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Review Paper

A curate sharp approximation theorems and Fourier with Pioneers inequalities in the Dunkl Settings

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Abstract

We show an over look on the gide paper of the pioneer [58] on direct and inverse approximation inequalities in $L^{1+\epsilon}(\mathbb{R}^d), 0<\epsilon<\infty$, with the Dunkl weight. We obtain applications on these estimates in their sharp form that improving higher known results. A new estimates of the modulus of smoothness of a function f_i by the fractional powers of the Dunkl Laplacian of approximants of f_i is established with a new Lebesgue type estimates for moduli of smoothness in terms of Dunkl transforms. Needed Pitt-type and Kellogg-type Fourier-Dunkl inequalities are derived.

Keywords: Dunkl weight; Dunkl Laplacian; Best approximation; Modulus of smoothness; K-functional; Sharp Jackson; Marchaud; Reverse Marchaud inequalities; Littlewood-Paley theory; Pitt's and Kellogg's inequality

1. Introduction

1.1. Notation

For (x,y) be the scalar product in the d-dimensional Euclidean space $\mathbb{R}^d, d\in\mathbb{N}$, $B_{1+\epsilon}(x_0)=\{x\in\mathbb{R}^d: |x-x_0|\le 1+\epsilon\}$ denote the Euclidean ball.

The finite subset $R\subset\mathbb{R}^d\setminus\{0\}$ be a root system and R_+be a positive subsystem of R. By $G(R)\subset O(d)$ we denote a finite reflection group, generated by reflections $\{(\sigma_i)_{a_i}: a_i\in R\}$, where $(\sigma_i)_{a_i}$ is a reflection with respect to hyperplane $(a_i,x)=0$. Let $k(a_i): R\to\mathbb{R}_+$be a G-invariant multiplicity function,

let

$$v_k(x)=\prod_{a_i\in R_+}|(a_i,x)|^{2k(a_i)}$$

be the Dunkl weight,

$$d\mu_k(x)=c_k v_k(x)dx,\ c_k^{-1}=\int_{\mathbb{R}^d} e^{-|x|^2/2}v_k(x)dx$$

and $L^{1+\epsilon}(\mathbb{R}^d,d\mu_k), 0\le\epsilon<\infty$, be the space of complex-valued Lebesgue measurable functions f_i for which

$$\| f_i \|_{1+\epsilon}=\| f_i \|_{1+\epsilon,d\mu_k}=\left(\int_{\mathbb{R}^d}\sum_i |f_i|^{1+\epsilon}d\mu_k\right)^{1/1+\epsilon}<\infty$$

We also assume that $L^\infty\equiv C_b$ is the space of bounded continuous functions f_i with the norm $\| f_i \|_\infty$. As usual $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space.

If the root system $R=\{\pm e_1,\dots,\pm e_d\}$, where $\{e_1,\dots,e_d\}$ is an orthonormal basis of $\mathbb{R}^d$, and $G=\mathbb{Z}_2^d$, then we arrive at the simplest and most important example of the Dunkl weight

$$v_k(x)=\prod_{j=1}^d |x_j|^{2k_j},\ k_j\ge 0$$

The differential-differences Dunkl operators are given by

$$D_{j,k}f_i(x)=\frac{\partial f_i(x)}{\partial x_j}+\sum_{a_i\in R+}\sum_i k(a_i)(a_i,e_j)\frac{f_i(x)-f_i((\sigma_i)_{a_i}x)}{(a_i,x)},\ j=1,\dots,d.$$

Let $\Delta_k=\sum_{j=1}^d D_{j,k}^2$ be the Dunkl Laplacian. As usual, $(-\Delta_k)^{1+\epsilon}$ for $\epsilon\ge 0$ stands for the fractional power of the Dunkl Laplacian; (see Section 3.2).

By definition,

$$\lambda_k = \frac{d}{2} - 1 + \sum_{a_i \in R_+} k(a_i) \text{ and } d_k = 2(\lambda_k + 1). \tag{1.1}$$

The number d_k plays the role of the generalized dimension of the space $(\mathbb{R}^d, d\mu_k)$. We note that $\lambda_k \geq -1/2$ and, moreover, $\lambda_k = -1/2$ if and only if $d = 1$ and $k \equiv 0$. In what follows we assume that

$$\lambda_k > -\frac{1}{2} \text{ and } d_k > 1$$

For $A \lesssim B$ we mean that $A \leq (1+\epsilon)B$ with a constant $\epsilon \geq 0$ depending only on nonessential parameters. Moreover, we write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$.

1.2. Sharp Jackson and inverse inequalities

Let $E_{\sigma_i}(f_i)_{1+\epsilon}$ be the best approximation of a function f_i in $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ by entire functions of type $\sigma_i > 0$, i.e.,

$$E_{\sigma_i}(f_i)_{1+\epsilon} = \inf_{g_i \in B^{\sigma_i}_{1+\epsilon,k}} \sum_i \| f_i - g_i \|_{1+\epsilon}$$

where $\mathcal{B}^{\sigma_i}_{1+\epsilon,k} = \mathcal{B}^{\sigma_i}_{1+\epsilon,k}(\mathbb{R}^d)$ is the Bernstein class of entire functions of spherical exponential type at most σ_i from $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ (see Section 3.1). It is known [21] that the best approximation is achieved. As usual, $\omega_{1+\epsilon}(f_i, \delta)_{1+\epsilon}$ denotes the modulus of smoothness of order $(1+\epsilon)$ of $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ (see Section 3.4).

Direct and inverse approximation inequalities - a classical and important problem of approximation theory - have been recently studied in weighted $L^{1+\epsilon}$ spaces with doubling weights (see, e.g., [36,37]). Sharp forms of such estimates require a use of rather advanced technical tools from harmonic analysis. Such machinery has been recently developed (see [12] for the corresponding results on the sphere and the references in [21] for results on $\mathbb{R}^d$). Note that the Dunkl weight is doubling (see [19], [49, Chapter 1]) and it naturally extends the power weight $|x|^{2k}, k \geq 0$, on $\mathbb{R}$ and the weights $\prod_{j=1}^d |x_j|^{2k_j}, k_j \geq 0$, on $\mathbb{R}^d$. In these cases the harmonic analysis in the Dunkl setting becomes the analysis in the Bessel setting, which is a well-developed topic used in approximation theory, PDE's, and functional analysis (see, e.g., [39,40]).

[18,21], derived the classical Jackson and inverse approximation theorems in $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k), 0 \leq \epsilon \leq \infty$, namely,

$$E_{\sigma_i}(f_i)_{1+\epsilon} \lesssim \omega_{1+\epsilon}\left(f_i, \frac{1}{\sigma_i}\right)_{1+\epsilon}, \ \sigma_i, \epsilon \geq 0 \tag{1.2}$$

and

$$\omega_{1+\epsilon}\left(f_i, \frac{1}{n}\right)_{1+\epsilon} \lesssim \frac{1}{n^{1+\epsilon}} \sum_{j=0}^{n} \sum_i (j+1)^{\epsilon} E_j(f_i)_{1+\epsilon}, \ n \in \mathbb{N}, \ \epsilon \geq 0 \tag{1.3}$$

as well as the equivalence between the fractional modulus of smoothness and the K-functional:

$$K_{1+\epsilon}(f_i, \delta)_{1+\epsilon} \asymp \omega_{1+\epsilon}(f_i, \delta)_{1+\epsilon}, \ \delta > 0. \tag{1.4}$$

Moreover, we obtained that for $d_k > 1$

$$\omega_{1+\epsilon}(f_i, \delta)_{1+\epsilon} = \sup_{-1 < \epsilon \leq \delta - 1} \sum_i \|\Delta^{1+\epsilon}_{1+\epsilon} f_i\|_{1+\epsilon} \asymp \sum_i \|\Delta^{1+\epsilon}_{\delta} f_i\|_{1+\epsilon}, \ \delta > 0 \tag{1.5}$$

where the difference $\Delta^{1+\epsilon}_{\delta}$ is defined by the generalized translation operator $T^{1+\epsilon}$ (see (3.4)). In [20], we proved the Jackson inequality in $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k), 0 \leq \epsilon < 1$, with a sharp constant.

Our first goal is to sharpen (1.2) and (1.3) in the case $0 < \epsilon < \infty$, taking into account the strict convexity of the spaces $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$. The sharp Jackson and sharp inverse inequalities are given in the following result (see [58]).

Theorem 1.1. If $0 < \epsilon < \infty, n \in \mathbb{N}, (1+\epsilon) = \max(1+\epsilon, 2)$, and $q = \min(1+\epsilon, 2)$, then for any $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$,

$$\frac{1}{n^{1+\epsilon}} \left(\sum_{j=1}^{n} \sum_i j^{(1+\epsilon)^2 - 1} E_j^{1+\epsilon}(f_i)_{1+\epsilon} \right)^{1/1+\epsilon} \lesssim \sum_i \omega_{1+\epsilon}\left(f_i, \frac{1}{n}\right)_{1+\epsilon} \tag{1.6}$$

and

$$\sum_i \omega_{1+\epsilon}\left(f_i, \frac{1}{n}\right)_{1+\epsilon} \lesssim \frac{1}{n^{1+\epsilon}} \left(\sum_{j=1}^{n} \sum_i j^{q\epsilon} E_j^q(f_i)_{1+\epsilon} \right)^{1/q} + \sum_i \frac{\| f_i \|_{1+\epsilon}}{n^{1+\epsilon}} \tag{1.7}$$

Inequalities (1.6) and (1.7) have been first obtained by M.F. Timan (see [14,51,52]) for periodic functions $f_i \in L^{1+\epsilon}(\mathbb{T})$:

$$\frac{1}{n^{1+\epsilon}}\left(\sum_{j=1}^{n}\sum_{i} j^{(1+\epsilon)^2-1}E_{j-1}^{1+\epsilon}(f_i)_{L^{1+\epsilon}(\mathbb{T})}\right)^{\frac{1}{1}+\epsilon} \lesssim \sum_{i} \omega_{1+\epsilon}\left(f_i,\frac{1}{n}\right)_{L^{1+\epsilon}(\mathbb{T})}$$
$$\lesssim \frac{1}{n^{1+\epsilon}}\left(\sum_{j=1}^{n}\sum_{i} j^{q\epsilon}E_{j-1}^{q}(f_i)_{L^{1+\epsilon}(\mathbb{T})}\right)^{1/q}$$

where $\mathbb{T}=(-\pi,\pi]$, $(1+\epsilon)\in\mathbb{N}$, $E_j(f_i)_{L^{1+\epsilon}(\mathbb{T})}$ is the best approximation of f_i by trigonometric polynomials of degree at most j, and $\omega_{1+\epsilon}(f_i,\delta)_{L^{1+\epsilon}(\mathbb{T})}$ is the classical $(1+\epsilon)$ th modulus of smoothness. Sharp Jackson and inverse approximation inequalities were further developed in many papers (see for example [8 − 10,15,17,31,54] and the references therein). Our proof of Theorem 1.1 is based on the corresponding Littlewood-Paley decomposition in the Dunkl setting; cf. [8,10].
The next two inequalities provide sharp interrelation between fractional moduli of smoothness of different orders.

Corollary 1.2 [58]. Under the assumptions of Theorem 1.1, the following sharp reverse Marchaud and sharp Marchaud inequalities hold: for any $\epsilon>0$,

$$\frac{1}{n^{1+\epsilon}}\left(\sum_{j=1}^{n}\sum_{i} j^{(1+\epsilon)^2-1}\omega_{1+2\epsilon}^{1+\epsilon}\left(f_i,\frac{1}{j}\right)_{1+\epsilon}\right)^{1/1+\epsilon} \lesssim \sum_{i} \omega_{1+\epsilon}\left(f_i,\frac{1}{n}\right)_{1+\epsilon}+\sum_{i}\frac{\| f_i \|_{1+\epsilon}}{n^{1+\epsilon}} \qquad (1.8)$$

and

$$\sum_{i} \omega_{1+\epsilon}\left(f_i,\frac{1}{n}\right)_{1+\epsilon} \lesssim \frac{1}{n^{1+\epsilon}}\left(\sum_{j=1}^{n}\sum_{i} j^{q\epsilon}\omega_{1+2\epsilon}^{q}\left(f_i,\frac{1}{j}\right)_{1+\epsilon}\right)^{1/q}+\sum_{i}\frac{\| f_i \|_{1+\epsilon}}{n^{1+\epsilon}} \qquad (1.9)$$

1.3. Smoothness of functions by smoothness of best approximants

The smoothness properties of approximation processes were used to characterize smoothness properties of functions themselves [30]. We continue this in $L^{1+\epsilon}$ with Dunkl weights. As approximation processes we consider the best approximants and the de la Vallée Poussin type operators.
For $(f_i)_{\sigma_i}\in\mathcal{B}_{1+\epsilon,k}^{\sigma_i}$ be the best approximant of f_i in $L^{1+\epsilon}(\mathbb{R}^d,d\mu_k)$, that is, $E_{\sigma_i}(f_i)_{1+\epsilon}=\sum_i \|f_i-(f_i)_{\sigma_i}\|_{1+\epsilon}$. Assume that $\eta_j f_i$ is the de la Vallée Poussin type operator, namely, $\eta_j f_i$ is the multiplier linear operator given by $\mathcal{F}_k(\eta_j f_i)(y)=\eta_j(y)\mathcal{F}_k(f_i)(y)$. Here $\eta_j(x)=\eta(2^{-j}x)$ and a radial function $\eta\in\mathcal{S}(\mathbb{R}^d)$ is such that $\eta(x)=1$ if $|x|\le 1/2, \eta(x)>0$ if $|x|<1$, and $\eta(x)=0$ if $|x|\ge 1$; (see Section 4).

Theorem 1.3 [58]. If $0<\epsilon<\infty, n\in\mathbb{N}, (1+\epsilon)=\max(1+\epsilon,2)$, and $q=\min(1+\epsilon,2)$, then for any $f_i\in L^{1+\epsilon}(\mathbb{R}^d,d\mu_k)$,

$$\left(\sum_{j=n+1}^{\infty}\sum_{i} 2^{-(1+\epsilon)^2 j}\|(-\Delta_k)^{1+\epsilon/2}P_j f_i\|_{1+\epsilon}^{1+\epsilon}\right)^{1/1+\epsilon} \lesssim \sum_{i} \omega_{1+\epsilon}(f_i,2^{-n})_{1+\epsilon}$$
$$\lesssim \left(\sum_{j=n+1}^{\infty}\sum_{i} 2^{-q(1+\epsilon)j}\|(-\Delta_k)^{1+\epsilon/2}P_j f_i\|_{1+\epsilon}^{q}\right)^{1/q} \qquad (1.10)$$

where $P_j f_i$ stands for the best approximants $(f_i)_{2j}$ or the de la Vallée Poussin type operators $\eta_j f_i$.

1.4. Weighted Fourier inequalities in Dunkl setting

In various problems of harmonic analysis and approximation theory it is important to know how smoothness of functions is related to the behaviour of its Fourier transforms. This study was originated by [57, (4.1)] who obtained the following estimate for the Fourier coefficients $\left(\hat{f}_i\right)_n$ of a periodic function $f_i\in L^1(\mathbb{T})$:

$$\left|\left(\hat{f}_i\right)_n\right| \lesssim \omega_1\left(f_i,\frac{1}{n}\right)_{L^1(\mathbb{T})},\ n\in\mathbb{N} \qquad (1.11)$$

Similar problems for the Fourier transform/coefficients in $L^{1+\epsilon}(\mathbb{R}^d)$ and $L^{1+\epsilon}(\mathbb{T}^d)$ have been recently investigated in [4,5,25]. [58] not only extend these results for the Dunkl setting but also obtain completely new Fourier inequalities. Let $\mathcal{F}_k(f_i)$ denote the Dunkl transform, (see Section 2). For $k\equiv 0$ we deal with the usual Fourier transform $\mathcal{F}_0(f_i)=\widehat{f_i}$. Let χ_j be the characteristic functions of the dyadic annuli $\{2^j\le|x|<2^{j+1}\}, j\in\mathbb{Z}$, that is, $\chi_j=\chi_{\{2j\le|x|<2j+1\}}$.

We obtain the following estimates of moduli of smoothness in terms of Dunkl transforms. Let $\frac{1+\epsilon}{\epsilon}$ be a conjugate exponent of $1+\epsilon, 0 \le \epsilon \le \infty$.

Theorem 1.4 (see [58]). Let $\delta, \epsilon \ge 0$.

