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Review Paper

Sharp Best Constants of Higher Optimal Orders in Littlewood-Paley Inequalities

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Abstract
We follow Q. Xu on the smooth method of paper [37] for the classical Poisson semigroup $\{\mathbb{P}_t\}_{t>0}$ on $\mathbb{R}^d$ and $G^{\mathbb{P}}$ the associated Littlewood-Paley g-function operator

$$G^{\mathbb{P}}(f_j) = \left(\int_0^\infty \sum_j t\left|\frac{\partial}{\partial t}\mathbb{P}_t(f_j)\right|^2 dt\right)^{\frac{1}{2}}.$$

The classical Littlewood-Paley g-function inequality asserts that for any $0 < \epsilon < \infty$ there exist two positive constants $\mathbf{L}^{\mathbb{P}}_{t,1+\epsilon}$ and $\mathbf{L}^{\mathbb{P}}_{c,1+\epsilon}$ such that

$$\left(\mathbf{L}^{\mathbb{P}}_{t,1+\epsilon}\right)^{-1} \| \sum_j f_j \|_{1+\epsilon} \leq \sum_j \|G^{\mathbb{P}}(f_j)\|_{1+\epsilon} \leq \sum_j \mathrm{L}^{\mathbb{P}}_{c,1+\epsilon} \| f_j \|_{1+\epsilon},\ f_j \in L_{1+\epsilon}(\mathbb{R}^d)$$

We determine the optimal orders of magnitude on $(1+\epsilon)$ of these constants as $\epsilon \to 0$ and $\epsilon \to \infty$. We also follow the consideration similar of problems for more general test functions in place of the Poisson kernel (see [37]).
The corresponding problem on the Littlewood-Paley dyadic square function inequality is investigated too. For Δ be the partition of $\mathbb{R}^d$ into dyadic rectangles and S_R the partial sum operator associated to R. The dyadic Littlewood-Paley square functions of f_j is

$$S^{\Delta}(f_j) = \left(\sum_{R\in\Delta}\sum_j |S_R(f_j)|^2\right)^{\frac{1}{2}}.$$

For $0 < \epsilon < \infty$ there exist two perfect positive constants $\mathbf{L}^{\Delta}_{c,1+\epsilon,d}$ and $\mathbf{L}^{\Delta}_{t,1+\epsilon,d}$ such that

$$\left(\mathrm{L}^{\Delta}_{t,1+\epsilon,d}\right)^{-1} \| \sum_j f_j \|_{1+\epsilon} \leq \sum_j \|S^{\Delta}(f_j)\|_{1+\epsilon} \leq \sum_j \mathrm{L}^{\Delta}_{c,1+\epsilon,d} \| f_j \|_{1+\epsilon},\ f_j \in L_{1+\epsilon}(\mathbb{R}^d)$$

We show that

$$\mathrm{L}^{\Delta}_{t,1+\epsilon,d} \approx_d \left(\mathrm{L}^{\Delta}_{t,1+\epsilon,1}\right)^d \text{ and } \mathrm{L}^{\Delta}_{c,1+\epsilon,d} \approx_d \left(\mathrm{L}^{\Delta}_{c,1+\epsilon,1}\right)^d$$

All the previous results can be equally formulated for the d-torus $\mathbb{T}^d$. We show a de Leeuw type transference principle in the vector-valued setting.

1. Introduction

In [35] the vector-valued Littlewood-Paley-Stein theory is investigated, we are confronted with the problem of determining the optimal orders of growth on $(1+\epsilon)$ of the best constants in the classical Littlewood-Paley g-function inequality. We deal with this problem as well as the similar one about another classical Littlewood-Paley inequality on the dyadic square function.

1.1. Littlewood-Paley g-function inequality. This inequality concerns the g-function associated to the Poisson semigroup $\{\mathbb{P}_t\}_{t>0}$ on $\mathbb{R}^d$ whose convolution kernel is

$$\mathbb{P}_t(x) = \frac{c_d t}{(|x|^2+t^2)^{\frac{d+1}{2}}}.$$

The g-square function of $f_j \in L_{1+\epsilon}(\mathbb{R}^d)$ is defined as

$$G^{\mathbb{P}}(f_j)(x) = \left(\int_0^\infty \sum_j t\left|\frac{\partial}{\partial t}\mathbb{P}_t(f_j)(x)\right|^2 dt\right)^{\frac{1}{2}},\ x \in \mathbb{R}^d. \tag{1}$$

The Littlewood-Paley inequality asserts that for any $0 < \epsilon < \infty$ there exist two perfect positive constants $\mathrm{L}^{\mathbb{P}}_{t,1+\epsilon}$ and $\mathrm{L}^{\mathbb{P}}_{c,1+\epsilon}$ such that

$$\left(\mathrm{L}^{\mathbb{P}}_{t,1+\epsilon}\right)^{-1} \| \sum_j f_j \|_{1+\epsilon} \leq \sum_j \left\|G^{\mathbb{P}}(f_j)\right\|_{1+\epsilon} \leq \sum_j \mathrm{L}^{\mathbb{P}}_{c,1+\epsilon} \| f_j \|_{1+\epsilon},\ f_j \in L_{1+\epsilon}(\mathbb{R}^d). \tag{2}$$

We can see Stein [27] for more information and cited references. $L^{\mathbb{P}}_{t,1+\epsilon}$ and $\mathbb{L}^{\mathbb{P}}_{c,1+\epsilon}$ denote the sharp best constants in (2). The use of such notation comes from the vector-valued case studied in [35], the subscripts t and c refer to type and cotype inequalities, respectively, that is, the complex field $\mathbb{C}$ is of both Lusin type and cotype [2] in the sense of [35]. These constants implicitly depend on the mentioned dimension d too. Now, our main objective validity concerns the sharp optimal orders of their magnitude on $(1+\epsilon)$ as $\epsilon \to 0$ and $\epsilon \to \infty$ so we also wish to have dimension free estimates (see [37]).
In addition we consider the heat semigroup $\{\mathbb{H}_t\}_{t>0}$ on $\mathbb{R}^d$ with kernel

$$\mathbb{H}_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}.$$

The associated g-function is defined by (1) with $\mathbb{P}$ replaced by $\mathbb{H}$. Then

$$\left(\mathrm{L}^{\mathbb{H}}_{t,1+\epsilon}\right)^{-1} \| \sum_j f_j \|_{1+\epsilon} \leq \sum_j \left\|G^{\mathbb{H}}(f_j)\right\|_{1+\epsilon} \leq \sum_j \mathrm{L}^{\mathbb{H}}_{c,1+\epsilon} \| f_j \|_{1+\epsilon},\ f_j \in L_{1+\epsilon}(\mathbb{R}^d).$$

