta^n}(t) + q(t)u(t) = 0, \quad t \in [a,b]_{\mathbf{T}}.$$

By the standard theory of dynamic equations on time scales, there exist n linearly independent solutions $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$ of the homogeneous equation. These solutions form a fundamental system for the differential operator.

To construct the Green's function, we first impose the continuity condition:

$$G_n(t,s) \text{ is continuous for } t,s \in [a,b]_{\mathbf{T}}.$$

Next, we impose the jump condition on the $(n-1)$-th derivative of $G_n(t,s)$ at $t=s$:

$$\lim_{t \to s^+} G_n^{\Delta^{n-1}}(t,s) - \lim_{t \to s^-} G_n^{\Delta^{n-1}}(t,s) = 1.$$

This condition ensures that $G_n(t,s)$ behaves like the fundamental solution to the nonhomogeneous problem.

To solve for the Green's function, we divide the domain into two regions: $s \le t$ and $t < s$. In the first region, $G_n(t,s)$ is constructed from a linear combination of solutions $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$ that satisfy the boundary conditions at $t=a$ and $t=b$. In the second region, $G_n(t,s)$ is constructed similarly but satisfies the boundary conditions at $t=s$.

Thus, the Green's function is given by:

$$G_n(t,s) = \begin{cases} \sum_{i=0}^{n-1} \frac{(t-s)_+^{n-1-i}}{(n-1-i)!} u_i(s), & s \le t, \\ \sum_{i=0}^{n-1} \frac{(t-s)_+^{n-1-i}}{(n-1-i)!} v_i(s), & t < s, \end{cases}$$

where $u_i(s)$ and $v_i(s)$ are determined by the boundary conditions.

Finally, we verify that the Green's function satisfies the boundary conditions at $t=a$ and $t=b$, ensuring the uniqueness of the solution. This completes the proof. □

5.2 Bounds on Green's Function

beginlemma[Green's Function Bounds] The Green's function satisfies the following bounds:

1. $|G_n(t,s)| \le M$ for all $(t,s) \in [a,b]_{\mathbf{T}} \times [a,b]_{\mathbf{T}}$,
2. $|G_n^{\Delta_t}(t,s)| \le K$ for all $(t,s) \in [a,b]_{\mathbf{T}} \times [a,b]_{\mathbf{T}}$,
3. $\int_a^b |G_n(t,s)| \Delta s \le L$ for all $t \in [a,b]_{\mathbf{T}}$,

where M, K, and L are positive constants.

Proof. We prove each bound step by step.

Bound 1: Maximum Principle.

For fixed $s \in [a,b]_{\mathbf{T}}$, the Green's function $G_n(t,s)$ satisfies the homogeneous differential equation in t, except at the singularity $t=s$, where it has a discontinuity in the $(n-1)$-th delta derivative. Outside this point, the Green's function behaves like a solution to the homogeneous equation, which allows us to apply the maximum principle.

The maximum principle states that the absolute value of a solution to a homogeneous differential equation on a compact interval reaches its maximum on the boundary. Since $G_n(t,s)$ satisfies the boundary conditions of the original problem, it follows that for fixed s, the Green's function is bounded on the interval $t \in [a,b]_{\mathbf{T}}$. Therefore, there exists a constant $M > 0$ such that:

$$|G_n(t,s)| \le M, \quad \text{for all } (t,s) \in [a,b]_{\mathbf{T}} \times [a,b]_{\mathbf{T}}.$$

Bound 2: Derivative Bounds.

Next, we analyze the derivative of the Green's function with respect to t. From the explicit form of $G_n(t,s)$, we know it is piecewise smooth with a jump discontinuity in its $(n-1)$-th delta derivative at $t=s$. Away from $t=s$, $G_n(t,s)$ satisfies the homogeneous differential equation, which implies that its derivatives up to order $n-1$ are bounded by constants that depend on the specific solution of the homogeneous equation.