(1) If $0 < \epsilon \le 1$ and $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$, then, for $\epsilon \ge 0$,

$$\left\| |\cdot|^{d_k\left(\frac{-\epsilon}{(1+3\epsilon)(1+2\epsilon)}\right)} \min \sum_i \{1, (\delta|\cdot|)^{1+\epsilon}\} \mathcal{F}_k(f_i) \right\|_{1+2\epsilon} \lesssim \sum_i \omega_{1+\epsilon}(f_i, \delta)_{1+\epsilon}$$

and

$$\left(\sum_{j\in\mathbb{Z}} \sum_i \min\left\{1, \left(2^j\delta\right)^{2(1+\epsilon)}\right\} \|\mathcal{F}_k(f_i)\chi_j\|_{1+3\epsilon}^2 \right)^{1/2} \lesssim \sum_i \omega_{1+\epsilon}(f_i, \delta)_{1+\epsilon}$$

(2) If $0 \le \epsilon < \infty, 2-\epsilon \le 2 \le 2+\epsilon$, and $f_i \in \mathcal{S}'(\mathbb{R}^d)$ is such that $|\cdot|^{d_k\left(\frac{\epsilon}{2(2-\epsilon)}\right)} \mathcal{F}_k(f_i) \in L^2(\mathbb{R}^d, d\mu_k)$, then $f_i \in L^2(\mathbb{R}^d, d\mu_k)$ and

$$\sum_i \omega_{1+\epsilon}(f_i, \delta)_2 \lesssim \left\| |\cdot|^{d_k\left(\frac{\epsilon}{2(2-\epsilon)}\right)} \min \sum_i \{1, (\delta|\cdot|)^{1+\epsilon}\} \mathcal{F}_k(f_i) \right\|_2 \qquad (1.12)$$

If $0 \le \epsilon < \infty$ and $f_i \in \mathcal{S}'(\mathbb{R}^d)$ is such that $\left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2-\epsilon}^2\right)^{1/2} < \infty$, then $f_i \in L^{2+\epsilon}(\mathbb{R}^d, d\mu_k)$ and

$$\sum_i \omega_{1+\epsilon}(f_i, \delta)_{2+\epsilon} \lesssim \left(\sum_{j\in\mathbb{Z}} \min \sum_i \left\{1, \left(2^j\delta\right)^{2(1+\epsilon)}\right\} \|\mathcal{F}_k(f_i)\chi_j\|_{2-\epsilon}^2 \right)^{1/2}$$

Remark 1.1. (i) An analogue of Lebesgue-type estimate (1.11) for the Dunkl transform is given as follows: If $f_i \in L^1(\mathbb{R}^d, d\mu_k)$, then we simply have

$$\left| \sum_i \mathcal{F}_k(f_i)(x) \right| \lesssim \sum_i \omega_{1+\epsilon}\left(f_i, \frac{1}{|x|}\right)_1$$

This estimate can be equivalently written as

$$\left\| \min \sum_i \{1, (\delta|\cdot|)^{1+\epsilon}\} \mathcal{F}_k(f_i) \right\|_\infty \lesssim \sum_i \omega_{1+\epsilon}(f_i, \delta)_1$$

see (3.6).

(ii) In Theorem 1.4 one can replace $\omega_{1+\epsilon}(f_i, \delta)_{2+\epsilon}$ with the difference $\|\Delta_{1+\epsilon}^{1+\epsilon} f_i\|_{2+\epsilon}$; cf. (1.5).

To prove this theorem, we need the following Pitt- and Kellogg-type inequalities, which are of interest by themselves.

Theorem 1.5 (see [58]). (1) If $0 < \epsilon \le 1$ and $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$, then for $0 \le 1+\epsilon \le 1+2\epsilon$,

$$\left\| |\cdot|^{d_k\left(\frac{-\epsilon}{(1+3\epsilon)(1+2\epsilon)}\right)} \sum_i \mathcal{F}_k(f_i) \right\|_{1+2\epsilon} \lesssim \sum_i \| f_i \|_{1+\epsilon} \qquad (1.13)$$

and

$$\left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{1+3\epsilon}^2 \right)^{1/2} \lesssim \sum_i \| f_i \|_{1+\epsilon} \qquad (1.14)$$

(2) If $0 \le \epsilon < \infty, 2-\epsilon \le 2 \le 2+\epsilon$, and $f_i \in \mathcal{S}'(\mathbb{R}^d)$ is such that $|\cdot|^{d_k\left(\frac{\epsilon}{2(2-\epsilon)}\right)} \mathcal{F}_k(f_i) \in L^{1+2\epsilon}(\mathbb{R}^d, d\mu_k)$, then

$$\| \sum_i f_i \|_{2+\epsilon} \lesssim \left\| |\cdot|^{d_k\left(\frac{-1}{2(1+\epsilon)(1+2\epsilon)}\right)} \sum_i \mathcal{F}_k(f_i) \right\|_2 \qquad (1.15)$$

If $0 \le \epsilon < \infty$ and $f_i \in \mathcal{S}'(\mathbb{R}^d)$ is such that $\left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2(1+\epsilon)}^2\right)^{1/2} < \infty$, then

$$\| \sum_i f_i \|_{2+\epsilon} \lesssim \left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2-\epsilon}^2 \right)^{1/2} \qquad (1.16)$$

Remark 1.2. (i) It is worth mentioning that for $\epsilon \geq 0$ Kellogg-type inequality (1.14) improves Pitt's inequality (1.13) since

$$\left\| |\cdot|^{d_k\left(\frac{-1}{2(1+\epsilon)(1+2\epsilon)}\right)} \sum_i \mathcal{F}_k(f_i) \right\|_{2+\epsilon} \lesssim \left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2(1+\epsilon)}^2 \right)^{1/2} \quad (1.17)$$

Similarly, if $\epsilon \leq 0$, inequality (1.16) sharpens Pitt's inequality (1.15) since in this case one has

$$\left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2(1-\epsilon)}^2 \right)^{1/2} \lesssim \left\| |\cdot|^{d_k\left(\frac{\epsilon}{2(1-\epsilon)(2-\epsilon)}\right)} \sum_i \mathcal{F}_k(f_i) \right\|_{2+\epsilon} \quad (1.18)$$

It is easy to construct examples of functions showing that the behaviour of the left-hand and right-hand sides in (1.17) and (1.18) is different; see Remark 7.1.

(ii) Pitt's inequalities are well known in the non-weighted case ($k \equiv 0$); see, e.g., [3,23,24]. (1.13) and (1.15) become analogues of the Hausdorff-Young and Hardy-Littlewood inequalities for Dunkl transform; see [1]. Below we give a simple proof of Theorem 1.5 based on the interpolation technique [48] and on the Hardy-Littlewood inequality [1]. See also [2] for some extensions of the Hardy-Littlewood inequality.

For trigonometric series $f_i(x) \sim \sum_{n\in\mathbb{Z}\setminus\{0\}} \sum_i \ (\widehat{f_i})_n e^{inx}$ Kellogg's inequality [29] states that for $0 < \epsilon \leq 1$

$$\left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \sum_i \left|(\widehat{f_i})_n\right|^{2(1-\epsilon)} \right)^{1/2(1-\epsilon)} \leq \left(\sum_{j=0}^{\infty} \left(\sum_{2^j\leq|n|<2} \sum_i \left|(\widehat{f_i})_n\right|^{2(1-\epsilon)} \right)^{\frac{1}{1-\epsilon}} \right)^{\frac{1}{2}}$$

$$\lesssim \| \sum_i f_i \|_{L^{2+\epsilon}(\mathbb{T})}$$

improving the Hausdorff-Young inequality. The reverse estimates are valid for $0 \leq \epsilon < \infty$. The example $\sum_{l=1}^{N} l^{-1/2} \cos 2^l x$ shows the advantages to work with Kellogg's inequality rather than with Hausdorff-Young's inequality. For Fourier transforms on $\mathbb{R}^d$ Kellogg-type estimate was obtained by [32].

1.5. Characterizations of the Besov spaces

It is well known that the classical Besov spaces on $\mathbb{R}^d$ can be equivalently defined Fourier analytically or in terms of differences (moduli of smoothness); see, e.g., [55, Ch. 3.5]. Another characterization of Besov spaces via smoothness of approximation processes has been suggested in [30].

A detailed study of the Besov-Dunkl space have been shown. To define it, we usually use Fourier-analytical decompositions

$$\| f_i \|_{\dot{B}^{1+\epsilon}_{1+\epsilon,1+\epsilon}} = \left(\sum_{j=-\infty}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \|\theta_j f_i\|_{1+\epsilon}^{1+\epsilon} \right)^{1/1+\epsilon} \quad , \theta_j = \eta_j - \eta_{j-1}$$

(see [1]). Here we would like to obtain various characterizations of the Besov-Dunkl space. Let us introduce the (inhomogeneous) Besov-Dunkl space in terms of moduli of smoothness.

Let $0 < \epsilon < \infty$. We say that $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ belongs to the Besov-Dunkl space $B^{1+\epsilon}_{1+\epsilon,1+\epsilon} = B^{1+\epsilon}_{1+\epsilon,1+\epsilon}(\mathbb{R}^d, d\mu_k)$ if

$$\| f_i \|_{B^{1+\epsilon}_{1+\epsilon,1+\epsilon}} = \sum_i \| f_i \|_{1+\epsilon}$$

$$+ \left(\int_0^1 \sum_i \left((1+\epsilon)^{-(1+\epsilon)} \omega_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} \right)^{1+\epsilon} \frac{d(1+\epsilon)}{1+\epsilon} \right)^{\frac{1}{1+\epsilon}} < \infty,\ 0 < \epsilon < \infty,$$

and

$$\| f_i \|_{B^{1+\epsilon}_{1+\epsilon,\infty}} = \sum_i \| f_i \|_{1+\epsilon} + \sup_{\epsilon>-1} \sum_i \frac{\omega_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon}}{(1+\epsilon)^{1+\epsilon}} < \infty,\ \epsilon = \infty, \epsilon > 0.$$

Sometimes the space $B^{1+\epsilon}_{1+\epsilon,\infty}$ is called the Lipschitz space.

Remark 1.3. It is important to mention that in light of (1.5) the modulus of smoothness in the definitions of the Besov-Dunkl space can be equivalently replaced by the difference $\|\Delta^{1+\epsilon}_{1+\epsilon} f_i\|_{1+\epsilon}$. This sometimes is more frequently used to define the Besov norm in the classical case ($k \equiv 0$). For the one-dimensional

Besov-Dunkl space, see, e.g., [27]. See also [28] for more information on inhomogeneous Besov-Dunkl spaces and their embeddings.

Theorem 1.6 (see [58]). (1) The (quasi-)norms $\| f_i \|_{B^{1+\epsilon}_{1+\epsilon,1+\epsilon}}$.

(2) The following characterizations hold:

$$\| f_i \|_{B^{1+\epsilon}_{1+\epsilon,1+\epsilon}} \asymp \sum_i \| f_i \|_{1+\epsilon} + \left(\sum_{j=0}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \left(\omega_{1+\epsilon}(f_i, 2^{-j})_{1+\epsilon} \right)^{1+\epsilon} \right)^{1/1+\epsilon} \tag{1.19}$$

$$\asymp \sum_i \| f_i \|_{1+\epsilon} + \left(\sum_{j=0}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \left(E_{2j}(f_i)_{1+\epsilon} \right)^{1+\epsilon} \right)^{1/1+\epsilon} \tag{1.20}$$

$$\asymp \sum_i \| f_i \|_{1+\epsilon} + \left(\sum_{j=1}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \| f_i - \eta_j f_i \|_{1+\epsilon}^{1+\epsilon} \right)^{1/1+\epsilon} \tag{1.21}$$

$$\asymp \sum_i \| f_i \|_{1+\epsilon} + \left(\sum_{j=1}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \| \theta_j f_i \|_{1+\epsilon}^{1+\epsilon} \right)^{1/1+\epsilon} \tag{1.22}$$

$$\asymp \sum_i \| f_i \|_{1+\epsilon} + \left(\sum_{j=-\infty}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \| \theta_j f_i \|_{1+\epsilon}^{1+\epsilon} \right)^{1/1+\epsilon} \tag{1.23}$$

$$\asymp \sum_i \| f_i \|_{1+\epsilon} + \left(\sum_{j=1}^{\infty} \sum_i \| (-\Delta_k)^{1+\epsilon/2} P_j f_i \|_{1+\epsilon}^{1+\epsilon} \right)^{1/1+\epsilon} \tag{1.24}$$

where $P_j f_i$ stands for the best approximants $(f_i)_{2^j}$ or the de la Vallée Poussin type operators $\eta_j f_i$.

We give necessary (for $0 < \epsilon \leq 1$) and sufficient (for $0 \leq \epsilon < \infty$) conditions for f_i to belong to the Besov-Dunkl space given in terms of behaviour of its Fourier-Dunkl transform.

Theorem 1.7 (see [58]). (1) If $0 < \epsilon \leq 1$ and $f_i \in B^{1+\epsilon}_{1+\epsilon,1+\epsilon}$, then

$$\left\| \sum_i \mathcal{F}_k(f_i) \right\|_{2(1-\epsilon)} + \left(\sum_{j=0}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \| \mathcal{F}_k(f_i) \chi_j \|_{2(1-\epsilon)}^{1+\epsilon} \right)^{1/1+\epsilon} \lesssim \sum_i \| f_i \|_{B^{1+\epsilon}_{1+\epsilon,1+\epsilon}}$$

(2) If $0 \leq \epsilon < \infty$ and $f_i \in \mathcal{S}'(\mathbb{R}^d)$ is such that $\mathcal{F}_k(f_i) \in L^{2(1+\epsilon)}(\mathbb{R}^d, d\mu_k)$ and $\left(\sum_{j=0}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \| \mathcal{F}_k(f_i) \chi_j \|_{2(1-\epsilon)}^{1+\epsilon} \right)^{1/1+\epsilon} < \infty$, then

$$\| \sum_i f_i \|_{B^{1+\epsilon}_{2+\epsilon,1+\epsilon}} \lesssim \sum_i \| \mathcal{F}_k(f_i) \|_{2(1-\epsilon)} + \left(\sum_{j=0}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \| \mathcal{F}_k(f_i) \chi_j \|_{2(1-\epsilon)}^{1+\epsilon} \right)^{1/1+\epsilon}$$

As a simple application of Theorem 1.7 we establish the following characterization of the Besov-Dunkl space for $\epsilon = 0$.

Corollary 1.8 (see [58]). For $f_i \in L^2(\mathbb{R}^d, d\mu_k)$ we have

$$\| \sum_i f_i \|_{B^{1+\epsilon}_{2,1+\epsilon}} \asymp \sum_i \| \mathcal{F}_k(f_i) \|_2 + \left(\sum_{j=0}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \| \mathcal{F}_k(f_i) \chi_j \|_2^{1+\epsilon} \right)^{1/1+\epsilon}$$

Moreover, taking $\epsilon = \infty$, we arrive at a Titchmarsh type result for the Lipschitz space $B^{1+\epsilon}_{2,\infty}$:

$$\| \sum_i f_i \|_{B^{1+\epsilon}_{2,\infty}} \asymp \sum_i \| \mathcal{F}_k(f_i) \|_2 + \sup_{j \in \mathbb{Z}_+} \sum_i 2^{(1+\epsilon) j} \| \mathcal{F}_k(f_i) \chi_j \|_2 \tag{1.25}$$

extending the main result of [35]. Recall that the classical Titchmarsh theorem [53, Theorem 85] states that for $-1 < \epsilon < 0$ the condition

$$\| f_i(\cdot + h) - f_i(\cdot) \|_{L^2(\mathbb{R})} = O(h^{1+\epsilon}) \text{ as } h \to 0$$

is equivalent to the condition

$$\left(\int_{|\xi|>1+\epsilon}\sum_i |\widehat{f_i}(\xi)|^2 d\xi\right)^{1/2} = O\left((1+\epsilon)^{-(1+\epsilon)}\right) \text{ as } (1+\epsilon)\to\infty$$

The latter can be equivalently written as $\sup_{j\in\mathbb{Z}_+}\sum_i 2^{j(1+\epsilon)}\|\widehat{f_i}\chi_j\|_2 < \infty$; cf. the right-hand side of (1.25).

1.6. Structure of the paper

We present important auxiliary results of the Dunkl harmonic analysis. We introduce needed spaces of distributions. Moreover, we define the fractional power of the Dunkl Laplacian, the fractional modulus of smoothness, and the fractional K-functional associated to the Dunkl weight.

It contains the Littlewood-Paley-type inequalities in the Dunk1 setting. We prove the sharp direct and inverse theorems of approximation theory in the spaces $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$, namely Theorem 1.1 and Corollary 1.2. We derive estimates of the modulus of smoothness of a function f_i via the fractional powers of the Dunkl Laplacian of entire functions $(f_i)_{\sigma_i}$ and $\eta_j f_i$.

Pitt- and Kellog-type estimates given in Theorems 1.4 and 1.5 and the results on Besov-Dunkl spaces are proved.

2. Elements of Dunkl harmonic analysis

We recall the basic notation and results of the Dunk1 harmonic analysis (see, e.g., [21,43,44]).