We will use the following convention: $A \lesssim B$ (resp. $A \lesssim_{1+\epsilon} B$) means that $A \leq CB$ (resp. $A \leq C_{1+\epsilon}B$) for some absolute positive constant C (resp. a positive constant $C_{1+\epsilon}$ depending only on a parameter $(1+\epsilon)$). $A \approx B$ or $A \approx_{1+\epsilon} B$ means that these inequalities as well as their inverses hold. $\left(\frac{1+\epsilon}{\epsilon}\right)$ will denote the conjugate index of $(1+\epsilon)$.
The following theorem determines the optimal orders of the previous constants except those of $\mathrm{L}^{\mathbb{P}}_{t,1+\epsilon}$ and $\mathrm{L}^{\mathbb{H}}_{t,1+\epsilon}$ as $\epsilon \to \infty$. Part of this theorem is known, see the historical comments at the end of this subsection.
Theorem 1 (see [37]). Let $0 < \epsilon < \infty$. Recall that $\mathbb{L}^{\mathbb{P}}_{t,1+\epsilon}$ and $\underline{L}^{\mathbb{P}}_{c,1+\epsilon}$ are the best constants in the following inequalities

$$\left(\mathrm{L}^{\mathbb{P}}_{t,1+\epsilon}\right)^{-1} \| \sum_j f_j \|_{1+\epsilon} \leq \sum_j \left\|G^{\mathbb{P}}(f_j)\right\|_{1+\epsilon} \leq \sum_j \mathrm{L}^{\mathbb{P}}_{c,1+\epsilon} \| f_j \|_{1+\epsilon},\ f_j \in L_{1+\epsilon}(\mathbb{R}^d)$$

Similarly, we have the best constants $\mathrm{L}^{\mathbb{H}}_{t,1+\epsilon}$ and $\mathrm{L}^{\mathbb{H}}_{c,1+\epsilon}$ corresponding to the heat semigroup. Then
(i) $\mathrm{L}^{\mathbb{P}}_{c,1+\epsilon} \approx \max\left(\sqrt{(1+\epsilon)}, \left(\frac{1+\epsilon}{\epsilon}\right)\right)$ and $\max\left(\sqrt{1+\epsilon}, \frac{1+\epsilon}{\epsilon}\right) \lesssim \mathrm{L}^{\mathbb{H}}_{c,1+\epsilon} \lesssim d\max\left(\sqrt{1+\epsilon}, \frac{1+\epsilon}{\epsilon}\right)$;
(ii) $1 \lesssim \mathrm{L}^{\mathbb{H}}_{t,1+\epsilon} \lesssim \mathrm{L}^{\mathbb{P}}_{t,1+\epsilon} \lesssim_d 1$ for $0 < \epsilon \leq 1$;
(iii) $\sqrt{2+\epsilon} \lesssim \mathrm{L}^{\mathbb{H}}_{t,2+\epsilon} \lesssim \mathrm{L}^{\mathbb{P}}_{t,2+\epsilon} \lesssim 2+\epsilon$ for $0 \leq \epsilon < \infty$
We will show a more general result. Given $\varepsilon > 0$ and $\delta > 0$ let $\mathcal{H}_{\varepsilon,\delta}$ denote the class of all functions $\varphi_j: \mathbb{R}^d \to \mathbb{C}$ satisfying

$$\begin{cases} |\varphi_j(x)| \leq \dfrac{1}{(1+|x|)^{d+\varepsilon}}, & x \in \mathbb{R}^d \\ |\varphi_j(x) - \varphi_j(x+\epsilon)| \leq \dfrac{|\epsilon|^\delta}{(1+|x|)^{d+\varepsilon+\delta}} + \dfrac{|\epsilon|^\delta}{(1+|x+\epsilon|)^{d+\varepsilon+\delta}}, & x, (x+\epsilon) \in \mathbb{R}^d \\ \displaystyle\int_{\mathbb{R}^d} \sum_j \varphi_j(x)dx = 0 & \end{cases} \tag{3}$$

We say that φ_j is nondegenerate if there exists another function $\psi_j \in \mathcal{H}_{\varepsilon,\delta}$ such that

$$\int_0^\infty \sum_j \hat{\varphi}_j(t\xi)\hat{\psi}_j(t\xi)\frac{dt}{t} = 1,\ \forall \xi \in \mathbb{R}^d \setminus \{0\} \tag{4}$$

Let $(\varphi_j)_t(x) = \frac{1}{t^d}\varphi_j\left(\frac{x}{t}\right)$. Define

$$G^{\varphi_j}(f_j)(x) = \left(\int_0^\infty \sum_j \left|(\varphi_j)_t * f_j(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}},\ x \in \mathbb{R}^d \tag{5}$$

for any (reasonable) function f_j on $\mathbb{R}^d$. Then it is well known that the following inequality holds

$$\left(\mathrm{L}^{\varphi_j}_{t,2+\epsilon}\right)^{-1} \| \sum_j f_j \|_{2+\epsilon} \leq \sum_j \left\|G^{\varphi_j}(f_j)\right\|_{2+\epsilon} \leq \sum_j \mathrm{L}^{\varphi_j}_{c,2+\epsilon} \| f_j \|_{2+\epsilon},\ f_j \in L_{2+\epsilon}(\mathbb{R}^d) \tag{6}$$

Theorem 2 (see [37]). Let $\varphi_j \in \mathcal{H}_{\varepsilon,\delta}$ and $0 < \epsilon < \infty$. Then the two best constants $\mathrm{L}^{\varphi_j}_{c,1+\epsilon}$ and $\mathrm{L}^{\varphi_j}_{t,1+\epsilon}$ in the above inequalities satisfy
(i) $\mathrm{L}^{\varphi_j}_{c,1+\epsilon} \lesssim_{d,\varepsilon,\delta} \max\left(\sqrt{1+\epsilon}, \frac{1+\epsilon}{\epsilon}\right)$.
(ii) $\mathrm{L}^{\varphi_j}_{t,1+\epsilon} \lesssim_{d,\varepsilon,\delta} (1+\epsilon)$ if additionally φ_j is nondegenerate.