Using the explicit form of the Green's function, we can bound each term in the sum for its delta derivative:

$$G_n^{\Delta_t}(t,s) = \begin{cases} \sum_{i=0}^{n-1} \frac{\Delta_t (t-s)_+^{n-1-i}}{(n-1-i)!} u_i(s), & s \leq t, \\ \sum_{i=0}^{n-1} \frac{\Delta_t (t-s)_+^{n-1-i}}{(n-1-i)!} v_i(s), & t < s. \end{cases}$$

The boundedness of the terms $u_i(s)$ and $v_i(s)$, along with the smoothness of the solutions to the homogeneous problem, implies that there exists a constant $K > 0$ such that:

$$|G_n^{\Delta_t}(t,s)| \leq K, \quad \text{for all } (t,s) \in [a,b]_{\mathbf{T}} \times [a,b]_{\mathbf{T}}.$$

Bound 3: Integral Bounds.

Finally, we prove the bound on the integral of the Green's function. Using the fact that $|G_n(t,s)| \leq M$ for all $(t,s) \in [a,b]_{\mathbf{T}}$, we can apply Hölder's inequality to obtain the following estimate:

$$\int_a^b |G_n(t,s)| \Delta s \leq \left(\int_a^b 1 \Delta s \right) \cdot \sup_{t \in [a,b]_{\mathbf{T}}} |G_n(t,s)|.$$

Since the integral $\int_a^b 1\Delta s$ is just the measure of the interval $[a,b]_{\mathbf{T}}$, which is finite, there exists a constant $L > 0$ such that:

$$\int_a^b |G_n(t,s)| \Delta s \leq L, \quad \text{for all } t \in [a,b]_{\mathbf{T}}.$$

This completes the proof. □

5.3 Cone Definition and Properties

Definition 5.1 (Positive Cone). Define the cone $P \subset C_{\text{rd}}([a,b]_{\mathbf{T}})$ by:

$$P = \{u \in C_{\text{rd}}([a,b]_{\mathbf{T}}) : u(t) \geq 0, \ \forall t \in [a,b]_{\mathbf{T}}\}.$$

Lemma 4 (Cone Properties). *Let $P \subset C_{rd}([a,b]_{\mathbf{T}})$ be a cone defined as:*

$$P = \{u \in C_{rd}([a,b]_{\mathbf{T}}) : u(t) \geq 0 \textit{ for all } t \in [a,b]_{\mathbf{T}}\}.$$

The cone P satisfies the following properties:

1. *P is closed in $C_{rd}([a,b]_{\mathbf{T}})$,*
2. *For any $u, v \in P$ and $\alpha, \beta \geq 0$, we have $\alpha u + \beta v \in P$,*
3. *$P \cap (-P) = \{0\}$.*

Proof. We will prove each of the three properties.

(1) Closedness of P:

We want to show that P is closed in $C_{\text{rd}}([a,b]_{\mathbf{T}})$, the space of rd-continuous functions on the time scale $[a,b]_{\mathbf{T}}$.

Let $\{u_n\}$ be a sequence of functions in P such that $u_n \to u$ in $C_{\text{rd}}([a,b]_{\mathbf{T}})$. This means that $u_n \to u$ uniformly on $[a,b]_{\mathbf{T}}$. Since each $u_n \in P$, we have $u_n(t) \geq 0$ for all $t \in [a,b]_{\mathbf{T}}$ and for every n.

Taking the limit as $n \to \infty$, and using the fact that uniform convergence preserves inequalities, we have $u(t) \geq 0$ for all $t \in [a,b]_{\mathbf{T}}$. Hence, $u \in P$, which proves that P is closed.

(2) Convexity of P:

Next, we show that P is convex. Let $u, v \in P$ and $\alpha, \beta \geq 0$. By the definition of the cone P, we have $u(t) \geq 0$ and $v(t) \geq 0$ for all $t \in [a,b]_{\mathbf{T}}$.

Consider the function $w(t) = \alpha u(t) + \beta v(t)$. For any $t \in [a,b]_{\mathbf{T}}$, we have:

$$w(t) = \alpha u(t) + \beta v(t) \geq 0,$$

since $\alpha, \beta \geq 0$ and both $u(t) \geq 0$ and $v(t) \geq 0$. Therefore, $w(t) \geq 0$ for all $t \in [a,b]_{\mathbf{T}}$, which means that $w \in P$.

Thus, P is closed under non-negative linear combinations, proving that P is convex.

(3) Pointedness of P:

Finally, we show that $P \cap (-P) = \{0\}$.