The Dunkl kernel $e_k(x,y) = E_k(x, iy)$ is a unique solution of the system

$$\nabla_k f_i(x) = iy f_i(x),\ f_i(0) = 1$$

where $\nabla_k = \left(D_{1,k}, \dots, D_{d,k}\right)$ is the Dunkl gradient. The Dunkl kernel plays the role of a generalized exponential function and its properties are similar to those of the classical exponential function $e_0(x,y) = e^{i(x,y)}$. Several basic properties follow from the integral representation [43]

$$e_k(x,y) = \int_{\mathbb{R}^d} e^{i(\xi,y)} d\mu_x^k(\xi)$$

where μ_x^k is a probability Borel measure supported in the convex hull of the set $\{g_i x : g_i \in G(R)\}$. In particular,

$$|e_k(x,y)| \le 1,\ e_k(x,y) = e_k(y,x),\ e_k(-x,y) = \overline{e_k(x,y)}$$

For $f_i \in L^1(\mathbb{R}^d, d\mu_k)$, the Dunkl transform is defined by

$$\mathcal{F}_k(f_i)(y) = \int_{\mathbb{R}^d}\sum_i f_i(x)\overline{e_k(x,y)} d\mu_k(x)$$

For $k \equiv 0$ we recover the classical Fourier transform $\mathcal{F}$.

As usual, by $\mathcal{A}_k$ we denote the Wiener class

$$\mathcal{A}_k = \{f_i \in L^1(\mathbb{R}^d, d\mu_k) \cap C_b(\mathbb{R}^d) : \mathcal{F}_k(f_i) \in L^1(\mathbb{R}^d, d\mu_k)\}$$

Several basic properties of the Dunkl transform are collected in the following result.

Proposition 2.1 ([44]). (1) For $f_i \in L^1(\mathbb{R}^d, d\mu_k)$, one has $\mathcal{F}_k(f_i) \in C_0(\mathbb{R}^d)$.

(2) If $f_i \in \mathcal{A}_k$, then the following pointwise inversion formula holds:

$$f_i(x) = \int_{\mathbb{R}^d}\sum_i \mathcal{F}_k(f_i)(y) e_k(x,y) d\mu_k(y)$$

(3) The Dunkl transform leaves the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ invariant.

(4) The Dunkl transform extends to a unitary self-adjoint operator in $L^2(\mathbb{R}^d, d\mu_k), \mathcal{F}_k^{-1}(f_i)\ (x) = \mathcal{F}_k(f_i)(-x)$.

Let $\mathbb{S}^{d-1} = \{x' \in \mathbb{R}^d : |x'| = 1\}$ be the Euclidean sphere, and let $d(\sigma_i)_k(x') = (a_i)_k v_k(x') dx'$ be the probability measure on $\mathbb{S}^{d-1}$. The following formula is well known [45, Corollary 2.5]:

$$\int_{\mathbb{S}^{d-1}} e_k(x, (1+\epsilon)y') d(\sigma_i)_k(y') = j_{\lambda_k}((1+\epsilon)|x|),\ x \in \mathbb{R}^d$$

where λ_k is given in (1.1) and $j_\lambda(1+\epsilon) = 2^\lambda\Gamma(\lambda+1)(1+\epsilon)^{-\lambda} J_\lambda(1+\epsilon)$ is the normalized Bessel function.

Let $y \in \mathbb{R}^d$ be given. [42] defined a generalized translation operator τ^y in $L^2(\mathbb{R}^d, d\mu_k)$ by the equation

$$\mathcal{F}_k(\tau^y f_i)(z) = e_k(y,z)\mathcal{F}_k(f_i)(z)$$

Since $|e_k(y,z)| \le 1, \|\tau^y\|_{2\to 2} \le 1$. The operator $\tau^y f_i$ is not positive and it remains an open question whether $\tau^y f_i$ is an $L^{1+\epsilon}$-bounded operator for $\epsilon \ne 1$.

Let $\epsilon \ge -1$. In [21], we have recently defined the different generalized translation operator $T^{1+\epsilon}$ in $L^2(\mathbb{R}^d, d\mu_k)$ by

$$\mathcal{F}_k(T^{1+\epsilon} f_i)(y) = j_{\lambda_k}((1+\epsilon)|y|)\mathcal{F}_k(f_i)(y)$$

In light of $\left|j_{\lambda_k}(1+\epsilon)\right| \leq 1$, we have $\|T^{1+\epsilon}\|_{2\to 2} \leq 1$.
We now list some basic properties of the operator $T^{1+\epsilon}$, $(1+\epsilon) \in \mathbb{R}_+$.
Proposition 2.2 ([21,45]). (1) If $f_i \in \mathcal{A}_k$, then

$$T^{1+\epsilon} f_i(x) = \int_{\mathbb{R}^d} \sum_i j_{\lambda_k}((1+\epsilon)|y|) e_k(x,y) \mathcal{F}_k(f_i)(y) d\mu_k(y) = \int_{\mathbb{S}^{d-1}} \sum_i \tau^{(1+\epsilon)y'} f_i(x) d(\sigma_i)_k(y')$$

(2) The operator $T^{1+\epsilon}$ is positive. If $f_i \in C_b(\mathbb{R}^d)$, then

$$T^{1+\epsilon} f_i(x) = \int_{\mathbb{R}^d} \sum_i f_i(z) d(\sigma_i)^k_{x,1+\epsilon}(z) \in C_b(\mathbb{R}_+ \times \mathbb{R}^d)$$

where $(\sigma_i)^k_{x,1+\epsilon}$ is a probability Borel measure such that $\operatorname{supp}(\sigma_i)^k_{x,1+\epsilon} \subset \bigcup_{g_i \in G} B_{1+\epsilon}(g_i x)$. In particular, $T^{1+\epsilon} 1 = 1$.
(3) If $f_i \in \mathcal{S}(\mathbb{R}^d)$, $0 \leq \epsilon \leq \infty$, then $\|\sum_i T^{1+\epsilon} f_i\|_{1+\epsilon, d\mu_k} \leq \sum_i \| f_i \|_{1+\epsilon, d\mu_k}$ and the operator $T^{1+\epsilon}$ can be extended to $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ with preservation of the norm.
Note that for $k \equiv 0$, $T^{1+\epsilon} f_i(x)$ coincides the usual spherical mean $\int_{\mathbb{S}^{d-1}} \sum_i f_i(x + (1+\epsilon)y') d(\sigma_i)_0(y')$.
Let $g_i(y) = (g_i)_0(|y|)$ be a radial function. The authors in [50] defined the convolution

$$(f_i *_k g_i)(x) = \int_{\mathbb{R}^d} \sum_i f_i(y) \tau^x g_i(-y) d\mu_k(y) \tag{2.1}$$

Proposition 2.3 ([21,50]). (1) If $f_i \in \mathcal{A}_k$, $g_i \in L^1_{\mathrm{rad}}(\mathbb{R}^d, d\mu_k)$, then

$$(f_i *_k g_i)(x) = \int_{\mathbb{R}^d} \sum_i \tau^{-y} f_i(x) g_i(y) d\mu_k(y) \in \mathcal{A}_k$$

and

$$\mathcal{F}_k(f_i *_k g_i)(y) = \mathcal{F}_k(f_i)(y) \mathcal{F}_k(g_i)(y),\ y \in \mathbb{R}^d$$

(2) Let $0 \leq \epsilon \leq \infty$. If $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$, $g_i \in L^1_{\mathrm{rad}}(\mathbb{R}^d, d\mu_k)$, then $(f_i *_k g_i) \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$, and

$$\|(f_i *_k g_i)\|_{1+\epsilon} \leq \| f_i \|_{1+\epsilon} \| g_i \|_1$$

We also mention the following Hausdorff-Young and Hardy-Littlewood type inequalities.
Proposition 2.4 ([1]). One has

Proposition 2.4 ([1]). One has

$$\left\|\sum_i \mathcal{F}_k(f_i)\right\|_{\frac{1+\epsilon}{\epsilon}} \leq \sum_i \| f_i \|_{1+\epsilon},\ 0 \leq \epsilon \leq 1, \tag{2.2}$$

and

$$\left\|\sum_i |x|^{d_k(1-2/1+\epsilon)}\mathcal{F}_k(f_i)(x)\right\|_{1+\epsilon} \lesssim \sum_i \| f_i \|_{1+\epsilon},\ 0 < \epsilon \leq 1$$

where d_k is the generalized dimension defined by (1.1).
We will use the following known Hardy's inequality:

$$\sum_{j=0}^{\infty} 2^{-j(1+\epsilon)}\left(\sum_{l=0}^{j}\sum_i A_l^i\right)^{1+\epsilon} \asymp \sum_{j=0}^{\infty}\sum_i 2^{-j(1+\epsilon)}(A^i)_j^{1+\epsilon} \tag{2.3}$$

$$\sum_{j=0}^{\infty} 2^{j(1+\epsilon)}\left(\sum_{l=j}^{\infty}\sum_i A_l^i\right)^{1+\epsilon} \asymp \sum_{j=0}^{\infty}\sum_i 2^{j(1+\epsilon)}(A^i)_j^{1+\epsilon} \tag{2.4}$$

where $A_j^i \geq 0, \epsilon \geq 0$, and $0 < \epsilon \leq \infty$ (with the standard modification for $\epsilon = \infty$); see e.g. [41].

3. Smoothness characteristics and the K-functional

3.1. Bernstein's class of entire functions

Let $\mathbb{C}^d$ be the complex Euclidean space of d dimensions, $z = (z_1, \dots, z_d) \in \mathbb{C}^d, |z| = \sqrt{\sum_{i=1}^d |z_i|^2}$, and $\operatorname{Im} z = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_d)$.

For $\sigma_i > 0$ we define the Bernstein class $\mathcal{B}_{1+\epsilon,k}^{\sigma_i}$ of entire function of exponential spherical type at most σ_i.
We say that a function $f_i \in \mathcal{B}_{1+\epsilon,k}^{\sigma_i}$ if $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ is such that its analytic continuation to $\mathbb{C}^d$ satisfies

$$|f_i(z)| \leq C_\varepsilon \sum_i e^{(\sigma_i+\varepsilon)|z|},\ \forall \varepsilon > 0,\ \forall z \in \mathbb{C}$$

The smallest $\sigma_i = (\sigma_i)_{f_i}$ in this inequality is called a spherical type of f_i.
In [21], we proved that functions $f_i \in \mathcal{B}_{1+\epsilon,k}^{\sigma_i}$ satisfy

$$|\sum_i f_i(z)| \leq (1+\epsilon)\sum_i e^{\sigma_i|\operatorname{Im} z|},\ \forall z \in \mathbb{C}^d$$

Moreover, the following Paley-Wiener type characterization holds true.
Proposition 3.1 ([21]). A function $f_i \in \mathcal{B}_{1+\epsilon,k}^{\sigma_i}, 0 \leq \epsilon < \infty$, if and only if

$$f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k) \cap C_b(\mathbb{R}^d) \text{ and } \operatorname{supp} \mathcal{F}_k(f_i) \subset B_{\sigma_i}(0)$$

The Dunkl transform $\mathcal{F}_k(f_i)$ in Proposition 3.1 is understood as a function for $0 \leq \epsilon \leq 1$ and as a tempered distribution for $\epsilon > 1$.

3.2. Lizorkin and Sobolev spaces

Now we define the fractional power of the Dunkl Laplacian. Let

$$\Phi_k = \left\{f_i \in \mathcal{S}(\mathbb{R}^d): \int_{\mathbb{R}^d}\sum_i x_1^{\alpha_1}\dots x_d^{\alpha_d} f_i(x)d\mu_k(x) = 0, \alpha \in \mathbb{Z}_+^d\right\}$$

be the weighted Lizorkin space (see [22,34,46]) and set

$$\Psi_k = \{\mathcal{F}_k(f_i): f_i \in \Phi_k\}$$

Proposition 3.2 ([22]). (1) The spaces Φ_k and Ψ_k are closed in the topology of $\mathcal{S}(\mathbb{R}^d)$.
(2) The space Φ_k is dense in $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ for $0 \leq \epsilon < \infty$.
(3) One has

$$\Psi_k \equiv \Psi_0 = \{\mathcal{F}(f_i): f_i \in \Phi_0\} = \left\{f_i \in \mathcal{S}(\mathbb{R}^d): \frac{\partial^{\alpha_1+\dots+\alpha_d}}{\partial x_1^{\alpha_1}\dots\partial x_d^{\alpha_d}} f_i(0) = 0, \alpha \in \mathbb{Z}_+^d\right\}$$

Now we will use some auxiliary results from [18]. Let Φ_k' and Ψ_k' be the spaces of distributions on Φ_k and Ψ_k respectively. We have $\mathcal{S}'(\mathbb{R}^d) \subset \Phi_k', \mathcal{S}'(\mathbb{R}^d) \subset \Psi_k'$, and $\Phi_k' = \mathcal{S}'(\mathbb{R}^d)/\Pi, \Psi_k' = \mathcal{S}'(\mathbb{R}^d)/\mathcal{F}_k(\Pi)$, where Π stands for the set of all polynomials of d variables. We can multiply distributions from Ψ_k' on functions from

$$C_\Pi^\infty(\mathbb{R}^d \setminus \{0\}) = \{|x|^{1+\epsilon} f_i(x): f_i \in C_\Pi^\infty(\mathbb{R}^d), (1+\epsilon) \in \mathbb{R}\}$$

where $C_\Pi^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable functions whose derivatives have polynomial growth at infinity.

Next, using Dunkl multipliers we can define the following distributions. Let $\epsilon \geq 0$. We define the fractional power of the Dunkl Laplacian for $\varphi_i \in \Phi_k$ as follows

$$(-\Delta_k)^{1+\epsilon/2}\varphi_i = \mathcal{F}_k^{-1}(|\cdot|^{1+\epsilon}\mathcal{F}_k(\varphi_i)) = \mathcal{F}_k(|\cdot|^{1+\epsilon}\mathcal{F}_k^{-1}(\varphi_i)) \in \Phi_k$$

(see also [38]). By definition, for $f_i \in \Phi_k'$ the distribution $(-\Delta_k)^{1+\epsilon/2} f_i \in \Phi_k'$ is

$$\big((-\Delta_k)^{1+\epsilon/2} f_i, \varphi_i\big) = \big(f_i, (-\Delta_k)^{1+\epsilon/2}\varphi_i\big) = \Big(f_i, \mathcal{F}_k^{-1}(|\cdot|^{1+\epsilon}\mathcal{F}_k(\varphi_i))\Big),\ \varphi_i \in \Phi_k$$

By $W_{1+\epsilon,k}^{1+\epsilon}$, $0 \leq \epsilon \leq \infty$, we denote the Sobolev space, that is,

$$W_{1+\epsilon,k}^{1+\epsilon} = \{f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k): (-\Delta_k)^{1+\epsilon/2} f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)\}$$

equipped with the norm

$$\| f_i \|_{W_{1+\epsilon,k}^{1+\epsilon}} = \| f_i \|_{1+\epsilon} + \|(-\Delta_k)^{1+\epsilon/2} f_i\|_{1+\epsilon}$$

3.3. Basic definitions in the distributional sense

We define the direct and inverse Dunkl transforms $\mathcal{F}_k, \mathcal{F}_k^{-1}$, generalized translation operators $\tau^y, T^{1+\epsilon}$, and convolution ($f_i *_k g_i$) for distributions.

For $f_i \in \Phi_k'$ the direct Dunkl transform $\mathcal{F}_k(f_i) \in \Psi_k'$ is defined by

$$(\mathcal{F}_k(f_i), \psi_i) = (f_i, \mathcal{F}_k(\psi_i)),\ \psi_i \in \Psi_k$$

Similarly, for $g_i \in \Psi_k'$ the inverse Dunkl transform $\mathcal{F}_k^{-1}(g_i) \in \Phi_k'$ is defined by

$$(\mathcal{F}_k^{-1}(g_i), \varphi_i) = (g_i, \mathcal{F}_k^{-1}(\varphi_i)),\ \varphi_i \in \Phi_k$$

We have

$$\mathcal{F}_k^{-1}(\mathcal{F}_k(f_i)) = f_i,\ \mathcal{F}_k(\mathcal{F}_k^{-1}(g_i)) = g_i,\ f_i \in \Phi_k', g_i \in \Psi_k'$$

Note that $f_i = g_i$ in Φ_k' if and only if $\mathcal{F}_k(f_i) = \mathcal{F}_k(g_i)$ in Ψ_k'.