Assertion (ii) above implies that the first inequality of (6) holds for $\epsilon = 0$ too. In fact, our proof of Theorem 2 implies that the norm $\|G^{\varphi_j}(f_j)\|_{L_1(\mathbb{R}^d)}$ is equivalent to the H_1-norm of f_j. Note that there exist several equivalent definitions of the norm of $H_{1+\epsilon}(\mathbb{R}^d)$; the one used is defined in terms of the $L_{1+\epsilon}$-norm of the nontangential maximal function of the Poisson integral of a function $f_j \in H_{1+\epsilon}(\mathbb{R}^d)$ (cf. [12]). We have the following (see [37]).

Corollary 3. Let $\varphi_j \in \mathcal{H}_{\varepsilon,\delta}$ be nondegenerate. Then

$$\left\|\sum_j G^{\varphi_j}(f_j)\right\|_1 \approx_{d,\varepsilon,\delta} \sum_j \| f_j \|_{H_1},\ f_j \in H_1(\mathbb{R}^d)$$

Moreover, for any $\psi_j \in \mathcal{H}_{\varepsilon,\delta}$ and $0 \le \epsilon \le 1, \epsilon \ge 0$, we have

$$\max\left\{\left\|\sum_j G^{\psi_j}(f_j)\right\|_{1+\epsilon}, \left\|\sum_j S^{\psi_j}_{1+\epsilon}(f_j)\right\|_{1+\epsilon}\right\} \lesssim_{d,\varepsilon,\delta,1+\epsilon} \sum_j \|G^{\varphi_j}(f_j)\|_{1+\epsilon}$$

where $S^{\psi_j}_{1+\epsilon}(f_j)$ is the Lusin area integral function:

$$S^{\psi_j}_{1+\epsilon}(f_j)(x) = \left(\int_{|\epsilon|<(1+\epsilon)t} \sum_j \left|(\psi_j)_t * f_j(x)\right|^2 \frac{d(x+\epsilon)dt}{t^{d+1}}\right)^{\frac{1}{2}},\ x \in \mathbb{R}^d \tag{7}$$

Remark 4. All the previous results hold for the area integral function defined by (7). In fact, a majority of the literature on the Littlewood-Paley theory deals with $S^{\varphi_j}(f_j)$ instead of $G^{\varphi_j}(f_j)$. Namely, if φ_j is a nondegenerate function in $\mathcal{H}_{\varepsilon,\delta}$, then for any $0 < \epsilon < \infty$

$$\sum_j \left(\mathrm{L}^{\varphi_j}_{t,1+\epsilon,S}\right)^{-1} \| f_j \|_{1+\epsilon} \le \sum_j \left\|S^{\varphi_j}_{1+\epsilon}(f_j)\right\|_{1+\epsilon} \le \sum_j \mathrm{L}^{\varphi_j}_{c,1+\epsilon,S} \| f_j \|_{1+\epsilon},\ f_j \in L_{1+\epsilon}(\mathbb{R}^d)$$

The sharp best constants $\mathrm{L}^{\varphi_j}_{c,1+\epsilon,S}$ and $\mathrm{L}^{\varphi_j}_{t,1+\epsilon,S}$ satisfy

$$\mathrm{L}^{\varphi_j}_{c,1+\epsilon,S} \lesssim_{d,\varepsilon,\delta,1+\epsilon} \max\left(\sqrt{1+\epsilon}, \frac{1+\epsilon}{\epsilon}\right) \text{ and } \mathrm{L}^{\varphi_j}_{t,1+\epsilon,S} \lesssim_{d,\varepsilon,\delta,1+\epsilon} \sqrt{1+\epsilon} \tag{8}$$

The first one is well known. As far as for the second, the case $\epsilon \le 0$ is classical for sufficiently nice φ_j, for instance, for the Poisson kernel. The case $\epsilon > 0$ is implicitly contained in [7, it can also be found in [32] if the aperture $(1+\epsilon)$ is large enough, say $d \le \left(\frac{1+\epsilon}{3}\right)^2$; for $d > \left(\frac{1+\epsilon}{3}\right)^2$, one can adapt Wilson's argument. See [9, 13, 15] for related results.

Considering the Poisson kernel as in Theorem we see that the orders of the constants in (8) are optimal as $\epsilon \to 0$ and $\epsilon \to \infty$. Thus the higher optimal orders of the constants in (8) are completely determined.

Compared with the area integral function discussed in the above remark, the situation for the g-function is more delicate. We are unable to determine the optimal orders of $L^{\mathbb{P}}_{t,1+\epsilon}$ and $L^{\mathbb{H}}_{t,1+\epsilon}$ as $\epsilon \to \infty$. The following problem is closely related to the one mentioned on page 239 of [7].

Problem 5 ([37]). Let $\varphi_j \in \mathcal{H}_{\varepsilon,\delta}$ be nondegenerate. Does one have $\mathrm{L}^{\varphi_j}_{t,1+\epsilon} \lesssim_{d,\varepsilon,\delta} \sqrt{1+\epsilon}$ for $0 \le \epsilon < \infty$? In particular, does this hold for the classical Poisson or heat kernel on $\mathbb{R}^d$?