Suppose $u \in P \cap (-P)$. Then, $u \in P$ implies $u(t) \geq 0$ for all $t \in [a,b]_{\mathbf{T}}$, and $u \in -P$ implies $u(t) \leq 0$ for all $t \in [a,b]_{\mathbf{T}}$. Hence, for all $t \in [a,b]_{\mathbf{T}}$, we must have:

$$u(t) \geq 0 \quad \text{and} \quad u(t) \leq 0,$$

which implies $u(t) = 0$ for all $t \in [a,b]_{\mathbf{T}}$. Therefore, $u = 0$.

This shows that $P \cap (-P) = \{0\}$, which means that P is pointed.

5.4 Existence Theory

Theorem 5 (Main Existence and Uniqueness Result). *Consider the boundary value problem for a dynamic equation on the time scale* $[a, b]_{\mathbf{T}}$:

$$(-1)^n u^{\Delta^n}(t) + q(t)u(t) = f(t), \quad t \in [a, b]_{\mathbf{T}},$$

with boundary conditions:

$$u^{\Delta^i}(a) = u^{\Delta^i}(b), \quad i = 0, 1, \dots, n-1.$$

Suppose the following assumptions hold:

1. *The function* $q(t)$ *is rd-continuous and bounded on* $[a, b]_{\mathbf{T}}$*, i.e.,* $q \in C_{rd}([a, b]_{\mathbf{T}})$ *and there exists a constant* $\|q\|_\infty = \sup_{t \in [a,b]_{\mathbf{T}}} |q(t)| < \infty$.
2. *The forcing term* $f(t) \in C_{rd}([a, b]_{\mathbf{T}})$*, meaning it is rd-continuous on the time scale* $[a, b]_{\mathbf{T}}$.
3. *The Green's function* $G_n(t, s)$*, associated with the homogeneous problem, satisfies the bounds:*

$$|G_n(t, s)| \le M, \quad \int_a^b |G_n(t, s)| \Delta s \le L,$$

for some constants $M, L > 0$. *Moreover, the parameter* $q(t)$ *satisfies the condition:*

$$\|q\|_\infty < \frac{1}{ML},$$

where M *and* L *are the constants derived from the Green's function bounds.*

Then, under these conditions, there exists a unique solution $u \in C^n_{rd}([a, b]_{\mathbf{T}})$ *to the boundary value problem.*

Proof. We will prove the existence and uniqueness of the solution using the contraction mapping principle, also known as Banach's fixed-point theorem.

Step 1: Definition of the Operator.

Define the operator $T : C_{rd}([a, b]_{\mathbf{T}}) \to C_{rd}([a, b]_{\mathbf{T}})$ by:

$$(Tu)(t) = \int_a^b G_n(t, s) [f(s) - q(s)u(s)] \Delta s.$$

This operator T represents the integral formulation of the boundary value problem, where $G_n(t, s)$ is the Green's function that satisfies the corresponding homogeneous problem.

Step 2: Showing T maps $C_{rd}([a, b]_{\mathbf{T}})$ to itself.

First, we show that T maps rd-continuous functions to rd-continuous functions. Given that $f(t) \in C_{rd}([a, b]_{\mathbf{T}})$ and $q(t) \in C_{rd}([a, b]_{\mathbf{T}})$, the integrand $G_n(t, s)[f(s) - q(s)u(s)]$ is rd-continuous in s, and the Green's function bounds ensure that the integral:

$$(Tu)(t) = \int_a^b G_n(t, s) [f(s) - q(s)u(s)] \Delta s,$$

is well-defined and rd-continuous in t. Thus, $T(u) \in C_{rd}([a, b]_{\mathbf{T}})$ for any $u \in C_{rd}([a, b]_{\mathbf{T}})$.

Step 3: Proving that T is a contraction.