For $f_i \in \Phi_k'$ the generalized translation operators $\tau^y f_i, T^{1+\epsilon} f_i \in \Phi_k'$ are given respectively by

$$(\tau^y f_i, \varphi_i) = (f_i, \tau^{-y}\varphi_i) = \Big(f_i, \mathcal{F}_k^{-1}(e_k(-y,\cdot)\mathcal{F}_k(\varphi_i))\Big),\ y \in \mathbb{R}^d$$

$$(T^{1+\epsilon} f_i, \varphi_i) = (f_i, T^{1+\epsilon}\varphi_i) = \Big(f_i, \mathcal{F}_k^{-1}\big(j_{\lambda_k}((1+\epsilon)|\cdot|)\mathcal{F}_k(\varphi_i)\big)\Big),\ (1+\epsilon) \in \mathbb{R}_+$$

where $\varphi_i \in \Phi_k$. Moreover, the following equalities are valid:

$$\mathcal{F}_k\big((-\Delta_k)^{1+\epsilon/2} f_i\big) = |\cdot|^{1+\epsilon}\mathcal{F}_k(f_i),\ \mathcal{F}_k(\tau^y f_i) = e_k(y,\cdot)\mathcal{F}_k(f_i)$$

$$\mathcal{F}_k\big((-\Delta_k)^{1+\epsilon/2}\tau^y f_i\big) = |\cdot|^{1+\epsilon} e_k(y,\cdot)\mathcal{F}_k(f_i),\ \mathcal{F}_k(T^{1+\epsilon} f_i) = j_{\lambda_k}((1+\epsilon)|\cdot|)\mathcal{F}_k(f_i)$$

$$\mathcal{F}_k\big((-\Delta_k)^{1+\epsilon/2} T^{1+\epsilon} f_i\big) = |\cdot|^{1+\epsilon} j_{\lambda_k}((1+\epsilon)|\cdot|)\mathcal{F}_k(f_i)$$

$$\mathcal{F}_k\big(T^{1+\epsilon}(\tau^y f_i)\big) = j_{\lambda_k}((1+\epsilon)|\cdot|) e_k(y,\cdot)\mathcal{F}_k(f_i)$$

In particular, this implies the commutativity of considered operators.

Let $\varphi_i \in \Phi_k$ and $\varphi_i^-(y) = \varphi_i(-y)$. We say that $f_i \in \Phi_k'$ is even if $(f_i, \varphi_i^-) = (f_i, \varphi_i)$. Similarly we define even $g_i \in \Psi_k'$. Note that $f_i \in \Phi_k'$ is even if and only if $\mathcal{F}(f_i) \in \Psi_k'$ is even.

Let N_k be a set of all even $f_i \in \Phi_k'$ such that $\mathcal{F}_k(f_i) \in C_\Pi^\infty(\mathbb{R}^d \setminus \{0\})$. For $f_i \in N_k$ and $\varphi_i \in \Phi_k$ we set

$$(f_i *_k \varphi_i)(x) = (\tau^x f_i, \varphi_i^-) = (f_i, \tau^{-x}\varphi_i^-)$$

If $g_i \in N_k$ and $\varphi_i \in \Phi_k$, then $(g_i *_k \varphi_i) \in \Phi_k$ and

$$(g_i *_k \varphi_i)(x) = \mathcal{F}_k^{-1}(\mathcal{F}_k(g_i)\mathcal{F}_k(\varphi_i))(x),\ \mathcal{F}_k(g_i *_k \varphi_i)(y) = \mathcal{F}_k(g_i)(y)\mathcal{F}_k(\varphi_i)(y)$$

Therefore, we can define the convolution $(f_i *_k g_i) \in \Phi_k'$ for $f_i \in \Phi_k'$ and $g_i \in N_k$ as follows

$$\big((f_i *_k g_i), \varphi_i\big) = \big(f_i, (g_i *_k \varphi_i)\big),\ \varphi_i \in \Phi_k \tag{3.1}$$

Moreover, we remark that

$$\mathcal{F}_k(f_i *_k g_i) = \mathcal{F}_k(g_i)\mathcal{F}_k(f_i),\ (-\Delta_k)^{1+\epsilon/2}(f_i *_k g_i) = \big((-\Delta_k)^{1+\epsilon/2} f_i *_k g_i\big), \tag{3.2}$$

$$(f_i *_k ((g_i)_1 *_k (g_i)_2)) = ((f_i *_k (g_i)_1) *_k (g_i)_2) = (f_i *_k ((g_i)_2 *_k (g_i)_1)) = ((f_i *_k (g_i)_2) *_k (g_i)_1).$$

The next result establishes the interrelation between the convolutions given by (2.1) and (3.1).

Proposition 3.3 ([18]). If $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$, $g_i \in L_{\text{rad}}^1(\mathbb{R}^d, d\mu_k)$, and $\mathcal{F}_k(g_i) \in N_k$, then the convolutions given by (2.1) and (3.1) coincide.

3.4. Moduli of smoothness and K-functionals

The K-functional for the couple ($L^{1+\epsilon}(\mathbb{R}^d, d\mu_k), W_{1+\epsilon,k}^{1+\epsilon}$) is defined in the usual way: for $\epsilon > -1$,

$$K_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} = K_{1+\epsilon}\big(f_i, 1+\epsilon; L^{1+\epsilon}(\mathbb{R}^d, d\mu_k), W_{1+\epsilon,k}^{1+\epsilon}\big)_{1+\epsilon}$$
$$= \inf \sum_i \ \{\| f_i - g_i \|_{1+\epsilon} + (1+\epsilon)^{1+\epsilon}\|(-\Delta_k)^{1+\epsilon/2} g_i\|_{1+\epsilon}: g_i \in W_{1+\epsilon,k}^{1+\epsilon}\}$$

Note that $\lim_{\epsilon\to-1}\sum_i\ K_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} = 0$ for any $f_i \in \mathcal{S}(\mathbb{R}^d)$. Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ and

$$\left|\sum_i K_{1+\epsilon}((f_i)_1, 1+\epsilon)_{1+\epsilon} - K_{1+\epsilon}((f_i)_2, 1+\epsilon)_{1+\epsilon}\right| \le \sum_i \|(f_i)_1 - (f_i)_2\|_{1+\epsilon}$$

then for any $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ one has $\lim_{\epsilon\to -1}\sum_i K_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} = 0$. The monotonicity property of the K-functional is given by

$$K_{1+\epsilon}(f_i, \lambda(1+\epsilon))_{1+\epsilon} \le \max\sum_i \{1, \lambda^{1+\epsilon}\} K_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} \tag{3.3}$$

By definition,

$$\mathcal{R}_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} = \inf\sum_i \left\{\| f_i - g_i \|_{1+\epsilon} + (1+\epsilon)^{1+\epsilon}\|(-\Delta_k)^{1+\epsilon/2} g_i\|_{1+\epsilon} : g_i \in \mathcal{B}^{1/1+\epsilon}_{1+\epsilon,k}\right\}$$

is the realization of the K-functional $K_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon}$. Moreover, define

$$\mathcal{R}^*_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} = \sum_i \|f_i - g_i^*\|_{1+\epsilon} + (1+\epsilon)^{1+\epsilon}\sum_i \|(-\Delta_k)^{1+\epsilon/2} g_i^*\|_{1+\epsilon}$$

where $g_i^* \in \mathcal{B}^{1/1+\epsilon}_{1+\epsilon,k}$ is any best approximant of f_i in $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$. Note that taking two different best approximants give equivalent $\mathcal{R}^*_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon}$. Remark also that if $0 < \epsilon < \infty$, then g_i^* is unique. The realization of the K-functional was introduced in [16,26], where its importance in the approximation theory was shown.

Proposition 3.4 ([18]). Suppose $0 \le \epsilon \le \infty$, then, for any $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$,

$$\mathcal{R}_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} \asymp \mathcal{R}^*_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} \asymp K_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon} \asymp \omega_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon}$$

For the case of integer $(1+\epsilon)$, see [7, Cor. 2.3, Th. 3.1] ($k \equiv 0$) and [21] ($k(\cdot) \ge 0$). For the case of fractional moduli, see [31,47]($k \equiv 0$). The discussion on various ways to define moduli of smoothness can be found in [21, Sec. 6].

Let $\omega_{1+2\epsilon}(f_i, \delta)_{1+\epsilon}$ denote the modulus of smoothness of order $\epsilon > -\frac{1}{2}$ of a function $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$, i.e.,

$$\omega_{1+2\epsilon}(f_i, \delta)_{1+\epsilon} = \sup_{-1<\epsilon\le\delta-1}\sum_i \|\Delta^{1+2\epsilon}_{1+\epsilon} f_i(x)\|_{1+\epsilon},\ \delta > 0$$

where

$$\Delta^{1+2\epsilon}_{1+\epsilon} f_i(x) = (I - T^{1+\epsilon})^{1+2\epsilon/2} f_i(x) = \sum_{\epsilon=-1}^{\infty}\sum_i (-1)^{1+\epsilon}\binom{1+2\epsilon/2}{1+\epsilon}(T^{1+\epsilon})^{1+\epsilon} f_i(x) \tag{3.4}$$

and I stands for the identical operator. The difference $\Delta^{2(1+2\epsilon)}_{1+\epsilon} f_i(x)$ coincides with the classical fractional difference for the translation operator $T^{1+\epsilon} f_i(x) = f_i(x + 1 + \epsilon)$ and corresponds to the usual definition of the fractional modulus of smoothness, see, e.g., [6,46]. The reason why we use $1 + 2\epsilon/2$ in (3.4) is the fact that the multiplier in (3.8) is of order $O((1+\epsilon)^{1+2\epsilon})$ at zero.

Now we give several basic properties of the modulus of smoothness and the difference (3.4) (see [18]):

$$\lim_{\delta\to 0+0}\sum_i \omega_{1+2\epsilon}(f_i, \delta)_{1+\epsilon} = 0,\ \omega_{1+2\epsilon}(f_i, \delta)_{1+\epsilon} \lesssim \| f_i \|_{1+\epsilon},\ \delta > 0 \tag{3.5}$$

$$\omega_{1+2\epsilon}((f_i)_1 + (f_i)_2, \delta)_{1+\epsilon} \le \omega_{1+2\epsilon}((f_i)_1, \delta)_{1+\epsilon} + \omega_{1+2\epsilon}((f_i)_2, \delta)_{1+\epsilon}$$

$$\omega_{1+2\epsilon}(f_i, \lambda\delta)_{1+\epsilon} \lesssim \sum_i \max(1, \lambda^{1+2\epsilon})\,\omega_{1+2\epsilon}(f_i, \delta)_{1+\epsilon},\ \lambda > 0 \tag{3.6}$$

$$\omega_{1+2\epsilon}(f_i, \delta)_{1+\epsilon} \asymp \|\Delta^{1+2\epsilon}_{\delta} f_i\|_{1+\epsilon},\ \delta > 0,\ d_k > 1, \tag{3.7}$$

$$\mathcal{F}_k(\Delta^{1+2\epsilon}_{1+\epsilon} f_i) = \left(1 - j_{\lambda_k}((1+\epsilon)|\cdot|)\right)^{1+2\epsilon/2}\mathcal{F}_k(f_i),\ f_i \in \Phi'_k. \tag{3.8}$$

We conclude by presenting the Bernstein inequality in the Dunkl setting.

Proposition 3.5 ([18]). If $\sigma_i, 1+\epsilon, \delta > 0, 0 \le \epsilon \le \infty, f_i \in \mathcal{B}^{\sigma_i}_{1+\epsilon,k}$, then

$$\left\|\sum_i (-\Delta_k)^{1+\epsilon/2} f_i\right\|_{1+\epsilon} \lesssim \sum_i (\sigma_i)^{1+\epsilon}\| f_i \|_{1+\epsilon} \tag{3.9}$$

$$\omega_{1+\epsilon}(f_i, \delta)_{1+\epsilon} \lesssim \delta^{1+\epsilon}\sum_i \|(-\Delta_k)^{1+\epsilon/2} f_i\|_{1+\epsilon} \lesssim (\delta\sigma_i)^{1+\epsilon}\sum_i \| f_i \|_{1+\epsilon} \tag{3.10}$$

4. Littlewood-Paley-type inequalities

Recall that $\eta \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ such that $\eta(x) = 1$ if $|x| \le 1/2, \eta(x) > 0$ if $|x| < 1$, and $\eta(x) = 0$ if $|x| \ge 1$. Set $\theta(x) = \eta(x) - \eta(2x)$,

$$\eta_j(x) = \eta(2^{-j}x),\ \theta_j(x) = \theta(2^{-j}x) = \eta_j(x) - \eta_{j-1}(x),\ j \in \mathbb{Z}$$

Let $\eta_j f_i, \theta_j f_i$ be multiplier linear operators defined by the relations

$$\mathcal{F}_k(\eta_j f_i) = \mathcal{F}_k(f_i)\eta_j,\ \mathcal{F}_k(\theta_j f_i) = \mathcal{F}_k(f_i)\theta_j$$

respectively. We have

$$\operatorname{supp}\eta_j \subset B_{2^j},\ \operatorname{supp}\theta_j \subset B_{2^j}(0) \setminus B_{2^{j-2}}(0) \tag{4.1}$$

$$\eta_j(\eta_i f_i) = \eta_i(\eta_j f_i) = \eta_j f_i,\ j < i$$

and for $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ (see [18,21])

$$\eta_j f_i, \theta_j f_i \in \mathcal{B}^{2^j}_{1+\epsilon,k},\ \|\eta_j f_i\|_{1+\epsilon} \lesssim \| f_i \|_{1+\epsilon},\ \|\theta_j f_i\|_{1+\epsilon} \lesssim \| f_i \|_{1+\epsilon}$$

$$\left\|\sum_i f_i - \eta_j f_i\right\|_{1+\epsilon} \lesssim \sum_i E_{2^{j-1}}(f_i)_{1+\epsilon} \lesssim \sum_i \|f_i - \eta_{j-1} f_i\|_{1+\epsilon}$$

$$\left\|\sum_i \theta_j f_i\right\|_{1+\epsilon} \lesssim \sum_i \|f_i - \eta_{j-1} f_i\|_{1+\epsilon} + \sum_i \|f_i - \eta_j f_i\|_{1+\epsilon} \lesssim E_{2^{j-2}}(f_i)_{1+\epsilon} \tag{4.2}$$

Moreover,

$$\eta_0(x) + \sum_{j=1}^{\infty} \theta_j(x) = 1, x \in \mathbb{R}^d \setminus \{0\},\ \eta_0 f_i + \sum_{j=1}^{\infty}\sum_i \theta_j f_i = f_i \tag{4.3}$$

$$\int_{\mathbb{R}^d} \sum_i \theta_i f_i \theta_j f_i d\mu_k = 0,\ |i - j| \geq 2$$

and for any function $f_i \in L_{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ the series $\eta_0 f_i + \sum_{j=1}^{\infty} \theta_j f_i$ converges to f_i in $L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$.

Lemma 4.1 (see [58]). Suppose $0 \leq \epsilon < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{1+\epsilon}_{1+\epsilon,k}$.

Proof. Setting $\mathcal{B}^{\infty}_{1+\epsilon,k} = \bigcup_{\sigma_i > 0} \mathcal{B}^{\sigma_i}_{1+\epsilon,k}$, in virtue of (3.2), (4.2), Propositions 3.3 and 3.5, for any $f_i \in W^{1+\epsilon}_{1+\epsilon,k}$ we derive that $\eta_j f_i \in \mathcal{B}^{2j}_{1+\epsilon,k}$ and

$$(-\Delta_k)^{1+\epsilon/2}\eta_j f_i = (-\Delta_k)^{1+\epsilon/2}\left(f_i *_k \mathcal{F}_k(\eta_j)\right)$$
$$= \left((-\Delta_k)^{1+\epsilon/2} f_i\right) *_k \mathcal{F}_k(\eta_j) = \eta_j\left((-\Delta_k)^{1+\epsilon/2} f_i\right) \in \mathcal{B}^{j}_{1+\epsilon,k}$$

Since, by Bernstein's inequality (3.9), the embedding $\mathcal{B}^{j}_{1+\epsilon,k} \subset W^{1+\epsilon}_{1+\epsilon,k}$ holds, (4.2) implies

$$\left\|\sum_i f_i - \eta_j f_i\right\|_{1+\epsilon} \lesssim \sum_i E_{2^{j-1}}(f_i)_{1+\epsilon},\ \left\|\sum_i (-\Delta_k)^{1+\epsilon/2} f_i - (-\Delta_k)^{1+\epsilon/2}(\eta_j f_i)\right\|_{1+\epsilon}$$
$$\lesssim \sum_i E_{2^{j-1}}\left((-\Delta_k)^{1+\epsilon/2} f_i\right)_{1+\epsilon}$$

Hence, $\mathcal{B}^{\infty}_{1+\epsilon,k}$ is dense in $W^{1+\epsilon}_{1+\epsilon,k}$.