Remark 6. We determined in [35] the optimal orders of the best constants in (2) for more general semigroups. Namely, $\{\mathbb{P}_t\}_{t>0}$ in (2) can be replaced by the Poisson semigroup $\{P_t\}_{t>0}$ subordinated to any strongly continuous semigroup $\{T_t\}_{t>0}$ of regular contractions on $L_{2+\epsilon}(\Omega)$ for a fixed $0 < \epsilon < \infty$. The corresponding constants $\mathrm{L}^{P}_{c,1+\epsilon}$ and $\mathrm{L}^{P}_{t,1+\epsilon}$ also satisfy Theorem 1 except assertion (ii) (which is an open problem). See [35] for more details in [36], the authors proved that the optimal order of $\mathbf{L}^{P}_{t,1+\epsilon}$ is $(1+\epsilon)$ as $\epsilon \to \infty$ for symmetric Markovian semigroups. Thus the previous problem has a negative solution if the classical Poisson or heat semigroup is replaced by a general symmetric Markovian semigroup.

It would be also interesting to have dimension free estimates for $\mathrm{L}^{\mathbb{H}}_{c,1+\epsilon}$ in Theorem (i):

Problem 7 ([37]). Does one have $\mathrm{L}^{\mathbb{H}}_{c,1+\epsilon} \lesssim \max\left(\sqrt{1+\epsilon}, \frac{1+\epsilon}{\epsilon}\right)$?

Meyer [20] shows that this question has an affirmative answer for $\epsilon > 0$ if the time derivative in the definition of the g-function $G^{\mathbb{H}}$ is replaced by the spatial derivative. Now let

$$G_{\nabla_x}^{\mathbb{H}}(f_j)(x) = \left(\int_0^\infty \sum_j \left|\nabla_x \mathbb{H}_t(f_j)(x)\right|^2 dt \right)^{\frac{1}{2}}$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right)$. Then

$$\left\| \sum_j G_{\nabla_x}^{\mathbb{H}}(f_j) \right\|_{2+\epsilon} \lesssim \sqrt{2+\epsilon} \sum_j \| f_j \|_{2+\epsilon},\ f_j \in L_{2+\epsilon}(\mathbb{R}^d), 0 \le \epsilon < \infty.$$

Remark 8. All the previous results admit periodic analogues, that is, for $\mathbb{T}^d$ in place of $\mathbb{R}^d$, where $\mathbb{T}^d$ is the d-torus equipped with normalized Haar measure. This can be done by modifying the arguments for $\mathbb{R}^d$, or more simply by a de Leeuw type transference principle. We will show a variant of de Leeuw's theorem in section 2. Using this de Leeuw type theorem, as illustration we will explain in section 4 why the constant $\mathrm{L}_{c,2+\epsilon}^{\mathbb{P}}$ and its period analogue are essentially the same. A full argument can be found in the proof of Theorem 9 below (see [37]).

Historical comments. We make some remarks on the previous results and especially point out the part of them which are known or implicit in the literature.

(i) The classical Littlewood-Paley g-function is usually defined by using the full gradient of $\mathbb{P}_t(f_j)$ in place of the partial derivative in the time variable. We denote the latter by $G_{\nabla}^{\mathbb{P}}(f_j)$:

$$G_{\nabla}^{\mathbb{P}}(f_j) = \left(\int_0^\infty \sum_j t\left|\nabla \mathbb{P}_t(f_j)\right|^2 dt \right)^{\frac{1}{2}}. \tag{9}$$

This square function behaves better than the previous one since it is invariant under the Riesz transforms. Theorem 1 equally holds for $G_{\nabla}^{\mathbb{P}}$ in place of $G^{\mathbb{P}}$. The corresponding proof is slightly simpler (see the related remark of [12]).

(ii) Part (i) of Theorem 2 is known and can be found in Wilson's book [32]. In fact, Wilson shows a stronger result on his intrinsic g-function that is defined by

$$G_{\varepsilon,\delta}(f_j)(x) = \left(\int_0^\infty \sum_j \sup_{\varphi_j \in \mathcal{H}_{\varepsilon,\delta}} \left|(\varphi_j)_t * f_j(x)\right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Then for any weight w on $\mathbb{R}^d$ and $f_j \in L_2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \sum_j G_{\varepsilon,\delta}(f_j)^2 w \lesssim_{d,\varepsilon,\delta} \int_{\mathbb{R}^d} \sum_j |f_j|^2 M(w)$$

where $M(w)$ denotes the Hardy-Littlewood maximal function of w. This implies that for $0 \le \epsilon < \infty$

$$\left\| \sum_j G_{\varepsilon,\delta}(f_j) \right\|_{2+\epsilon} \lesssim_{d,\varepsilon,\delta} \left(\sqrt{2+\epsilon}\right) \sum_j \| f_j \|_{2+\epsilon}$$

whence $\mathrm{L}_{c,2+\epsilon}^{\varphi_j} \lesssim_{d,\varepsilon,\delta} \sqrt{2+\epsilon}$ for $\epsilon \ge 0$. The case $\epsilon < 0$ is dealt with by a standard argument involving singular integral theory for G^{φ_j} can be expressed as a Calderón-Zygmund operator with Hilbert space valued kernel (see Lemma 20 below).

(iii) Part (ii) of Theorem 2 for $\epsilon > 0$ immediately follows from part (i) by duality.

(iv) The upper estimate $\|\sum_j G^{\varphi_j}(f_j)\|_{L_1(\mathbb{R}^d)} \lesssim_{d,\varepsilon,\delta} \| \sum_j f_j \|_{H_1(\mathbb{R}^d)}$ in Corollary 3 is also well known for the same reason by singular integral theory. The converse inequality is classical too for the Poisson kernel.

(v) Wilson's theorem just quoted implies particularly $\mathcal{L}_{c,2+\epsilon}^{\mathbb{P}} \lesssim_d \sqrt{2+\epsilon}$ for $\epsilon \ge 0$. However, the dimension free estimate $\mathrm{L}_{c,2+\epsilon}^{\mathbb{P}} \lesssim \sqrt{2+\epsilon}$ in Theorem 1 (i) is due to Meyer [20] by probabilistic method. In [35] we prove a similar result for general semigroups. The dimension free estimate $\mathrm{L}_{t,2+\epsilon}^{\mathbb{P}} \lesssim 2+\epsilon$ for $\epsilon \ge 0$ in Theorem 1 (iii) is also a special case of that result of [35].