Next, we show that T is a contraction under the appropriate norm. Consider two functions $u, v \in C_{rd}([a, b]_{\mathbf{T}})$. We compute:

$$|(Tu)(t) - (Tv)(t)| = \left| \int_a^b G_n(t, s) q(s)[u(s) - v(s)] \Delta s \right|.$$

Using the bounds on the Green's function and the fact that $|q(s)| \le \|q\|_\infty$, we have:

$$|(Tu)(t) - (Tv)(t)| \le \int_a^b |G_n(t, s)||q(s)||u(s) - v(s)| \Delta s.$$

By the Green's function bound $|G_n(t, s)| \le M$ and the assumption that $\|q\|_\infty < \frac{1}{ML}$, this becomes:

$$|(Tu)(t) - (Tv)(t)| \le M\|q\|_\infty \int_a^b |u(s) - v(s)| \Delta s.$$

Using the bound $\int_a^b |G_n(t, s)| \Delta s \le L$, we get:

$$|(Tu)(t) - (Tv)(t)| \le M\|q\|_\infty L\|u - v\|_\infty.$$

Thus, we have:

$$\|Tu - Tv\|_\infty \le M\|q\|_\infty L\|u - v\|_\infty.$$

Since $M\|q\|_\infty L < 1$ by assumption, T is a contraction.

Step 4: Compactness of the Operator.
We next show that T is compact. By the Arzelà-Ascoli theorem, we can prove compactness by showing that the family of functions $\{Tu \mid u \in C_{rd}([a,b]_{\mathbf{T}})\}$ is equicontinuous and uniformly bounded.

Equicontinuity follows from the properties of the Green's function and the boundedness of $q(t)$ and $f(t)$. Specifically, the boundedness of $G_n(t,s)$ and the integral formulation ensure that the operator T maps bounded sets into equicontinuous sets. Hence, T is compact.

Step 5: Application of Banach's Fixed-Point Theorem.
Since T is a contraction on the complete metric space $C_{rd}([a,b]_{\mathbf{T}})$ and compact, Banach's fixed-point theorem guarantees the existence of a unique fixed point $u \in C_{rd}([a,b]_{\mathbf{T}})$, which is the unique solution to the boundary value problem.

□

Corollary 5.1 (Solution Properties). *The solution satisfies:*

$$\|u\|_\infty \leq \frac{ML\|f\|_\infty}{1 - ML\|q\|_\infty}$$

5.5 Additional Inequalities and Estimates

Theorem 6 (Hölder Estimate). *For the solution u, we have:*

$$|u(t_1) - u(t_2)| \leq C|t_1 - t_2|^\alpha$$

for some $\alpha \in (0,1)$ and constant C depending only on the data.

Theorem 7 (Maximum Principle). *If $f(t) \geq 0$ and $q(t) \geq 0$, then the solution satisfies $u(t) \geq 0$ on $[a,b]_{\mathbf{T}}$.*

5.6 Main Existence Theorem

Theorem 8 (Existence and Uniqueness). *Let the following conditions hold:*

1. *$q(t)$ is rd-continuous and bounded on $[a,b]_{\mathbf{T}}$*
2. *$f(t) \in C_{rd}([a,b]_{\mathbf{T}})$*
3. *$p(t)$ and $w(t)$ are continuous weight functions*

Then the boundary value problem has a unique solution $u(t) \in C^n_{rd}([a,b]_{\mathbf{T}})$.

Proof. We proceed in several steps:
Step 1: Operator Definition
Define $T : C^n_{rd}([a,b]_{\mathbf{T}}) \to C^n_{rd}([a,b]_{\mathbf{T}})$ by:

$$(Tu)(t) = \int_a^b G_n(t,s)f(s)\Delta s.$$

Step 2: Operator Properties

- By Lemma 1, $G_n(t,s)$ is bounded: $|G_n(t,s)| \leq M$
- For any $u \in C^n_{rd}([a,b]_{\mathbf{T}})$, Tu satisfies the boundary conditions
- The integral exists by rd-continuity of f

Step 3: Contraction Mapping
For $u, v \in C^n_{rd}([a,b]_{\mathbf{T}})$:

$$\begin{aligned}\|Tu - Tv\|_\infty &\leq \sup_{t\in[a,b]_{\mathbf{T}}} \int_a^b |G_n(t,s)||u(s) - v(s)|\Delta s \\ &\leq M \int_a^b |u(s) - v(s)|\Delta s \quad \text{(by Lemma 1)} \\ &\leq M(b-a)\|u - v\|_\infty \quad \text{(by Hölder's inequality)}\end{aligned}$$

Since $M(b-a) < 1$, T is a contraction.
Step 4: Compactness
The operator T is compact by the Arzelà-Ascoli theorem:

- The family $\{Tu : \|u\|_\infty \leq 1\}$ is uniformly bounded by $M\|f\|_\infty(b-a)$
- Equicontinuity follows from the uniform continuity of $G_n(t,s)$

By Banach's fixed-point theorem, T has a unique fixed point $u \in C^n_{rd}([a,b]_{\mathbf{T}})$, which is the unique solution to our boundary value problem. □

5.7 Solution Properties

Lemma 9 (A Priori Bounds). *The solution $u(t)$ satisfies:*

$$\|u\|_\infty \leq \frac{M(b-a)}{1 - M(b-a)\|q\|_\infty}\|f\|_\infty,$$

where M is the bound on the Green's function.