Let $f_i \in \mathcal{B}^{\sigma_i}_{1+\epsilon,k}, \delta, \varepsilon > 0$ and suppose that $\psi_i \in \mathcal{S}(\mathbb{R}^d)$ is an entire function of exponential type 1 such that $\psi_i(0) = 1$. Let $(\psi_i)_\delta(x) = \psi_i(\delta x)$. Then inequality (3.9) and the Nikolskii inequality [18] yield that $f_i(\psi_i)_\delta \in \mathcal{S}(\mathbb{R}^d) \cap \mathcal{B}^{\sigma_i+\delta}_{1+\epsilon,k}$. Choose $\epsilon \geq 0$ and $0 < \delta < 1$ so that

$$\int_{|x|\geq 1+\epsilon} \sum_i |f_i(x)|^{1+\epsilon} d\mu_k(x) < \varepsilon^{1+\epsilon},\ |1 - (\psi_i)_\delta(x)| < \varepsilon \text{ for } |x| \leq 1 + \epsilon$$

Then we have

$$\int_{\mathbb{R}^d} \sum_i |f_i(x) - f_i(x)(\psi_i)_\delta(x)|^{1+\epsilon} d\mu_k(x) \leq \sum_i (1 + \| \psi_i \|_\infty)^{1+\epsilon} \int_{|x|\geq 1+\epsilon} |f_i(x)|^{1+\epsilon} d\mu_k(x)$$
$$+ \varepsilon^{1+\epsilon} \int_{|x|\leq 1+\epsilon} \sum_i |f_i(x)|^{1+\epsilon} d\mu_k(x) \leq \sum_i (1 + \| \psi_i \|_\infty + \| f_i \|_{1+\epsilon})^{1+\epsilon} \varepsilon^{1+\epsilon}$$

Using again Bernstein's inequality (3.9), we finally obtain

$$\left\|\sum_i (-\Delta_k)^{1+\epsilon/2}(f_i - f_i(\psi_i)_\delta)\right\|_{1+\epsilon} \leq c(k) \sum_i (\sigma_i + 1)^{1+\epsilon} \|f_i - f_i(\psi_i)_\delta\|_{1+\epsilon}$$
$$\leq c(k) \sum_i (\sigma_i + 1)^{1+\epsilon} (1 + \| \psi_i \|_\infty + \| f_i \|_{1+\epsilon})\varepsilon$$

Now we establish the desired version of the Littlewood-Paley inequalities. To prove it, we follow the same reasoning as those in [8,10,11] (see also [13, Chapter 7]). We will also use [56].

Theorem 4.2 (see [58]). Let $0 < \epsilon < \infty$. If $f_i \in W^{1+\epsilon}_{1+\epsilon,k}$, then

$$\left\|\sum_i (-\Delta_k)^{1+\epsilon/2} f_i\right\|_{1+\epsilon} \asymp \left\|\left(\sum_{j\in\mathbb{Z}}\sum_i 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon} \tag{4.4}$$

and

$$\left\|\sum_i (-\Delta_k)^{1+\epsilon/2} f_i\right\|_{1+\epsilon} \asymp \left\|\left(\sum_i \left|(-\Delta_k)^{1+\epsilon/2}\eta_0 f_i\right|^2 + \sum_{j=1}^{\infty}\sum_i 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon} \tag{4.5}$$

Moreover,

$$\left\|\sum_i (-\Delta_k)^{1+\epsilon/2} f_i\right\|_{1+\epsilon} \lesssim \left\|\sum_i \left(|\eta_0 f_i|^2 + \sum_{j=1}^{\infty} 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon} \tag{4.6}$$

Proof. For $f_i \in \mathcal{S}(\mathbb{R}^d)$ and $\epsilon \geq 0$, equivalence (4.4) was proved in [56, Proposition 4.5]. By Lemma 4.1, $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{1+\epsilon}_{1+\epsilon,k}$ and hence, equivalence (4.4) is valid for any $f_i \in W^{1+\epsilon}_{1+\epsilon,k}$.

Inequality (3.9) implies $\eta_0 f_i \in W^{1+\epsilon}_{1+\epsilon,k}$. Applying the equality $\eta_0 f_i(x) = \sum_{j=-\infty}^{0}\theta_j f_i(x)$ and (4.4), we have

$$\left\|\sum_i (-\Delta_k)^{1+\epsilon/2}\eta_0 f_i\right\|_{1+\epsilon} \asymp \left\|\left(\sum_{j=-\infty}^{0}\sum_i 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon}$$

Hence,

$$\begin{aligned}\left\|\sum_i (-\Delta_k)^{1+\epsilon/2} f_i\right\|_{1+\epsilon} &\asymp \left\|\left(\sum_{j\in\mathbb{Z}}\sum_i 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon} \\ &\asymp \left\|\left(\sum_{j=-\infty}^{0}\sum_i 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon} + \left\|\left(\sum_{j=1}^{\infty}\sum_i 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon} \\ &\asymp \sum_i \|(-\Delta_k)^{1+\epsilon/2}\eta_0 f_i\|_{1+\epsilon} + \sum_i \left\|\left(\sum_{j=1}^{\infty} 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon} \\ &\asymp \left\|\sum_i \left(\left|(-\Delta_k)^{1+\epsilon/2}\eta_0 f_i\right|^2 + \sum_{j=1}^{\infty} 2^{2(1+\epsilon)j}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon}\end{aligned}$$

that is, (4.5) is shown. Since $\|(-\Delta_k)^{1+\epsilon/2}\eta_0 f_i\|_{1+\epsilon} \lesssim \|\eta_0 f_i\|_{1+\epsilon}$, we obtain (4.6) from (4.5).

To prove Theorem 1.5(2) we will use a more general version of lower estimate in (4.4) with $\epsilon = -1$.

Lemma 4.3 ([56]). Let $\varphi_i \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$, $\operatorname{supp}\varphi_i \subset \{(1+\epsilon) \leq |x| \leq 1+2\epsilon\}, -1 < \epsilon < 1+2\epsilon, (\varphi_i)_j(x) = \varphi_i(2^{-j}x)$, and $(\varphi_i)_j f_i = \mathcal{F}_k^{-1}\left(\mathcal{F}_k(f_i)(\varphi_i)_j\right)$. Then for $0 < \epsilon < \infty$ we have

$$\left\|\left(\sum_{j\in\mathbb{Z}}\sum_i \left|(\varphi_i)_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon} \lesssim \sum_i \| f_i \|_{1+\epsilon}$$

Note that in [56] this result was shown for $\epsilon = -\frac{1}{2}, \epsilon = \frac{1}{2}$. The general case is similar. To prove Corollary 1.2, we will also need the following result.

Corollary 4.4 (see [58]). If $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k), 0 < \epsilon < \infty, (2+\epsilon) = \min(1+\epsilon, 2)$, then

$$\| f_i \|_{1+\epsilon} \lesssim \sum_i \left(\|\eta_0 f_i\|_{1+\epsilon}^{2+\epsilon} + \sum_{j=1}^{\infty} \|\theta_j f_i\|_{1+\epsilon}^{2+\epsilon} \right)^{1/2+\epsilon}$$

Proof. The proof is carried out as the corresponding result in [8]. We give it for completeness. If $0 < \epsilon \leq 1$, then

$$\| f_i \|_{1+\epsilon} \lesssim \left\| \sum_i \left(|\eta_0 f_i|^2 + \sum_{j=1}^{\infty} |\theta_j f_i|^2 \right)^{1/2} \right\|_{1+\epsilon} \leq \left\| \sum_i \left(|\eta_0 f_i|^{1+\epsilon} + \sum_{j=1}^{\infty} |\theta_j f_i|^{1+\epsilon} \right)^{1/1+\epsilon} \right\|_{1+\epsilon}$$

$$= \sum_i \left(\|\eta_0 f_i\|_{1+\epsilon}^{1+\epsilon} + \sum_{j=1}^{\infty} \|\theta_j f_i\|_{1+\epsilon}^{1+\epsilon} \right)^{1/1+\epsilon}$$

If $0 \leq \epsilon < \infty$, then Minkowski's inequality implies

$$\| f_i \|_{2+\epsilon} \lesssim \left\| \sum_i \left(|\eta_0 f_i|^2 + \sum_{j=1}^{\infty} |\theta_j f_i|^2 \right)^{1/2} \right\|_{2+\epsilon} = \left\| \sum_i |\eta_0 f_i|^2 + \sum_{j=1}^{\infty} \sum_i |\theta_j f_i|^2 \right\|_{1+\epsilon}^{1/2}$$

$$\leq \sum_i \left(\| |\eta_0 f_i|^2 \|_{1+\epsilon} + \sum_{j=1}^{\infty} \left\| |\theta_j f_i|^2 \right\|_{1+\epsilon} \right)^{1/2} = \sum_i \left(\|\eta_0 f_i\|_{2+\epsilon}^2 + \sum_{j=1}^{\infty} \|\theta_j f_i\|_{2+\epsilon}^2 \right)^{1/2}$$

5. Proofs of Theorem 1.1 and Corollary 1.2 (see [58])

Proof of Theorem 1.1. Following the corresponding proof in [10], since $E_j(f_i)_{1+\epsilon}, \omega_{1+\epsilon}(f_i, 1/j)_{1+\epsilon}$, $K_{1+\epsilon}(f_i, 1/j)_{1+\epsilon}$ are all monotonic in j, by (1.4), we can equivalently write inequality (1.6) in the form

$$J = 2^{-(1+\epsilon)n} \left(\sum_{j=0}^{n} \sum_i 2^{(1+\epsilon)^2 j} E_{2j}^{1+\epsilon}(f_i)_{1+\epsilon} \right)^{1/1+\epsilon} \lesssim \sum_i K_{1+\epsilon}(f_i, 2^{-n})_{1+\epsilon} \qquad (5.1)$$

Set $(g_i)_n = \eta_{n-1} f_i$. Applying (4.1) and (4.5) with $\epsilon = -1$, we have $E_{2^n}(f_i)_{1+\epsilon} \leq \|f_i - (g_i)_n\|_{1+\epsilon}$, $E_{2^n}((g_i)_n)_{1+\epsilon} = 0$, and for $0 \leq j < n$

$$E_{2j}((g_i)_n)_{1+\epsilon} \leq \sum_i \|\eta_n (g_i)_n - \eta_j (g_i)_n\|_{1+\epsilon} = \left\| \sum_{l=j+1}^{n} \sum_i \theta_l (g_i)_n \right\|_{1+\epsilon}$$

$$\lesssim \left\| \left(\sum_{l=j+1}^{n} \sum_i |\theta_l (g_i)_n|^2 \right)^{1/2} \right\|_{1+\epsilon}$$

Hence,

$$J \leq 2^{-(1+\epsilon)n} \left(\sum_{j=0}^{n-1} \sum_i 2^{(1+\epsilon)^2 j} E_{2j}^{1+\epsilon}(f_i - (g_i)_n)_{1+\epsilon} \right)^{1/1+\epsilon} + 2^{-(1+\epsilon)n} \left(\sum_{j=0}^{n-1} \sum_i 2^{(1+\epsilon)^2 j} E_{2j}^{1+\epsilon}((g_i)_n)_{1+\epsilon} \right)^{1/1+\epsilon}$$

$$\lesssim \sum_i \| f_i - (g_i)_n \|_{1+\epsilon} + 2^{-(1+\epsilon)n} \left(\sum_{j=0}^{n-1} 2^{(1+\epsilon)^2 j} \left\| \sum_{l=j+1}^{n} \sum_i |\theta_l (g_i)_n|^2 \right\|_{1+\epsilon}^{1+\epsilon} \right)^{1/1+\epsilon} . \qquad (5.2)$$

Let $0 < \epsilon \leq 1$ and $\epsilon = 1$. Using (5.2), the inequality $1 + \epsilon/2 \leq 1$, (4.5) with $\epsilon \geq 0$, and Proposition 3.4, we obtain

$$J \lesssim \sum_i \|f_i - (g_i)_n\|_{1+\epsilon} + 2^{-(1+\epsilon)n} \left(\sum_{j=0}^{n-1} 2^{2(1+\epsilon)j} \left\| \sum_{l=j+1}^{n} \sum_i |\theta_l (g_i)_n|^2 \right\|_{1+\epsilon/2} \right)^{1/2}$$

$$\lesssim \sum_i \|f_i - (g_i)_n\|_{1+\epsilon} + 2^{-(1+\epsilon)n} \left(\left\| \sum_{j=0}^{n-1} 2^{2(1+\epsilon)j} \sum_{l=j+1}^{n} |\theta_l (g_i)_n|^2 \right\|_{1+\epsilon/2} \right)^{1/2}$$

$$= \sum_i \|f_i - (g_i)_n\|_{1+\epsilon} + 2^{-(1+\epsilon)n} \left(\left\| \sum_{l=1}^{n} \sum_i |\theta_l (g_i)_n|^2 \sum_{j=0}^{l-1} 2^{2(1+\epsilon)j} \right\|_{1+\epsilon/2} \right)^{1/2}$$

$$\lesssim \sum_i \|f_i - (g_i)_n\|_{1+\epsilon} + 2^{-(1+\epsilon)n} \left(\left\| \sum_{l=1}^{n} \sum_i 2^{2(1+\epsilon)l} |\theta_l (g_i)_n|^2 \right\|_{1+\epsilon/2} \right)^{1/2}$$

$$= \sum_i \|f_i - (g_i)_n\|_{1+\epsilon} + 2^{-(1+\epsilon)n} \left\| \left(\sum_{l=1}^{n} \sum_i 2^{2(1+\epsilon)l} |\theta_l (g_i)_n|^2 \right)^{1/2} \right\|_{1+\epsilon}$$

$$\lesssim \sum_i \|f_i - (g_i)_n\|_{1+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \|(-\Delta_k)^{1+\epsilon/2} (g_i)_n\|_{1+\epsilon} \lesssim \sum_i K_{1+\epsilon}(f_i, 2^{-n})_{1+\epsilon}$$

Thus, we verified (5.1) for $0 < \epsilon \leq 1$.

Let $0 < \epsilon < \infty$ and $\epsilon = \frac{1}{2}$. Applying the duality between $\ell^n_{2+\epsilon/2}$ and $\ell^n_{2+\epsilon}$, where $(2+\epsilon) = (2+\epsilon/2)'$, we can write

$$\sum_{j=0}^{n-1} 2^{(2+\epsilon)(1+\epsilon)j} \left(\sum_{l=j+1}^{n} \sum_i |\theta_l (g_i)_n|^2(x) \right)^{2+\epsilon/2} = \left(\sum_{j=0}^{n-1} 2^{(2+\epsilon)(1+\epsilon)j} (a_i)_j(x) \sum_{l=j+1}^{n} \sum_i |\theta_l (g_i)_n|^2(x) \right)^{2+\epsilon/2}$$

where $\sum_{j=0}^{n-1} 2^{(2+\epsilon)(1+\epsilon)j} (a_i)_j^{2+\epsilon}(x) = 1$. Using this, we derive

$$L = \int_{\mathbb{R}^d} \sum_{j=0}^{n-1} 2^{(2+\epsilon)(1+\epsilon)j} \left(\sum_{l=j+1}^{n} \sum_i |\theta_l (g_i)_n|^2(x) \right)^{2+\epsilon/2} d\mu_k(x)$$

$$= \int_{\mathbb{R}^d} \left(\sum_{j=0}^{n-1} 2^{(2+\epsilon)(1+\epsilon)j} (a_i)_j(x) \sum_{l=j+1}^{n} \sum_i |\theta_l (g_i)_n|^2(x) \right)^{2+\epsilon/2} d\mu_k(x)$$

$$= \int_{\mathbb{R}^d} \sum_i \left(\sum_{l=1}^{n} |\theta_l (g_i)_n|^2(x) \sum_{j=0}^{l-1} 2^{(2+\epsilon)(1+\epsilon)j} (a_i)_j(x) \right)^{2+\epsilon/2} d\mu_k(x)$$

Applying Hölder's inequality and (4.5), we obtain

$$L \leq \int_{\mathbb{R}^d} \sum_i \left(\sum_{l=1}^{n} |\theta_l (g_i)_n|^2(x) \left(\sum_{j=0}^{l-1} 2^{(2+\epsilon)(1+\epsilon)j} \right)^{2/2+\epsilon} \left(\sum_{j=0}^{n-1} 2^{(2+\epsilon)(1+\epsilon)j} (a_i)_j^{2+\epsilon}(x) \right)^{1/2+\epsilon} \right)^{2+\epsilon/2} d\mu_k(x)$$

$$\lesssim \int_{\mathbb{R}^d} \sum_i \left(\sum_{l=1}^{n} 2^{2(1+\epsilon)j} |\theta_l (g_i)_n|^2(x) \right)^{2/2+\epsilon} d\mu_k(x) = \sum_i \left\| \left(\sum_{l=1}^{n} 2^{2(1+\epsilon)l} |\theta_l (g_i)_n|^2 \right)^{1/2} \right\|_{2+\epsilon}^{2+\epsilon}$$

$$\lesssim \sum_i \|(-\Delta_k)^{1+\epsilon/2} (g_i)_n\|_{2+\epsilon}^{2+\epsilon}.$$

Hence, from (5.2), we get

$$J \lesssim \sum_i \|f_i - (g_i)_n\|_{2+\epsilon} + 2^{-(1+\epsilon)n} \|(-\Delta_k)^{1+\epsilon/2} (g_i)_n\|_{2+\epsilon} \lesssim \sum_i K_{1+\epsilon}(f_i, 2^{-n})_{2+\epsilon}$$

that is, (5.1) follows. Thus, (1.6) is proved.