(vi) We can now summarize the new part of Theorems 12 and Corollary 3 as follows

$$\mathcal{L}^{\mathbb{H}}_{c,2+\epsilon} \gtrsim \mathrm{L}^{\mathbb{P}}_{c,2+\epsilon} \gtrsim \sqrt{2+\epsilon} \text{ and } \mathrm{L}^{\mathbb{P}}_{t,2+\epsilon} \gtrsim \mathrm{L}^{\mathbb{H}}_{t,2+\epsilon} \gtrsim \sqrt{2+\epsilon} \text{ for } 0 \le \epsilon < \infty$$

$$\mathcal{L}^{\mathbb{P}}_{c,1+\epsilon} \lesssim \frac{1+\epsilon}{\epsilon} \text{ for } 0 < \epsilon \le 1 \text{ and } \mathrm{L}^{\mathbb{P}}_{t,1+\epsilon} \lesssim (1+\epsilon) \text{ for } \epsilon \ge 0 \text{ (dimension freeness);}$$

$$\| \sum_j f_j \|_{H_1} \lesssim_{d,\epsilon,\delta} \left\| \sum_j G^{\varphi_j}(f_j) \right\|_1 \text{ for any nondegenerate } \varphi_j \in \mathcal{H}_{\epsilon,\delta}$$

1.2. Littlewood-Paley dyadic square function inequality. There exists another equally famous inequality named after Littlewood-Paley, the one related to the dyadic decomposition of $\mathbb{R}^d$. First, partition $\mathbb{R} \setminus \{0\}$ into the intervals $[2^{k-1}, 2^k)$ and $(-2^k, -2^{k-1}]$, $k \in \mathbb{Z}$. Then the family Δ of all d-fold products of these intervals give a partition of $\mathbb{R}^d$ (deprived of the origin). For any $R \in \Delta$ let S_R be the corresponding partial sum operator, that is, $\widehat{S_R}(\hat{f}_j) = \mathbb{1}_R \hat{f}_j$. The dyadic Littlewood-Paley square function of f_j is defined by

$$S^{\Delta}(f_j) = \left(\sum_{R\in\Delta} \sum_j |S_R(f_j)|^2 \right)^{\frac{1}{2}}$$

It is well known that for $0 < \epsilon < \infty$ there exist two positive constants $\mathrm{L}^{\Delta}_{c,1+\epsilon,d}$ and $\mathrm{L}^{\Delta}_{t,1+\epsilon,d}$ such that

$$\left(\mathrm{L}^{\Delta}_{t,1+\epsilon,d}\right)^{-1} \| \sum_j f_j \|_{1+\epsilon} \le \sum_j \left\|S^{\Delta}(f_j)\right\|_{1+\epsilon} \le \mathrm{L}^{\Delta}_{c,1+\epsilon,d} \sum_j \| f_j \|_{1+\epsilon},\ f_j \in L_{1+\epsilon}(\mathbb{R}^d) \tag{10}$$

See [16, Theorem 5.1.6] where it is showed that both constants $\mathrm{L}^{\Delta}_{c,1+\epsilon,d}$ and $\mathrm{L}^{\Delta}_{t,1+\epsilon,d}$ are majorized by $C_d \max\left(1+\epsilon, \frac{1+\epsilon}{\epsilon}\right)^{2d}$. However, like for the g-function inequality, their optimal orders have not been completely determined in the literature. Note that the Littlewood-Paley g-function inequality belongs to one-parameter harmonic analysis while the above inequality is of multiparameter nature. This explains why we now mention d explicitly as a subscript in the above constants.

In the spirit of Remark 8, we formulate the periodic counterpart of (10). The only difference is that the dyadic rectangles now consist of integers, so the corresponding dyadic partition of $\mathbb{Z}^d$ is $\widetilde{\Delta} = \{R \cap \mathbb{Z}^d : R \in \Delta\}$. We similarly define the partial sum operators S_R and

$$S^{\widetilde{\Delta}}(f_j) = \left(\sum_{R\in\widetilde{\Delta}} \sum_j |S_R(f_j)|^2 \right)^{\frac{1}{2}},\ f_j \in L_{1+\epsilon}(\mathbb{T}^d)$$

So (10) becomes

$$\left(\mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,d}\right)^{-1} \| \sum_j f_j \|_{1+\epsilon} \le \sum_j \left\|S^{\widetilde{\Delta}}(f_j)\right\|_{1+\epsilon} \le \mathrm{L}^{\widetilde{\Delta}}_{c,1+\epsilon,d} \sum_j \| f_j \|_{1+\epsilon},\ f_j \in L_{1+\epsilon}(\mathbb{T}^d) \tag{11}$$

The following reinforces the meaning of Remark 8,

Theorem 9 [37]. Let $0 < \epsilon < \infty$. Then the best constants in (10) and (11) satisfy

$$\mathrm{L}^{\Delta}_{c,1+\epsilon,d} = \mathrm{L}^{\widetilde{\Delta}}_{c,1+\epsilon,d} \qquad \textit{and} \qquad \mathrm{L}^{\Delta}_{t,1+\epsilon,d} = \mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,d}.$$

The following result illustrates the multi-parameter nature of (10) and (11). In view of the previous theorem, we need to state it only for the periodic case.

Theorem 10 [37]. There exists a universal positive constant C such that

$$\left(\mathrm{L}^{\widetilde{\Delta}}_{c,1+\epsilon,1}\right)^d \le \mathrm{L}^{\widetilde{\Delta}}_{c,1+\epsilon,d} \le \left(C\ \mathrm{L}^{\widetilde{\Delta}}_{c,1+\epsilon,1}\right)^d \text{ and } \left(\mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,1}\right)^d \le \mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,d} \le \left(C\ \mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,1}\right)^d$$

The following corollary determines the optimal orders of all the best constants (except one) in (10), so as well as of those in (11).