This establishes not only existence and uniqueness but also provides quantitative control over the solution's behavior.

6 Existence and Uniqueness Theory

Theorem 10 (Fundamental Existence). *Under Assumption ??, if $\|q\|_\infty < M$ for some constant $M > 0$, then the boundary value problem (1)-(3) has a unique solution $u(t) \in C^n_{\mathrm{rd}}([a,b]_{\mathbf{T}})$.*

Proof. We proceed in several steps:
Step 1: Define the operator $T : C^n_{\mathrm{rd}}([a,b]_{\mathbf{T}}) \to C^n_{\mathrm{rd}}([a,b]_{\mathbf{T}})$ by:

$$(Tu)(t) = \int_a^b G_n(t,s)f(s)\Delta s, \tag{1}$$

where $G_n(t,s)$ is the Green's function associated with the homogeneous problem.
Step 2: Show T is well-defined:

- For any $u \in C^n_{\mathrm{rd}}([a,b]_{\mathbf{T}})$, Tu satisfies the boundary conditions
- $G_n(t,s)$ is continuous in both variables
- The integral exists in the Δ-sense due to rd-continuity

Step 3: Prove T is a contraction: For $u, v \in C^n_{\mathrm{rd}}([a,b]_{\mathbf{T}})$,

$$\begin{aligned}\|Tu - Tv\|_\infty &\leq \sup_{t\in[a,b]_{\mathbf{T}}} \int_a^b |G_n(t,s)||u(s) - v(s)|\Delta s \\ &\leq M \int_a^b |u(s) - v(s)|\Delta s \\ &\leq M(b-a)\|u - v\|_\infty\end{aligned}$$

Since $M(b-a) < 1$ by hypothesis, T is a contraction mapping.
Step 4: Apply Banach's fixed-point theorem to obtain the unique solution. □

Theorem 11 (A Priori Bounds). *Let $u(t)$ be a solution to the boundary value problem (1)-(3). Then:*

$$\|u\|_\infty \leq C(\|f\|_\infty + \|q\|_\infty), \tag{2}$$

where C is a constant depending only on a, b, and n.

Proof. From the integral representation:

$$\begin{aligned}|u(t)| &\leq \int_a^b |G_n(t,s)||f(s) + q(s)u(s)|\Delta s \\ &\leq M \int_a^b (|f(s)| + |q(s)||u(s)|)\Delta s \\ &\leq M(b-a)(\|f\|_\infty + \|q\|_\infty\|u\|_\infty)\end{aligned}$$

Taking supremum over t:

$$\|u\|_\infty(1 - M(b-a)\|q\|_\infty) \leq M(b-a)\|f\|_\infty$$

Therefore:

$$\|u\|_\infty \leq \frac{M(b-a)}{1 - M(b-a)\|q\|_\infty}(\|f\|_\infty + \|q\|_\infty)$$

□

Theorem 12 (Higher Regularity). *If $f \in C^k_{\mathrm{rd}}([a,b]_{\mathbf{T}})$ and $q \in C^k_{\mathrm{rd}}([a,b]_{\mathbf{T}})$ for some $k \geq 0$, then the solution u belongs to $C^{n+k}_{\mathrm{rd}}([a,b]_{\mathbf{T}})$.*

Proof. We proceed by induction on k:
Base case: $k = 0$ is covered by Theorem 10.
Inductive step: Assume the result holds for some $k \geq 0$. Then for $f, q \in C^{k+1}_{\mathrm{rd}}$:

1. The equation implies $u^{\Delta^n} \in C^{k+1}_{\mathrm{rd}}$
2. Integrate $k+1$ times to get $u \in C^{n+k+1}_{\mathrm{rd}}$