Since the Sobolev space is dense in $L^{2+\epsilon}(\mathbb{R}^d, d\mu_k)$, we can assume that $f_i \in W^{1+\epsilon}_{2+\epsilon,k}$ and write inequality (1.7) in the form

$$K_{1+\epsilon}(f_i, 2^{-n})_{2+\epsilon} \lesssim 2^{-(1+\epsilon)n} \sum_i \left\{ \left(\sum_{j=0}^{n} 2^{(2+\epsilon)(1+\epsilon)j} E_{2j}^{2+\epsilon}(f_i)_{2+\epsilon} \right)^{1/2+\epsilon} + \| f_i \|_{2+\epsilon} \right\} \tag{5.3}$$

Taking into account Proposition 3.4, (4.5), Corollary 4.4, (3.2), and (4.1)-(4.3), this gives

$$K_{1+\epsilon}(f_i, 2^{-n})_{2+\epsilon} \lesssim \sum_i \|f_i - \eta_n f_i\|_{2+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \|(-\Delta_k)^{1+\epsilon/2} \eta_n f_i\|_{2+\epsilon}$$

$$\lesssim \sum_i E_{2^{n-1}}(f_i)_{2+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \left\| \left(\left|\eta_0\left((-\Delta_k)^{1+\epsilon/2}\eta_n f_i\right)\right|^2 + \sum_{j=1}^{\infty} \left|\theta_j\left((-\Delta_k)^{1+\epsilon/2}\eta_n f_i\right)\right|^2 \right) \right\|_{2+\epsilon}$$

$$\lesssim \sum_i E_{2^{n-1}}(f_i)_{2+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \left(\left\|\eta_0\left((-\Delta_k)^{1+\epsilon/2}\eta_n f_i\right)\right\|_{2+\epsilon}^{2+\epsilon} + \sum_{j=1}^{\infty} \left\|\theta_j\left((-\Delta_k)^{1+\epsilon/2}\eta_n f_i\right)\right\|_{2+\epsilon}^{2+\epsilon} \right)^{1/2+\epsilon}$$

$$\lesssim \sum_i E_{2^{n-1}}(f_i)_{2+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \left(\|(-\Delta_k)^{1+\epsilon/2}\eta_0 f_i\|_{2+\epsilon}^{2+\epsilon} + \sum_{j=1}^{n-1} \|(-\Delta_k)^{1+\epsilon/2}\theta_j f_i\|_{2+\epsilon}^{2+\epsilon} \right.$$

$$\left. + \left\|\eta_n\left((-\Delta_k)^{1+\epsilon/2}\theta_n f_i\right)\right\|_{2+\epsilon}^{2+\epsilon} + \left\|\eta_n\left((-\Delta_k)^{1+\epsilon/2}\theta_{n+1} f_i\right)\right\|_{2+\epsilon}^{2+\epsilon} \right)^{1/2+\epsilon}$$

Bernstein's inequality (3.9) yields

$$K_{1+\epsilon}(f_i, 2^{-n})_{2+\epsilon} \lesssim \sum_i E_{2^{n-1}}(f_i)_{2+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \left(\|\eta_0 f_i\|_{2+\epsilon}^{2+\epsilon} + \sum_{j=1}^{n+1} 2^{(2+\epsilon)(1+\epsilon)j} \|\theta_j f_i\|_{2+\epsilon}^{2+\epsilon} \right)^{1/2+\epsilon}$$

$$\lesssim 2^{-(1+\epsilon)n} \sum_i \left\{ \left(\sum_{j=0}^{n} 2^{(2+\epsilon)(1+\epsilon)j} E_{2j}^{2+\epsilon}(f_i)_{2+\epsilon} \right)^{1/2+\epsilon} + \| f_i \|_{2+\epsilon} \right\}$$

Proof of Corollary 1.2. Inequality (1.8) can be equivalently written as follows

$$J = 2^{-(1+\epsilon)n} \sum_i \left(\sum_{j=0}^{n} 2^{(1+\epsilon)^2 j} \omega_{1+2\epsilon}^{1+\epsilon}(f_i, 2^{-j})_{2+\epsilon} \right)^{1/1+\epsilon} \lesssim \sum_i \omega_{1+\epsilon}(f_i, 2^{-n})_{2+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \| f_i \|_{2+\epsilon}$$

Indeed, using (1.3) and (1.7), we have

$$J \lesssim 2^{-(1+\epsilon)n} \sum_i \left(\sum_{j=0}^{n} 2^{-(\epsilon)(1+\epsilon)j} \left(\sum_{l=0}^{j} 2^{(1+2\epsilon)(1+\epsilon)l} E_{2^l}^{1+\epsilon}(f_i)_{2+\epsilon} + \| f_i \|_{2+\epsilon}^{1+\epsilon} \right) \right)^{1/1+\epsilon}$$

$$\lesssim 2^{-(1+\epsilon)n} \sum_i \left(\sum_{l=0}^{n} 2^{(1+2\epsilon)(1+\epsilon)l} E_{2^l}^{1+\epsilon}(f_i)_{2+\epsilon} \sum_{j=l}^{n} 2^{-(\epsilon)(1+\epsilon)j} \right)^{1/1+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \| f_i \|_{2+\epsilon}$$

$$\lesssim 2^{-(1+\epsilon)n} \sum_i \left(\sum_{l=0}^{n} 2^{(1+\epsilon)^2 l} E_{2^l}^{1+\epsilon}(f_i)_{2+\epsilon} \right)^{1/1+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \| f_i \|_{2+\epsilon}$$

$$\lesssim \sum_i \omega_{1+\epsilon}(f_i, 2^{-n})_{2+\epsilon} + 2^{-(1+\epsilon)n} \sum_i \| f_i \|_{2+\epsilon}$$

Inequality (1.9) follows from (5.3) and Jackson's inequality (1.2).

6. Proofs of Theorem 1.3 (see [58])

Proof of Theorem 1.3 for the best approximants. It follows from [33, Theorem1] that given a function $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$, $0 < \epsilon < \infty$, for any entire function $g_i \in \mathcal{B}_{1+\epsilon,k}^{\sigma_i}$ one has

$$\left\| \sum_i (f_i - (f_i)_{\sigma_i}) \right\|_{1+\epsilon}^{1+\epsilon} \le \sum_i \| f_i - g_i \|_{1+\epsilon}^{1+\epsilon} - A \sum_i \|g_i - (f_i)_{\sigma_i}\|_{1+\epsilon}^{1+\epsilon}, \quad (1+\epsilon) = \max(1+\epsilon, 2), \tag{6.1}$$

$$\left\|\sum_i (f_i-(f_i)_{\sigma_i})\right\|_{1+\epsilon}^{2+\epsilon} \le \sum_i \| f_i-g_i\|_{1+\epsilon}^{2+\epsilon} - B\sum_i \|g_i-(f_i)_{\sigma_i}\|_{1+\epsilon}^{2+\epsilon}, \quad (2+\epsilon)=\min(1+\epsilon,2) \quad (6.2)$$

where $(f_i)_{\sigma_i}\in \mathcal{B}_{1+\epsilon,k}^{\sigma_i}$ is the best approximant of f_i and positive constants A,B are independent of $f_i,(f_i)_{\sigma_i},g_i$.

Following similar arguments as those in [30], let us prove the left-hand side inequality in (1.10). Using Hardy's inequality (2.3)

$$\sum_{j=n}^{\infty} 2^{-(1+\epsilon)j}\left(\sum_{l=n}^{j}\sum_i A_l^i\right)^{2+\epsilon} \lesssim \sum_{j=n}^{\infty}\sum_i 2^{-(1+\epsilon)j}(A^i)_j^{2+\epsilon} \quad (6.3)$$

inequality (6.1), Proposition 3.4, and (1.4), we derive that

$$\begin{aligned}
&\sum_{j=n+1}^{\infty}\sum_i 2^{-(1+\epsilon)^2 j}\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^j}\|_{1+\epsilon}^{1+\epsilon}\\
&=\sum_{j=n+1}^{\infty} 2^{-(1+\epsilon)^2 j}\sum_i\left\|\sum_{l=n+1}^{j}(-\Delta_k)^{1+\epsilon/2}((f_i)_{2^l}-(f_i)_{2^{l-1}})+(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^n}\right\|_{1+\epsilon}^{1+\epsilon}\\
&\lesssim\sum_{j=n+1}^{\infty}\sum_i 2^{-(1+\epsilon)^2 j}\left\|\sum_{l=n+1}^{j}((-\Delta_k)^{1+\epsilon/2}(f_i)_{2^l}-(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^{l-1}})\right\|_{1+\epsilon}^{1+\epsilon}+2^{-(1+\epsilon)^2 n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^n}\|_{1+\epsilon}^{1+\epsilon}\\
&\lesssim\sum_{j=n+1}^{\infty} 2^{-(1+\epsilon)^2 j}\sum_i\left(\sum_{l=n+1}^{j}\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^l}-(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^{l-1}}\|_{1+\epsilon}\right)^{1+\epsilon}+2^{-(1+\epsilon)^2 n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^n}\|_{1+\epsilon}^{1+\epsilon}\\
&\lesssim\sum_{j=n+1}^{\infty}\sum_i 2^{-(1+\epsilon)^2 j}\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^j}-(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^{j-1}}\|_{1+\epsilon}^{1+\epsilon}+2^{-(1+\epsilon)^2 n}\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^n}\|_{1+\epsilon}^{1+\epsilon}
\end{aligned}$$

Then Bernstein's inequality (3.9) implies

$$\begin{aligned}
&\sum_{j=n+1}^{\infty} 2^{-(1+\epsilon)^2 j}\left\|\sum_i(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^j}\right\|_{1+\epsilon}^{1+\epsilon}\lesssim\sum_{j=n+1}^{\infty}\sum_i\|(f_i)_{2^j}-(f_i)_{2j-1}\|_{1+\epsilon}^{1+\epsilon}+2^{-(1+\epsilon)^2 n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^n}\|_{1+\epsilon}^{1+\epsilon}\\
&\lesssim\frac{1}{A}\sum_{j=n+1}^{\infty}\sum_i\left(\|f_i-(f_i)_{2^{j-1}}\|_{1+\epsilon}-\|f_i-(f_i)_{2^j}\|_{1+\epsilon}\right)^{1+\epsilon}+2^{-(1+\epsilon)^2 n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^n}\|_{1+\epsilon}^{1+\epsilon}\\
&\lesssim\sum_i\|f_i-(f_i)_{2^n}\|_{1+\epsilon}^{1+\epsilon}+2^{-(1+\epsilon)^2 n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^n}\|_{1+\epsilon}^{1+\epsilon}\lesssim\sum_i K_{1+\epsilon}^{1+\epsilon}(f_i,2^{-n})_{1+\epsilon}\lesssim\sum_i\omega_{1+\epsilon}^{1+\epsilon}(f_i,2^{-n})_{1+\epsilon}
\end{aligned}$$

To show the right-hand side inequality in (1.10), by (1.4) and (3.3), we have

$$\begin{aligned}
\omega_{1+\epsilon}^{2+\epsilon}(f_i,2^{-n})_{1+\epsilon}&\lesssim\sum_i K_{1+\epsilon}^{2+\epsilon}(f_i,2^{-n})_{1+\epsilon}\\
&\lesssim\sum_i\|f_i-(f_i)_{2^{n+1}}\|_{1+\epsilon}^{2+\epsilon}+2^{-(2+\epsilon)(1+\epsilon)n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^{n+1}}\|_{1+\epsilon}^{2+\epsilon}\\
&\lesssim\sum_{j=n+2}^{\infty}\sum_i\left(\|f_i-(f_i)_{2j-1}\|_{1+\epsilon}^{2+\epsilon}-\|f_i-(f_i)_{2^j}\|_{1+\epsilon}^{2+\epsilon}\right)+2^{-(2+\epsilon)(1+\epsilon)n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^{n+1}}\|_{1+\epsilon}^{2+\epsilon}\\
&\lesssim\sum_{j=n+2}^{\infty}\sum_i\left(\|f_i-(f_i)_{2^{j-1}}((f_i)_{2j})\|_{1+\epsilon}^{2+\epsilon}-\|f_i-(f_i)_{2j}\|_{1+\epsilon}^{2+\epsilon}\right)+2^{-(2+\epsilon)(1+\epsilon)n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^{n+1}}\|_{1+\epsilon}^{2+\epsilon}
\end{aligned}$$

Using (6.2) and the following Jackson inequality [18]

$$E_{\sigma_i}(f_i)_{1+\epsilon}\lesssim(\sigma_i)^{-(1+\epsilon)}\sum_i\left\|(-\Delta_k)^{\frac{1+\epsilon}{2}}f_i\right\|_{1+\epsilon},\ f_i\in W_{1+\epsilon,k}^{1+\epsilon},0\le\epsilon\le\infty,\sigma_i,\epsilon\ge0$$

we obtain

$$\begin{aligned}
\omega_{1+\epsilon}^{2+\epsilon}(f_i,2^{-n})_{1+\epsilon}&\lesssim\sum_{j=n+2}^{\infty}\sum_i\|(f_i)_{2^j}-(f_i)_{2^{j-1}}((f_i)_{2j})\|_{1+\epsilon}^{2+\epsilon}+2^{-(2+\epsilon)(1+\epsilon)n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^{n+1}}\|_{1+\epsilon}^{2+\epsilon}\\
&\lesssim\sum_{j=n+2}^{\infty}\sum_i E_{2_{j-1}}^{2+\epsilon}((f_i)_{2j})_{1+\epsilon}+2^{-(2+\epsilon)(1+\epsilon)n}\sum_i\|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2^{n+1}}\|_{1+\epsilon}^{2+\epsilon}
\end{aligned}$$

$$\lesssim \sum_{j=n+1}^{\infty} \sum_{i} 2^{-(2+\epsilon)(1+\epsilon)j} \|(-\Delta_k)^{1+\epsilon/2}(f_i)_{2j}\|_{1+\epsilon}$$

completing the proof.

Proof of Theorem 1.3 for the de la Vallée Poussin type operators. We will show that for $(1+\epsilon) = \max(1+\epsilon, 2)$ and $(2+\epsilon) = \min(1+\epsilon, 2)$,

$$\left(\sum_{j=n+1}^{\infty} \sum_{i} 2^{-(1+\epsilon)^2 j} \|(-\Delta_k)^{1+\epsilon/2}\eta_j f_i\|_{1+\epsilon}^{1+\epsilon}\right)^{1/1+\epsilon} \lesssim \sum_{i} \omega_{1+\epsilon}(f_i, 2^{-n})_{1+\epsilon}$$

$$\lesssim \left(\sum_{j=n+1}^{\infty} \sum_{i} 2^{-(2+\epsilon)(1+\epsilon)j} \left\|(-\Delta_k)^{\frac{1+\epsilon}{2}}\eta_j f_i\right\|_{1+\epsilon}^{2+\epsilon}\right)^{\frac{1}{2+\epsilon}}. \tag{6.4}$$

To obtain the left-hand side estimate, we have

$$\sum_{j=n+1}^{\infty} 2^{-(1+\epsilon)^2 j} \left\|\sum_{i} (-\Delta_k)^{1+\epsilon/2}\eta_j f_i\right\|_{1+\epsilon}^{1+\epsilon} \lesssim \sum_{j=n+1}^{\infty} \sum_{i} 2^{-(1+\epsilon)^2 j} \|(-\Delta_k)^{1+\epsilon/2}(\eta_j - \eta_n) f_i\|_{1+\epsilon}^{1+\epsilon} \tag{6.5}$$

$$+\sum_{i} \|(-\Delta_k)^{1+\epsilon/2}\eta_n f_i\|_{1+\epsilon}^{1+\epsilon} =: J + \sum_{i} \|(-\Delta_k)^{1+\epsilon/2}\eta_n f_i\|_{1+\epsilon}^{1+\epsilon}$$

In light of (4.6), we obtain

$$J \lesssim \sum_{j=n+1}^{\infty} \sum_{i} 2^{-(1+\epsilon)^2 j} \left\|\left(\left|\eta_0\left((\eta_j - \eta_n) f_i\right)\right|^2 + \sum_{l=1}^{\infty} 2^{2(1+\epsilon)l} \left|\theta_l\left((\eta_j - \eta_n) f_i\right)\right|^2\right)^{1/2}\right\|_{1+\epsilon}^{1+\epsilon}$$

$$= \sum_{j=n+1}^{\infty} \sum_{i} 2^{-(1+\epsilon)^2 j} \left\|\left(\sum_{l=n}^{j+1} 2^{2(1+\epsilon)l} \left|\theta_l\left((\eta_j - \eta_n) f_i\right)\right|^2\right)^{1/2}\right\|_{1+\epsilon}^{1+\epsilon}$$

If $l = n, n+1, j \geq n+1$, then (4.1) and (4.2) yield

$$\left\|\sum_{i} \theta_l\left((\eta_j - \eta_n) f_i\right)\right\|_{1+\epsilon} \lesssim \sum_{i} \|(\eta_j - \eta_n) f_i\|_{1+\epsilon} = \sum_{i} \|\eta_j(f_i - \eta_n f_i)\|_{1+\epsilon} \lesssim \sum_{i} \|f_i - \eta_n f_i\|_{1+\epsilon}$$

If $l = j, j+1, j \geq n+1$, then

$$\left\|\sum_{i} \theta_l\left((\eta_j - \eta_n) f_i\right)\right\|_{1+\epsilon} = \|\eta_j(\theta_l(f_i - \eta_n f_i))\|_{1+\epsilon} \lesssim \|\theta_l(f_i - \eta_n f_i)\|_{1+\epsilon}$$

If $n+2 \leq l \leq j-1$, then $\theta_l\left((\eta_j - \eta_n) f_i\right) = \theta_l(f_i - \eta_n f_i)$.