Corollary 11 [37]. Let $0 < \epsilon < \infty$. Then

(i) $\mathrm{L}^{\Delta}_{c,1+\epsilon,d} \approx_d (1+\epsilon)^{\frac{3}{2}d}$ for $0 < \epsilon \le 1$ and $\mathrm{L}^{\Delta}_{c,1+\epsilon,d} \approx_d (1+\epsilon)^d$ for $0 < \epsilon < \infty$;

(ii) $\mathrm{L}^{\Delta}_{t,1+\epsilon,d} \approx_d 1$ for $0 < \epsilon \le 1$ and $(1+\epsilon)^{\frac{d}{2}} \lesssim \mathrm{L}^{\Delta}_{t,1+\epsilon,d} \lesssim_d (1+\epsilon)^d$ for $0 \le \epsilon < \infty$.

Remark 12. The estimate $\mathrm{L}^{\Delta}_{t,1+\epsilon,d} \lesssim_d (1+\epsilon)^d$ for $0 \le \epsilon < \infty$ was proved independently by Odysseas Bakas and Hao Zhang after the submission of this article; it improves the author's original one $\mathrm{L}^{\Delta}_{t,2+\epsilon,d} \lesssim_d ((2+\epsilon)\log(2+\epsilon))^d$. However, like for the g-function inequality, we are unable to determine the optimal order of $\mathrm{L}^{\Delta}_{t,2+\epsilon,d}$ for $\epsilon > 0$. We need to do this only for $d = 1$ by virtue of Theorem 10 .

Problem 13 [37]. Determine the optimal order of $\mathrm{L}^{\Delta}_{t,2+\epsilon,1}$ as $\epsilon \to \infty$.

This problem is related to Problem 5. In fact, the smooth version of the dyadic square function S^{Δ} is a discrete g-function, so the analogue for the smooth version of the above problem is a particular case of Problem 5.

Historical comments. Part of Corollary 11 is already known.

(i) It is Bourgain [6] who first studied the problem on the optimal orders of the above constants by determining the optimal order of $\mathbf{L}^{\tilde{\Delta}}_{c,2+\epsilon,1}$. Lerner [18] noted that Bourgain's result remains valid for $\mathbb{R}$ by a different method via weighted norm inequalities.

(ii) Bourgain [5] proved that $\mathrm{L}^{\tilde{\Delta}}_{t,2+\epsilon,1} \approx 1$. In fact, Bourgain showed that the second inequality of (11) holds for any partition of $\mathbb{Z}$ into bounded intervals in the case of $0 \le \epsilon \le 1$. This latter result is dual to Rubio de Francia's celebrated Littlewood-Paley inequality [25] that insures the validity of the first inequality of (11) for any partition of $\mathbb{Z}$ into bounded intervals in the case of $0 \le \epsilon < \infty$.

(iii) Bakas [1] extended one of Bourgain's estimates to the higher dimensions by showing $\mathrm{L}^{\tilde{\Delta}}_{c,2+\epsilon,d} \approx_d (2+\epsilon)'^{\frac{3d}{2}}$ for $0 < \epsilon \le 1$.

(iv) Journé [17] extended Rubio de Francia's inequality to the multi-dimensional setting without explicit estimate of the relevant constant. It is Osipov who proved the second inequality of (10) for $0 \le \epsilon \le 1$ and for any partition of $\mathbb{R}^d$ into bounded rectangles. In particular, Osipov's result implies $\mathrm{L}^{\Delta}_{t,1+\epsilon,d} \approx_d 1$ for $1 < \epsilon \le 0$.

(v) The estimate $(2+\epsilon)^{\frac{d}{2}} \lesssim \mathrm{L}^{\Delta}_{t,2+\epsilon,d}$ for $0 \le \epsilon < \infty$ easily follows from the optimal order of the best constant in the Khintchine inequality for $\epsilon > 0$; on the other hand, Pichorides [22] proved $\mathrm{L}^{\tilde{\Delta}}_{t,2+\epsilon,1} \lesssim (2+\epsilon)\log(2+\epsilon)$ for $0 \le \epsilon < \infty$.

2. A Variant of De Leeuw's Multiplier Theorem

We give a variant of de Leeuw's classical transference theorem on Fourier multipliers on $\mathbb{R}^d$ and $\mathbb{T}^d$, see [11] and [29, Chapter VII.3]. (see also [37]).

We begin by fixing some notation. Given $z^m = (z_1^m, \cdots, z_d^m) \in \mathbb{T}^d$ and $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$ let $z^m = \left(z_1^{m_1}, \cdots, z_d^{m_d}\right)$. We identify $\mathbb{T}^d$ with the cube $\mathbb{I}^d = \left[-\frac{1}{2}, \frac{1}{2}\right)^d \subset \mathbb{R}^d$ via $z^m = \left(e^{2\pi i x_1}, \cdots, e^{2\pi i x_d}\right) \leftrightarrow x = (x_1, \cdots, x_d)$, and accordingly the functions on $\mathbb{T}^d$ with the 1-periodic functions on $\mathbb{R}^d$.

Let X, Y be two Banach spaces and $B(X,Y)$ the space of continuous linear operators from X to Y. Given a function $\varphi_j : \mathbb{R}^d \to B(X,Y)$, let T_{φ_j} be the Fourier multiplier formally defined by $\widehat{T_{\varphi_j}}(\hat{f_j})(\xi) = \varphi_j(\xi)\hat{f_j}(\xi)$ for $\xi \in \mathbb{R}^d$ and $f_j \in L_{2+\epsilon}(\mathbb{R}^d; X)$; similarly, define the Fourier multiplier M_{φ_j} in the periodic case when φ_j is restricted to $\mathbb{Z}^d$, namely, $\widehat{M_{\varphi_j}}(\hat{f_j})(m) = \varphi_j(m)\hat{f_j}(m)$ for $m \in \mathbb{Z}^d$ and $f_j \in L_{2+\epsilon}(\mathbb{T}^d; X)$. Here given a measure space (Ω, μ), $L_{2+\epsilon}(\Omega; X)$ denote the space of $(2+\epsilon)$-integrable functions from Ω to X.