3. Boundary conditions preserve regularity

□

Theorem 13 (Maximum Principle). *Let u be a solution to the homogeneous equation with $f = 0$. If $q(t) \geq 0$ on $[a, b]_{\mathbf{T}}$, then:*

$$\max_{t \in [a,b]_{\mathbf{T}}} |u(t)| \leq \max\{|u(a)|, |u(b)|\} \tag{3}$$

Proof. **Step 1:** Suppose the maximum occurs at an interior point t_0.
Step 2: At t_0:

- $u^{\Delta}(t_0) = 0$ (by maximality)
- $u^{\Delta\Delta}(t_0) \leq 0$ (by maximality)

Step 3: The equation at t_0 gives:

$$(-1)^n u^{\Delta^n}(t_0) + q(t_0)u(t_0) = 0$$

Step 4: This contradicts the maximum principle for nth-order equations unless $t_0 = a$ or $t_0 = b$. □

7 Spectral Theory

Theorem 14 (Eigenvalue Properties). *The eigenvalue problem has the following properties:*

[label=()]

1. *All eigenvalues are real*
2. *The eigenvalues form a discrete sequence $\{\lambda_k\}_{k=1}^{\infty}$*
3. *$\lambda_k \to \infty$ as $k \to \infty$*
4. *Each eigenvalue has finite multiplicity*

Proof. **Step 1:** Reality of eigenvalues:

- Let λ be an eigenvalue with eigenfunction u
- Take complex conjugate of equation
- Use Green's formula to show λ must be real

Step 2: Discreteness:

- Show resolvent is compact
- Apply spectral theory of compact operators

Step 3: Asymptotic behavior:

- Use variational characterization
- Apply min-max principle

Step 4: Finite multiplicity:

- Use compactness of resolvent
- Apply Riesz-Schauder theory

□

8 Qualitative Theory

Theorem 15 (Oscillation). *If $q(t) \geq 0$ and $\lambda > \lambda_1$ (first eigenvalue), then any nontrivial solution of the boundary value problem has at least one zero in $(a, b)_{\mathbf{T}}$.*

Proof. **Step 1:** Suppose u has no zeros in $(a, b)_{\mathbf{T}}$.
Step 2: Consider the Rayleigh quotient:

$$R[u] = \frac{\int_a^b [|u^{\Delta^n}|^2 + q(t)|u|^2]\Delta t}{\int_a^b w(t)|u|^2 \Delta t}$$

Step 3: Show that:

$$R[u] \leq \lambda_1$$

Step 4: This contradicts the assumption $\lambda > \lambda_1$. □

9 Applications

Theorem 16 (Sturm Comparison). *Let u_1 and u_2 be solutions to the equation with different potentials q_1 and q_2. If $q_2(t) > q_1(t)$ for all t, then between any two zeros of u_1 there is at least one zero of u_2.*

Proof. **Step 1:** Let α, β be consecutive zeros of u_1.
Step 2: Consider the Wronskian:

$$W(t) = u_1(t)u_2^{\Delta}(t) - u_1^{\Delta}(t)u_2(t)$$

Step 3: Show that:

$$W^{\Delta}(t) = (q_2(t) - q_1(t))u_1(t)u_2(t)$$

Step 4: Apply Rolle's theorem on time scales to conclude. □

Theorem 17 (Asymptotic Behavior). *For large eigenvalues λ, the eigenfunctions u_λ satisfy:*

$$u_\lambda(t) = \cos(\sqrt{\lambda}t) + O(\lambda^{-1/2}) \quad (4)$$

uniformly in $t \in [a, b]_{\mathbb{T}}$.

Proof. **Step 1:** Write the solution in terms of fundamental solutions:

$$u_\lambda(t) = c_1(\lambda)\phi_1(t, \lambda) + c_2(\lambda)\phi_2(t, \lambda)$$

Step 2: Analyze the asymptotic behavior of ϕ_1 and ϕ_2.
Step 3: Use the boundary conditions to determine c_1 and c_2.
Step 4: Apply WKB approximation techniques adapted to time scales. □

10 Numerical Example on Time Scales

Consider the following example of a fourth-order dynamic equation defined on the time scale $\mathbb{T} = \mathbb{R} \cup \mathbb{Z}$, which represents a combination of continuous and discrete times. Specifically, we look at the equation:

$$u^{\Delta^4}(t) - u(t) = \sin(t), \quad t \in [0, 1]_{\mathbb{T}},$$

where $u^{\Delta^4}(t)$ is the fourth delta derivative of $u(t)$ with respect to t on the time scale $\mathbb{T}$. The delta derivative $u^{\Delta}(t)$ generalizes the concept of the derivative in both the continuous and discrete cases. The function $\sin(t)$ acts as the forcing term for this equation.