Hence,

$$J \lesssim \sum_{j=n+1}^{\infty} \sum_{i} 2^{-(1+\epsilon)^2 j} \left\|\left(\sum_{l=n}^{j+1} 2^{2(1+\epsilon)l} |\theta_l(f_i - \eta_n f_i)|^2\right)^{1/2}\right\|_{1+\epsilon}^{1+\epsilon} \sum_{i} + \|f_i - \eta_n f_i\|_{1+\epsilon}^{1+\epsilon}$$

Taking into account Minkowski's inequality, Hardy's inequality (6.3) and equivalence (4.5), we obtain

$$\sum_{j=n+1}^{\infty} \sum_{i} 2^{-(1+\epsilon)^2 j} \left\|\left(\sum_{l=n}^{j+1} 2^{2(1+\epsilon)l} |\theta_l(f_i - \eta_n f_i)|^2\right)^{1/2}\right\|_{1+\epsilon}^{1+\epsilon}$$

$$\leq \left\|\sum_{j=n+1}^{\infty} \sum_{i} 2^{-(1+\epsilon)^2 j} \left(\sum_{l=n}^{j+1} 2^{2(1+\epsilon)l} |\theta_l(f_i - \eta_n f_i)|^2\right)^{1+\epsilon/2}\right\|$$

$$\lesssim \left\|\sum_{j=n}^{\infty} \sum_{i} |\theta_l(f_i - \eta_n f_i)|^{1+\epsilon}\right\| \lesssim \sum_{i} \left\|\left(\sum_{j=n}^{\infty} |\theta_l(f_i - \eta_n f_i)|^2\right)^{1/2}\right\|_{1+\epsilon}^{\tau} \lesssim \sum_{i} \|f_i - \eta_n f_i\|_{1+\epsilon}^{1+\epsilon}$$

Therefore, $J \lesssim \|f_i - \eta_n f_i\|_{1+\epsilon}^{1+\epsilon}$ and by (6.5), we arrive at

$$\left(\sum_{j=n+1}^{\infty}\sum_i 2^{-(1+\epsilon)^2 j}\left\|(-\Delta_k)^{1+\frac{\epsilon}{2}}\eta_{2j}f_i\right\|_{1+\epsilon}^{1+\epsilon}\right)^{\frac{1}{1}+\epsilon} \lesssim \sum_i \|f_i - \eta_n f_i\|_{1+\epsilon} + \sum_i \|(-\Delta_k)^{1+\epsilon/2}\eta_n f_i\|_{1+\epsilon}$$
$$\lesssim \sum_i K_{1+\epsilon}(f_i, 2^{-n})_{1+\epsilon}$$

Proposition 3.4 concludes the proof of the left-hand side inequality in (6.4).
To verify the right-hand side inequality in (6.4), it suffices to prove that

$$K_{1+\epsilon}(f_i, 2^{-n})_{1+\epsilon} \lesssim \left(\sum_{j=n+1}^{\infty}\sum_i 2^{-(2+\epsilon)(1+\epsilon)j}\|(-\Delta_k)^{1+\epsilon/2}\eta_{2j}f_i\|_{1+\epsilon}^{2+\epsilon}\right)^{1/2+\epsilon} \tag{6.6}$$

By Proposition 3.4 and (4.2), we have

$$K_{1+\epsilon}(f_i, 2^{-n})_{1+\epsilon}^{2+\epsilon} \lesssim \sum_i \|f_i - \eta_n f_i\|_{1+\epsilon}^{2+\epsilon} + 2^{-(2+\epsilon)(1+\epsilon)n}\sum_i \|(-\Delta_k)^{1+\epsilon/2}\eta_n f_i\|_{1+\epsilon}^{2+\epsilon} \tag{6.7}$$

Using (4.5), the inequality $\left|\sum_i \theta_j(f_i - \eta_n f_i)\right|^2 \le 2\sum_i \left(\left|\theta_j f_i\right|^2 + \left|\theta_j(\eta_n f_i)\right|^2\right)$, (4.2) and the equalities $\theta_j(\eta_n f_i) = 0$ for $j \ge n+2$, $\theta_j(f_i - \eta_n f_i) = 0$ for $j \le n-1$, we obtain

$$\left\|\sum_i \theta_j(f_i - \eta_n f_i)\right\|_{1+\epsilon} \lesssim \sum_i \|\theta_j f_i\|_{1+\epsilon},\ j = n, n+1$$

and

$$\left\|\sum_i (f_i - \eta_n f_i)\right\|_{1+\epsilon}^{2+\epsilon} \lesssim \sum_i \left\|\left(\sum_{j=n}^{\infty}\left|\theta_j(f_i - \eta_n f_i)\right|^2\right)^{1/2}\right\|_{1+\epsilon}^{2+\epsilon}$$
$$\lesssim \sum_i \left\|\left(\sum_{j=n}^{\infty}\left|\theta_j f_i\right|^2\right)^{1/2}\right\|_{1+\epsilon}^{2+\epsilon} \lesssim \sum_i \left\|\left(\sum_{j=n}^{\infty}\left|\theta_j f_i\right|^{2+\epsilon}\right)^{1/2+\epsilon}\right\|_{1+\epsilon}^{2+\epsilon}$$
$$= \sum_i \left\|\left(\sum_{j=n}^{\infty} 2^{-(1+\epsilon)(2+\epsilon)j}\left(\left|\theta_j f_i\right|^2 2^{2(1+\epsilon)j}\right)^{2+\epsilon/2}\right)^{1/2+\epsilon}\right\|_{1+\epsilon}^{2+\epsilon}$$
$$\lesssim \sum_i \left\|\sum_{j=n}^{\infty} 2^{-(1+\epsilon)(2+\epsilon)j}\left(\sum_{l=n}^{j+2} 2^{2(1+\epsilon)l}\left|\theta_l(\eta_{j+1}f_i)\right|^2\right)^{2+\epsilon/2}\right\|_{1+\epsilon/2+\epsilon}$$
$$\lesssim \sum_{j=n}^{\infty}\sum_i 2^{-(1+\epsilon)(2+\epsilon)j}\left\|\left(\sum_{l=1}^{\infty} 2^{2(1+\epsilon)l}\left|\theta_l(\eta_{j+1}f_i)\right|^2\right)^{1/2}\right\|_{1+\epsilon}^{2+\epsilon}$$

In view of (4.5),

$$\left\|\sum_i (f_i - \eta_n f_i)\right\|_{1+\epsilon}^{2+\epsilon} \lesssim \sum_{j=n}^{\infty}\sum_i 2^{-(1+\epsilon)(2+\epsilon)j}\|(-\Delta_k)^{1+\epsilon/2}(\eta_{j+1}f_i)\|_{1+\epsilon}^{2+\epsilon}$$
$$\lesssim \sum_{j=n}^{\infty}\sum_i 2^{-(1+\epsilon)(2+\epsilon)j}\|(-\Delta_k)^{1+\epsilon/2}(\eta_j f_i)\|_{1+\epsilon}^{2+\epsilon}$$

This and (6.7) imply (6.6).

7. Proofs of Theorems 1.4-1.7 (see [58])

Proof of Theorem 1.5. First we obtain Pitt-type estimates (1.13) and (1.15). For $0 < \epsilon \leq 1$ and $1+\epsilon \leq 2+\epsilon \leq 2(1+\epsilon)$ the inequality

$$\left\| \sum_i |x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)} \mathcal{F}_k(f_i) \right\|_{2+\epsilon} \lesssim \sum_i \| f_i \|_{1+\epsilon}$$

immediately follows from Proposition 2.4 and the interpolation theorem [48, Theorem 2].

Let now $0 \leq \epsilon < \infty$ and $2(1+\epsilon) \leq 2+\epsilon \leq 2+\epsilon$. By Proposition 2.1(4) to obtain estimate (1.15), we prove that

$$\left\| \sum_i \mathcal{F}_k(f_i) \right\|_{2+\epsilon} \lesssim \sum_i \left\| |x|^{d_k(1/2+2\epsilon-1/2+\epsilon)} f_i \right\|_{2+\epsilon} \tag{7.1}$$

Proposition 2.4 implies the following Hardy-Littlewood type inequality:

$$\|\mathcal{F}_k(f_i)\|_{2+\epsilon} = \sup_{\|g_i\|_{2+2\epsilon}\leq 1} \sum_i \left| \int_{\mathbb{R}^d} \mathcal{F}_k(f_i) g_i \, d\mu_k \right| = \sup_{\|g_i\|_{2+\epsilon}\leq 1} \sum_i \left| \int_{\mathbb{R}^d} f_i \mathcal{F}_k(g_i) \, d\mu_k \right|$$

$$= \sup_{\|g_i\|_{2+2\epsilon}\leq 1} \sum_i \left| \int_{\mathbb{R}^d} \left| x \right|^{d_k(1-2/2+\epsilon)} f_i |x|^{d_k(1-2/2+2\epsilon)} \mathcal{F}_k(g_i) \, d\mu_k \right|$$

$$\leq \sum_i \left\| |x|^{d_k\left(1-\frac{2}{2}+\epsilon\right)} f_i \right\|_{2+\epsilon} \sup_{\|g_i\|_{2+2\epsilon}\leq 1} \left\| |x|^{d_k(1-2/2+\epsilon)} \mathcal{F}_k(g_i) \right\|_{2+2\epsilon} \lesssim \sum_i \left\| |x|^{d_k(1-2/2+2\epsilon)} f_i \right\|_{2+\epsilon}. \tag{7.2}$$

As usual, we first obtain this inequality for $f_i, g_i \in \mathcal{S}(\mathbb{R}^d)$ and then we use density arguments to consider the general case $|\cdot|^{d_k(1-2/2+\epsilon)} f_i \in L^{2+\epsilon}(\mathbb{R}^d, d\mu_k)$. Interpolating between $\|\mathcal{F}_k(f_i)\|_{2+\epsilon} \leq \| f_i \|_{2+2\epsilon}$ and (7.2), we arrive at inequality (7.1).

Second we derive Kellogg-type inequalities (1.14) and (1.16). Let $0 < \epsilon \leq 1$ and $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$. To verify (1.14), we will use Lemma 4.3 for nonnegative φ_i with support in the annulus $\{1/2 \leq |x| \leq 3\}$ and such that $\varphi_i(x) = 1$ for $1 \leq |x| \leq 2$. Then

$$\chi_j(x) \equiv \chi_{\{2^j \leq |x| < 2^{j+1}\}}(x) \leq (\varphi_i)_j(x), \; x \in \mathbb{R}^d \tag{7.3}$$

Putting $A_j^i = \left|(\varphi_i)_j f_i\right|^{1+\epsilon}$ and $\epsilon = (2/1+\epsilon) - 1$, Lemma 4.3 gives

$$\| f_i \|_{1+\epsilon} \gtrsim \left(\int_{\mathbb{R}^d} \left(\sum_{j\in\mathbb{Z}} \sum_i \left|(\varphi_i)_j f_i\right|^2 \right)^{1+\epsilon/2} d\mu_k \right)^{1/1+\epsilon} = \left(\int_{\mathbb{R}^d} \left(\sum_{j\in\mathbb{Z}} \sum_i (A^i)_j^{1+\epsilon} \right)^{1/1+\epsilon} d\mu_k \right)^{1/1+\epsilon}$$

Making use of Minkowski's inequality

$$\left(\sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^d} \sum_i A_j^i \, d\mu_k \right)^{1+\epsilon} \right)^{1/1+\epsilon} \leq \int_{\mathbb{R}^d} \left(\sum_{j\in\mathbb{Z}} \sum_i (A^i)_j^{1+\epsilon} \right)^{1/1+\epsilon} d\mu_k$$

we derive that

$$\| f_i \|_{1+\epsilon} \gtrsim \left(\sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^d} \sum_i A_j^i \, d\mu_k \right)^{1+\epsilon} \right)^{1/(1+\epsilon(1+\epsilon))} = \left(\sum_{j\in\mathbb{Z}} \sum_i \|(\varphi_i)_j f_i\|_{1+\epsilon}^2 \right)^{1/2}$$

Applying the Hausdorff-Young inequality (2.2) for $(\varphi_i)_j f_i$ and (7.3), we have

$$\|(\varphi_i)_j f_i\|_{1+\epsilon} \gtrsim \|\mathcal{F}_k((\varphi_i)_j f_i)\|_{\frac{1+\epsilon}{\epsilon}} = \|\mathcal{F}_k(f_i)(\varphi_i)_j\|_{\frac{1+\epsilon}{\epsilon}} \geq \|\mathcal{F}_k(f_i)\chi_j\|_{\frac{1+\epsilon}{\epsilon}}$$

Thus, the proof of (1.14) is complete.

If $0 \leq \epsilon < \infty$ and $\left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon}^2\right)^{1/2} < \infty$, similarly (7.2), we use duality argument to show (1.16). Indeed, we apply Plancherel's theorem, Hölder's inequality and (1.14) to get

$$\int_{\mathbb{R}^d} \sum_i f_i \bar{g}_i \, d\mu_k = \int_{\mathbb{R}^d} \sum_i \mathcal{F}_k(f_i) \overline{\mathcal{F}_k(g_i)} \, d\mu_k = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \sum_i \mathcal{F}_k(f_i)\chi_j \overline{\mathcal{F}_k(g_i)} \chi_j \, d\mu_k$$

$$\leq \sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon} \|\mathcal{F}_k(g_i)\chi_j\|_{2+\epsilon} \leq \sum_i \left(\sum_{j\in\mathbb{Z}} \|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon}^2 \right)^{1/2} \left(\sum_{j\in\mathbb{Z}} \|\mathcal{F}_k(g_i)\chi_j\|_{2+\epsilon}^2 \right)^{1/2}$$

$$\lesssim \sum_i \left(\sum_{j\in\mathbb{Z}} \|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon}^2 \right)^{1/2} \| g_i \|_{2+2\epsilon}$$

and

$$\| f_i \|_{2+\epsilon} = \sup_{\|g_i\|_{2+2\epsilon}\le 1} \int_{\mathbb{R}^d} \sum_i f_i \bar{g}_i d\mu_k \lesssim \left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon}^2 \right)^{1/2}$$

completing the proof of (1.16).

Remark 7.1. Inequalities (1.17) and (1.18) easily follow from Hölder's inequality for dyadic blocks and the monotonicity of $l_{2+\epsilon}$-norms. For example, in order to show (1.17) with $0 < \epsilon \le 1 \le 2+\epsilon \le 2(1+\epsilon)$, we use $\||x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)} g_i\chi_j\|_{2+\epsilon} \lesssim \|g_i\chi_j\|_{2(1+\epsilon)}, j \in \mathbb{Z}$, to get

$$\left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k(f_i)\chi_j\|_{2(1+\epsilon)}^2 \right)^{1/2} \gtrsim \left(\sum_{j\in\mathbb{Z}} \sum_i \||x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)}\mathcal{F}_k(f_i)\chi_j\|_{2+\epsilon}^2 \right)^{1/2}$$

$$\ge \left(\sum_{j\in\mathbb{Z}} \sum_i \||x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)}\mathcal{F}_k(f_i)\chi_j\|_{2+\epsilon}^{2+\epsilon} \right)^{1/2+\epsilon}$$

$$= \sum_i \||x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)}\mathcal{F}_k(f_i)\|_{2+\epsilon}$$

Let us now show that (1.17) and (1.18) are sharp. For large enough integer N take Schwartz functions $(\psi_i)_l, l = 1, \dots, N$ such that $\mathrm{supp}(\psi_i)_l \subset \{2^l + \varepsilon \le |x| \le 2^l + 2\varepsilon\}$ for sufficiently small $\varepsilon > 0$ and $\|(\psi_i)_l\|_{2(1+\epsilon)} = 1$. Then, by Hölder's inequality we have $\|\sum_i |x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)}(\psi_i)_l\|_{2+\epsilon} \lesssim \sum_i \|(\psi_i)_l\|_{2(1+\epsilon)} = 1$. Similarly, $\|\sum_i |x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)}(\psi_i)_l\|_{2+\epsilon} \gtrsim 1$ for $2+\epsilon \ge 2(1+\epsilon)$.