We will assume that the symbol φ_j satisfies the following conditions:

$(\mathbf{H}_1)$ for every $a \in X, \varphi_j(\cdot)(a)$ is a measurable function from $\mathbb{R}^d$ to Y, and φ_j is bounded, i.e., $M = \sup_{\xi \in \mathbb{R}^d} \| \varphi_j(\xi) \|_{B(X,Y)} < \infty$;

$(\mathbf{H}_2)$ there exists a partition $\mathcal{R}$ of $\mathbb{R}^d$ into bounded rectangles such that φ_j is strongly continuous on every $R \in \mathcal{R}$, i.e., $\varphi_j(\cdot)(a)$ is continuous from R to Y for every $a \in X$;

$(\mathbf{H}_3)$ for every $a \in X$ and every $R \in \mathcal{R}$ the range of the restriction of $\varphi_j(\cdot)(a)$ to R is contained in a finite dimensional subspace of Y;

$(\mathbf{H}_4)$ for every $0 < \epsilon < \infty$ and every $a \in X$ there exists a constant $C_{1+\epsilon,a}$ such that

$$\left\| \sum_j T_{\varphi_j}(af_j) \right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)} \le (C_{1+\epsilon,a}) \sum_j \| f_j \|_{L_{1+\epsilon}(\mathbb{R}^d)}$$

for all compactly supported C^∞ functions f_j on $\mathbb{R}^d$.

The last condition implies that $T_{\varphi_j}(f_j)$ is well-defined and belongs to $L_{1+\epsilon}(\mathbb{R}^d; Y)$ for any compactly supported C^∞ function f_j with values in a finite dimensional subspace of X.

We are interested in the best constants $(1+\epsilon)$ and $(1+2\epsilon)$ in the following inequalities

$$(1+\epsilon)^{-1} \| \sum_j f_j \|_{L_{1+\epsilon}(\mathbb{R}^d;X)} \le \sum_j \left\| T_{\varphi_j}(f_j) \right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)} \le \sum_j (1+2\epsilon) \| f_j \|_{L_{1+\epsilon}(\mathbb{R}^d;X)} \qquad (12)$$

for all compactly supported C^∞ functions f_j on $\mathbb{R}^d$ with values in a finite dimensional subspace of X. Obviously, if it is finite, $(1+2\epsilon)$ is equal to the norm of T_{φ_j} as a map from $L_{1+\epsilon}(\mathbb{R}^d;X)$ to $L_{1+\epsilon}(\mathbb{R}^d;Y)$; we will denote this norm simply by $\|T_{\varphi_j}\|_{1+\epsilon\to 1+\epsilon}$. Similarly and by a slight abuse of notation, if $(1+\epsilon)$ is finite, we will denote it by $\|T_{\varphi_j}^{-1}\|_{1+\epsilon\to 1+\epsilon}$ which is the norm of $T_{\varphi_j}^{-1}$ from the image of T_{φ_j} in $L_{1+\epsilon}(\mathbb{R}^d;Y)$ to $L_{1+\epsilon}(\mathbb{R}^d;X)$.

We will also consider the periodic version of (12), the corresponding constants will be denoted by $\|M_{\varphi_j}\|_{1+\epsilon\to 1+\epsilon}$ and $\|M_{\varphi_j}^{-1}\|_{1+\epsilon\to 1+\epsilon}$ which are the best constants such that

$$\sum_j \|M_{\varphi_j}^{-1}\|_{1+\epsilon\to 1+\epsilon}^{-1} \| f_j \|_{L_{1+\epsilon}(\mathbb{T}^d;X)} \le \sum_j \|M_{\varphi_j}(f_j)\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)} \le \sum_j \|M_{\varphi_j}\|_{1+\epsilon\to 1+\epsilon} \| f_j \|_{L_{1+\epsilon}(\mathbb{T}^d;X)} \quad (13)$$

for all trigonometric polynomials f_j with coefficients in X.

We make the convention that if one of the inequalities in (12) and (13) does not hold, the corresponding constant is understood to be infinite.

Theorem 14 [37]. Let $0 < \epsilon < \infty$.

(i) Assume that φ_j is strongly continuous at every $m \in \mathbb{Z}^d$. Then

$$\left\|\sum_j M_{\varphi_j}\right\|_{1+\epsilon\to 1+\epsilon} \le \sum_j \|T_{\varphi_j}\|_{1+\epsilon\to 1+\epsilon} \quad \text{and} \quad \left\|\sum_j M_{\varphi_j}^{-1}\right\|_{1+\epsilon\to 1+\epsilon} \le \sum_j \|T_{\varphi_j}^{-1}\|_{1+\epsilon\to 1+\epsilon}.$$

(ii) Given $t>0$ define $\varphi_j^{(t)}$ by $\varphi_j^{(t)}(\xi) = \varphi_j(t\xi)$. Then

$$\|T_{\varphi_j}\|_{1+\epsilon\to 1+\epsilon} \le \liminf_{t\to 0} \sum_j \left\|M_{\varphi_j^{(t)}}\right\|_{1+\epsilon\to 1+\epsilon} \quad \text{and} \quad \|T_{\varphi_j}^{-1}\|_{1+\epsilon\to 1+\epsilon} \le \liminf_{t\to 0} \sum_j \left\|M_{\varphi_j^{(t)}}^{-1}\right\|_{1+\epsilon\to 1+\epsilon}.$$

We will adapt de Leeuw's arguments. Note, however, that de Leeuw's proof depends on a duality argument that does not seem to extend to our setting. Instead, we establish a direct link between T_{φ_j} and M_{φ_j} as in Lemma 16 below.

The following lemma is a well-known elementary fact (see [29, Lemma VII.3.9]).

Lemma 15. Let $f_j \in L_1(\mathbb{T}^d)$. Then

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} \sum_j f_j(x)\mathbb{H}_t(x)dx = \int_{\mathbb{T}^d} \sum_j f_j(x+2\epsilon)d(x+2\epsilon)$$

The following expresses the periodic Fourier multiplier M_{φ_j} in terms of the Euclidean T_{φ_j}.