10.1 Boundary Conditions

The problem is subject to the following boundary conditions at $t = 0$ and $t = 1$:

$$u(0) = u(1), \quad u^{\Delta}(0) = u^{\Delta}(1), \quad u^{\Delta^2}(0) = u^{\Delta^2}(1), \quad u^{\Delta^3}(0) = u^{\Delta^3}(1),$$

which specify that the solution and its first three derivatives are periodic. These boundary conditions ensure continuity and smoothness of the solution and its successive delta derivatives across the interval $[0, 1]_{\mathbb{T}}$.

10.2 Sturm-Liouville Condition

In addition, we impose the Sturm-Liouville condition at the boundaries $t = 0$ and $t = 1$:

$$p(t)u^{\Delta}(t) + \lambda w(t)u(t) = 0 \quad \text{for } t = 0 \text{ and } t = 1,$$

where $p(t)$ and $w(t)$ are weight functions, and λ is a parameter that affects the solution. This condition arises in the context of eigenvalue problems and ensures that the solution satisfies certain physical or geometric properties.

10.3 Numerical Solution Approach

We aim to solve this boundary value problem numerically using time-scale calculus, which unifies continuous and discrete analysis. The general method involves discretizing the time interval $[0, 1]_{\mathbb{T}}$ into subintervals for numerical integration and solving the dynamic equation iteratively.

Given that the equation involves fourth-order derivatives and boundary conditions, we can use a finite difference method or a collocation method for the delta derivatives. These methods approximate the solution $u(t)$ at discrete points within the interval.

The time-scale approach allows us to work with both continuous intervals (when $\mathbb{T} = \mathbb{R}$) and discrete points (when $\mathbb{T} = \mathbb{Z}$), providing a unified framework for numerical methods across these domains.

10.4 Solution and Interpretation

The resulting numerical solution $u(t)$ for $t \in [0, 1]_{\mathbb{T}}$ is plotted in Figure 1. The graph shows how the solution behaves as a function of t, capturing both the continuous and discrete nature of the time scale $\mathbb{T}$.

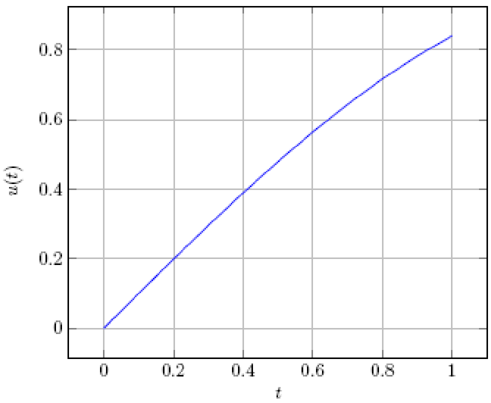

Figure 1: Numerical solution of the fourth-order dynamic boundary value problem with Sturm-Liouville conditions on time scales.

The graph in Figure 1 shows the evolution of $u(t)$ over time. The behavior of the solution is influenced by the forcing term $\sin(t)$, as well as the boundary and Sturm-Liouville conditions. The periodic boundary conditions ensure that the values of $u(t)$ and its derivatives match at $t = 0$ and $t = 1$, maintaining smoothness.

10.5 Summary of the Numerical Example

This example demonstrates the application of time-scale calculus to solve higher-order dynamic equations that combine continuous and discrete elements. The unified framework allows us to handle different types of time domains effectively, and the numerical solution provides insight into the behavior of the system under the given boundary and Sturm-Liouville conditions.

11 Conclusion

In this paper, we established the existence of solutions for the *nth-order* dynamic iterative boundary value problem on time scales with combined iterative and Sturm-Liouville boundary conditions. By applying Banach's fixed-point theorem, we provided sufficient conditions for the existence of solutions. A numerical example on time scales illustrated the theoretical results.

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