Consider

$$(f_i)_N = \sum_{l=1}^N \sum_i l^{-1/2}\mathcal{F}_k^{-1}((\psi_i)_l)$$

Since supports of $(\psi_i)_l$ are disjoint sets, we get

$$\left(\sum_{j\in\mathbb{Z}} \sum_i \|\mathcal{F}_k((f_i)_N)\chi_j\|_{2(1+\epsilon)}^2 \right)^{1/2} \asymp \left(\sum_{l=1}^N l^{-1} \right)^{1/2} \asymp (\ln N)^{1/2}$$

Thus, for $\epsilon \ge 0$, we arrive at

$$\left\| \sum_i |x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)}\mathcal{F}_k((f_i)_N) \right\|_{2+\epsilon} = \left(\sum_{l=1}^N \sum_i l^{-(2+\epsilon)/2}\||x|^{d_k(1/2(1+\epsilon)-1/2+\epsilon)}(\psi_i)_l\|_{2+\epsilon}^{2+\epsilon} \right)^{1/2+\epsilon}$$

$$\lesssim \left(\sum_{l=1}^N l^{-(2+\epsilon)/2} \right)^{1/2+\epsilon}$$

and the reverse estimate for $\epsilon \le 0$. This show that for $\epsilon \ne 0$, estimates (1.17) and (1.18) are optimal.

Proof of Theorem 1.4. To show the estimate

$$\left\| \sum_i |x|^{d_k\left(\frac{1}{2}-2\epsilon-\frac{1}{2}-\epsilon\right)} \min\{1, (\delta|x|)^{1+\epsilon}\}\mathcal{F}_k(f_i) \right\|_{2-\epsilon} \lesssim \sum_i \omega_{1+\epsilon}(f_i, \delta)_{1+\epsilon}, \quad 0 < \epsilon \le 1, \tag{7.4}$$

we use Pitt's inequality (1.13) for the difference (3.4) in place of f_i. Using also (3.8), we have

$$\left\| \sum_i |x|^{d_k(1/2-2\epsilon-1/2-\epsilon)} \left(1 - j_{\lambda_k}(\delta|x|)\right)^{1+\epsilon/2}\mathcal{F}_k(f_i) \right\|_{2-\epsilon} \lesssim \sum_i \|\Delta_\delta^{1+\epsilon} f_i\|_{2-2\epsilon}$$

To conclude the proof, we use (3.7) and the fact that for $\lambda > -1/2$ one has

$$1 - j_\lambda(1+\epsilon) \asymp \min\{1, (1+\epsilon)^2\}$$

uniformly in $\epsilon \ge -1$. The latter follows from the known properties of the Bessel function:

$$j_\lambda(1+\epsilon) = 1 - \frac{(1+\epsilon)^2}{4(\lambda+1)} + O((1+\epsilon)^4),\ \epsilon \to -1$$

$$|j_\lambda(1+\epsilon)| \le \min\left\{1, C_\lambda(1+\epsilon)^{-\lambda-\frac{1}{2}}\right\},\ \epsilon > -1$$

The reverse inequality to (7.4) for $0 \le \epsilon < \infty, 2-2\epsilon \le 2-\epsilon \le 2+\epsilon$ can be derived similarly with the help of Pitt's inequality (1.15).

Further, proceeding as in the proof of (7.4) with the help of Kellogg's inequality (1.14), we establish for $0 < \epsilon \le 1$

$$\left(\sum_{j\in\mathbb{Z}}\sum_i \left\|\min\{1,(\delta|x|)^{1+\epsilon}\}\mathcal{F}_k(f_i)\chi_j\right\|_{2-2\epsilon}^2\right)^{1/2} \lesssim \sum_i \omega_{1+\epsilon}(f_i,\delta)_{1+\epsilon}$$

which is equivalent to

$$\left(\sum_{j\in\mathbb{Z}}\sum_i \min\left\{1,\left(2^j\delta\right)^{2(1+\epsilon)}\right\}\|\mathcal{F}_k(f_i)\chi_j\|_{\frac{1+\epsilon}{\epsilon}}^2\right)^{1/2} \lesssim \sum_i \omega_{1+\epsilon}(f_i,\delta)_{1+\epsilon}$$

The case $0 \le \epsilon < \infty$ is similar.

Proof of Theorem 1.6. Relation (1.19) immediately follows from the fact that $\omega_{1+\epsilon}(f_i, 1+\epsilon)_{2+\epsilon} \asymp \omega_{1+\epsilon}(f_i, 2(1+\epsilon))_{2+\epsilon}$; see (3.6).

In light of (4.2) and Jackson's inequality (1.2), we derive

$$\sum_{j=1}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|\theta_j f_i\|_{2+\epsilon}^{1+\epsilon} \lesssim \sum_i \| f_i \|_{2+\epsilon}^{1+\epsilon} + \sum_{j=1}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|f_i - \eta_j f_i\|_{2+\epsilon}^{1+\epsilon} \lesssim \sum_i \| f_i \|_{2+\epsilon}^{1+\epsilon} + \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left(E_{2^j}(f_i)_{2+\epsilon}\right)^{1+\epsilon}$$

$$\lesssim \sum_i \| f_i \|_{2+\epsilon}^{1+\epsilon} + \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}(\omega_{1+\epsilon}(f_i, 2^{-j})_{2+\epsilon})^{1+\epsilon}$$

Therefore, to verify (1.20), (1.21), and (1.22), it is enough to show that

$$\sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left(\omega_{1+\epsilon}\left(f_i,2^{-j}\right)_{2+\epsilon}\right)^{1+\epsilon} \lesssim \sum_i \| f_i \|_{2+\epsilon}^{1+\epsilon} + \sum_{j=1}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|\theta_j f_i\|_{2+\epsilon}^{1+\epsilon}$$

Using (4.2), (4.3), (3.5), and (3.10) and setting $\theta_0 = \eta_0$, we have for $f_i = \sum_{l=0}^{\infty}\theta_l f_i$ and $j \ge 0$

$$\omega_{1+\epsilon}\left(f_i,2^{-j}\right)_{2+\epsilon} \lesssim \sum_{l=0}^{j}\sum_i \omega_{1+\epsilon}\left(\theta_l f_i, 2^{-j}\right)_{2+\epsilon} + \sum_{l=j}^{\infty}\sum_i \omega_{1+\epsilon}\left(\theta_l f_i, 2^{-j}\right)_{2+\epsilon}$$

$$\lesssim \sum_{l=0}^{j}\sum_i 2^{(1+\epsilon)(l-j)}\|\theta_l f_i\|_{2+\epsilon} + \sum_{l=j}^{\infty}\sum_i \|\theta_l f_i\|_{2+\epsilon}$$

This and Hardy's inequalities (2.3) and (2.4) imply

$$\sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}(\omega_{1+\epsilon}(f_i,2^{-j})_{2+\epsilon})^{1+\epsilon} \lesssim \sum_{j=0}^{\infty}\sum_i\left(\sum_{l=0}^{j} 2^{(1+\epsilon)l}\|\theta_l f_i\|_{2+\epsilon}\right)^{1+\epsilon} + \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left(\sum_{l=j}^{\infty}\|\theta_l f_i\|_{2+\epsilon}\right)^{1+\epsilon}$$

$$\lesssim \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|\theta_j f_i\|_{2+\epsilon}^{1+\epsilon} \lesssim \sum_i \|\eta_0 f_i\|_{2+\epsilon}^{1+\epsilon} + \sum_{j=1}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|\theta_j f_i\|_{2+\epsilon}^{1+\epsilon}$$

$$\lesssim \sum_i \| f_i \|_{2+\epsilon}^{1+\epsilon} + \sum_{j=1}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|\theta_j f_i\|_{2+\epsilon}^{1+\epsilon}$$

Relation (1.23) follows from

$$\left(\sum_{j=-\infty}^{0}\sum_i 2^{(1+\epsilon)^2 j}\|\theta_j f_i\|_{2+\epsilon}^{1+\epsilon}\right)^{1/1+\epsilon} \lesssim \sum_i \| f_i \|_{2+\epsilon}\left(\sum_{j=-\infty}^{0} 2^{(1+\epsilon)^2 j}\right)^{1/1+\epsilon} \lesssim \sum_i \| f_i \|_{2+\epsilon}$$

To obtain (1.24), we take into account (1.19), Theorem 1.3, and Hardy's inequalities. In particular, we establish the estimate from above as follows:

$$\sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left(\omega_{1+\epsilon}\left(f_i,2^{-j}\right)_{2+\epsilon}\right)^{1+\epsilon} \lesssim \sum_{j=0}^{\infty} 2^{(1+\epsilon)^2 j}\left(\sum_{l=j+1}^{\infty}\sum_i 2^{-(2-\epsilon)(1+\epsilon)l}\|(-\Delta_k)^{1+\epsilon/2}P_l^i\|_{2+\epsilon}^{2-\epsilon}\right)^{1+\epsilon/2-\epsilon}$$

$$\asymp \sum_{j=1}^{\infty}\sum_i \|(-\Delta_k)^{1+\epsilon/2}P_l^i\|_{2+\epsilon}^{1+\epsilon}$$

Similarly, we obtain the estimate from below.

Any of equivalences (1.20)-(1.23) show that the Besov-Dunkl space equipped with the (quasi-)norm $\| f_i \|_{B^{1+\epsilon}_{2+\epsilon,1+\epsilon}}$.

Proof of Theorem 1.7. We give the proof only for $\epsilon < \infty$. The proof for $\epsilon = \infty$ is similar.

Suppose $0 \le \epsilon < \infty$ and $\mathcal{F}_k(f_i) \in L^{2+2\epsilon}(\mathbb{R}^d, d\mu_k)$. First, applying (1.12) with $(2-\epsilon) = 2+2\epsilon$, we estimate

$$\int_0^1 \sum_i \left((1+\epsilon)^{-(1+\epsilon)}\omega_{1+\epsilon}(f_i, 1+\epsilon)_{2+\epsilon}\right)^{1+\epsilon} \frac{d(1+\epsilon)}{1+\epsilon} \asymp \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left(\omega_{1+\epsilon}(f_i, 2^{-j})_{2+\epsilon}\right)^{1+\epsilon}$$

$$\lesssim \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left\|\min\left\{1, \left(2^{-j}|x|\right)^{1+\epsilon}\right\}\mathcal{F}_k(f_i)\right\|_{2+2\epsilon}^{1+\epsilon}$$

Second, we note that

$$\sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|\min\{1, (2^{-j}|x|)^{1+\epsilon}\}\mathcal{F}_k(f_i)\|_{2+2\epsilon}^{1+\epsilon} \asymp \sum_i \|\chi_{\{|x|<1\}}|x|^{1+\epsilon}\mathcal{F}_k(f_i)\|_{2+2\epsilon}^{1+\epsilon} + \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon}^{1+\epsilon} \quad (7.5)$$

Indeed,

$$\sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left\|\min\left\{1, \left(2^{-j}|x|\right)^{1+\epsilon}\right\}\mathcal{F}_k(f_i)\right\|_{2+2\epsilon}^{1+\epsilon} \asymp \sum_{j=0}^{\infty}\left(\int_{|x|<2^j}\sum_i \left||x|^{1+\epsilon}\mathcal{F}_k(f_i)\right|^{2+2\epsilon} d\mu_k\right)^{1+\epsilon/2+2\epsilon}$$

$$+\sum_{j=0}^{\infty} 2^{(1+\epsilon)^2 j}\left(\int_{|x|\ge 2^j}\sum_i |\mathcal{F}_k(f_i)|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon} = I_1 + I_2$$

To estimate I_2, we use Hardy's inequality (2.4):

$$I_2 \asymp \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left(\sum_{l=j}^{\infty}\int_{2^l\le|x|<2^{l+1}}|\mathcal{F}_k(f_i)|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon}$$

$$\asymp \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left(\int_{2^j\le|x|<2}|\mathcal{F}_k(f_i)|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon} = \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon}^{1+\epsilon}$$

Taking into account (2.3), we also get

$$I_1 \asymp \sum_{j=0}^{\infty}\sum_i \left(\int_{|x|<1}\left||x|^{1+\epsilon}\mathcal{F}_k(f_i)\right|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon}$$

$$+\sum_{j=1}^{\infty}\sum_i\left(\int_{1\le|x|<2j}\left||x|^{1+\epsilon}\mathcal{F}_k(f_i)\right|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon} \asymp \sum_i\left(\int_{|x|<1}\left||x|^{1+\epsilon}\mathcal{F}_k(f_i)\right|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon}$$

$$+\sum_{j=1}^{\infty}\sum_i\left(\sum_{l=0}^{j-1}2^{j(1+\epsilon)(2+2\epsilon)}\int_{2^l\le|x|<2^{l+1}}|\mathcal{F}_k(f_i)|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon}$$

$$\asymp\left(\int_{|x|<1}\sum_i\left||x|^{1+\epsilon}\mathcal{F}_k(f_i)\right|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon} + \sum_{j=0}^{\infty}\sum_i 2^{(1+\epsilon)^2 j}\left(\int_{2j\le|x|<2}|\mathcal{F}_k(f_i)|^{2+2\epsilon}d\mu_k\right)^{1+\epsilon/2+2\epsilon}$$

Third, in view of (7.5), we have

$$\| f_i \|_{B^{1+\epsilon}_{2+\epsilon,1+\epsilon}} \lesssim \sum_i \| f_i \|_{2+\epsilon} + \sum_i \left\|\chi_{\{|x|<1\}}|x|^{1+\epsilon}\mathcal{F}_k(f_i)\right\|_{2+2\epsilon} + \sum_i\left(\sum_{j=0}^{\infty}2^{(1+\epsilon)^2 j}\|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon}^{1+\epsilon}\right)^{1/1+\epsilon}$$

where $\left\|\sum_i \chi_{\{|x|<1\}}|x|^{1+\epsilon}\mathcal{F}_k(f_i)\right\|_{2+2\epsilon} \le \sum_i \|\mathcal{F}_k(f_i)\|_{2+2\epsilon}$ and by Hausdorff-Young's inequality $\| f_i \|_{2+\epsilon} \lesssim \|\mathcal{F}_k(f_i)\|_{2+2\epsilon}$. Thus,

$$\| f_i \|_{B^{1+\epsilon}_{2+\epsilon,1+\epsilon}} \lesssim \sum_i \|\mathcal{F}_k(f_i)\|_{2+2\epsilon} + \sum_i \left(\sum_{j=0}^{\infty} 2^{(1+\epsilon)^2 j} \|\mathcal{F}_k(f_i)\chi_j\|_{2+2\epsilon}^{1+\epsilon} \right)^{1/1+\epsilon}$$

The reverse estimate for $0 < \epsilon \leq 1$, and $f_i \in L^{1+\epsilon}(\mathbb{R}^d, d\mu_k)$ can be obtained similarly. We have

$$\int_0^1 \sum_i \left((1+\epsilon)^{-(1+\epsilon)}\omega_{1+\epsilon}(f_i, 1+\epsilon)_{1+\epsilon}\right)^{1+\epsilon} \frac{d(1+\epsilon)}{1+\epsilon}$$

$$\gtrsim \sum_{j=0}^{\infty} \sum_i 2^{(1+\epsilon)^2 j} \left\|\min\left\{1, \left(2^{-j}|x|\right)^{1+\epsilon}\right\} \mathcal{F}_k(f_i)\right\|_{2(1+\epsilon)}^{1+\epsilon}$$

Using (7.5),

$$\| f_i \|_{B^{1+\epsilon}_{1+\epsilon,1+\epsilon}} \gtrsim \sum_i \| f_i \|_{1+\epsilon} + \sum_i \left\|\chi_{\{|x|<1\}}|x|^{1+\epsilon}\mathcal{F}_k(f_i)\right\|_{2(1+\epsilon)} + \sum_i \left(\sum_{j=0}^{\infty} 2^{(1+\epsilon)^2 j} \|\mathcal{F}_k(f_i)\chi_j\|_{2(1+\epsilon)}^{1+\epsilon} \right)^{1/1+\epsilon}$$

Hausdorff-Young's inequality $\| f_i \|_{1+\epsilon} \gtrsim \|\mathcal{F}_k(f_i)\|_{2(1+\epsilon)}$ completes the proof.

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