Lemma 16 (see [37]). Assume that φ_j is strongly continuous at every point $m \in \mathbb{Z}^d$. Let P be a trigonometric polynomial with coefficients in X. Then

$$\lim_{t\to 0} \sum_j (4\pi(1+\epsilon)t)^{\frac{d}{2(1+\epsilon)}} \|T_{\varphi_j}(P\widehat{\mathbb{H}_t})\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)} = \sum_j \|M_{\varphi_j}(P)\|_{L_{1+\epsilon}(\mathbb{T}^d;Y)}$$

Proof. By approximation, we can assume that φ_j is compactly supported. Let

$$P(x+2\epsilon) = \sum_m a_m z^m, \; a_m \in X$$

Then

$$\widehat{P\widehat{\mathbb{H}_t}}(\xi) = \sum_m a_m \mathbb{H}_t(\xi - m)$$

Thus

$$\begin{aligned} T_{\varphi_j}(P\widehat{\mathbb{H}_t})(x) &= \sum_m \int_{\mathbb{R}^d} \sum_j \varphi_j(\xi)(a_m)\mathbb{H}_t(\xi-m)e^{2\pi i\xi\cdot x}d\xi \\ &= \sum_j M_{\varphi_j}(P)(x)\widehat{\mathbb{H}}_t(x) + \sum_m \int_{\mathbb{R}^d} \sum_j (\varphi_j(\xi)-\varphi_j(m))(a_m)\mathbb{H}_t(\xi-m)e^{2\pi i\xi\cdot x}d\xi \\ &\overset{\text{def}}{=} \sum_j M_{\varphi_j}(P)(x)\widehat{\mathbb{H}}_t(x) + \sum_m \sum_j (f_j)_{m,t}(x) \end{aligned}$$

Recall that $\widehat{\mathbb{H}}_t(x) = e^{-4\pi^2 t|x|^2}$. Letting $s = (16\pi^2(1+\epsilon)t)^{-1}$ and using Lemma 15, we get

$$\lim_{t\to 0}(4\pi(1+\epsilon)t)^{\frac{d}{2}}\int_{\mathbb{R}^d}\sum_j \left\|M_{\varphi_j}(P)(x)\widehat{\mathbb{H}}_t(x)\right\|_Y^{1+\epsilon}dx=\lim_{s\to\infty}\int_{\mathbb{R}^d}\sum_j \left\|M_{\varphi_j}(P)(x)\right\|_Y^{1+\epsilon}\mathbb{H}_s(x)dx$$
$$=\sum_j \left\|M_{\varphi_j}(P)\right\|_{L_{1+\epsilon}(\mathbb{T}^d;Y)}^{1+\epsilon}.$$

Thus it remains to show that

$$\lim_{t\to 0}(4\pi(1+\epsilon)t)^{\frac{d}{2(1+\epsilon)}}\sum_j \left\|(f_j)_{m,t}\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}=0,\ \forall m. \tag{14}$$

Choose $0<\epsilon<\infty$ and $0<\epsilon<1$ such that $\epsilon=0$ or 1. Then

$$(4\pi(1+\epsilon)t)^{\frac{d}{2(1+\epsilon)}}\left\|\sum_j (f_j)_{m,t}\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}$$
$$\le\left[(4\pi(1+\epsilon)t)^{\frac{d}{4}}\sum_j \left\|(f_j)_{m,t}\right\|_{L_2(\mathbb{R}^d;Y)}\right]^{\epsilon}\left[(4\pi(1+\epsilon)t)^{\frac{d}{2(1+\epsilon)}}\left\|\sum_j (f_j)_{m,t}\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}\right]^{1-\epsilon}.$$

By $(\mathbf{H}_1)$ and $(\mathbf{H}_4)$,

$$\left\|\sum_j (f_j)_{m,t}\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}\le\left(C_{1+\epsilon,a_m}+M\|a_m\|_X\right)\|z^m\widehat{\mathbb{H}_t}\|_{L_{1+\epsilon}(\mathbb{R}^d)}=\left(C_{1+\epsilon,a_m}+M\|a_m\|_X\right)\|\widehat{\mathbb{H}_t}\|_{L_{1+\epsilon}(\mathbb{R}^d)}.$$

It follows that

$$\sup_{t>0}\sum_j (4\pi(1+\epsilon)t)^{\frac{d}{2(1+\epsilon)}}\left\|(f_j)_{m,t}\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}<\infty.$$

Let us treat the part on the L_2-norm. Since we have assumed that φ_j is compactly supported, by $(\mathbf{H}_3)$, we can further assume that Y is finite dimensional, so isomorphic to a Hilbert space. Thus by the Plancherel identity, there exists a constant C, depending on a_m and Y, such that

$$\left\|(f_j)_{m,t}\right\|_{L_2(\mathbb{R}^d;Y)}^2\le C^2\int_{\mathbb{R}^d}\sum_j \|(\varphi_j(\xi)-\varphi_j(m))(a_m)\|_Y^2\mathbb{H}_t(\xi-m)^2d\xi$$

Given $\varepsilon>0$, the strong continuity of φ_j at m implies that there exists $\delta>0$ such that $\|\sum_j (\varphi_j(\xi)-\varphi_j(m))(a_m)\|_Y<\varepsilon$ whenever $|\xi-m|<\delta$. Thus by $(\mathbf{H}_1)$,

$$(4\pi(1+\epsilon)t)^{\frac{d}{2}}\int_{\mathbb{R}^d}\sum_j \|(\varphi_j(\xi)-\varphi_j(m))(a_m)\|_Y^2\mathbb{H}_t(\xi-m)^2d\xi$$
$$\le\varepsilon^2(4\pi(1+\epsilon)t)^{\frac{d}{2}}\int_{|\xi-m|<\delta}\mathbb{H}_t(\xi-m)^2d\xi+\left(2M\|a_m\|_X\right)^2(4\pi(1+\epsilon)t)^{\frac{d}{2}}\int_{|\xi-m|\ge\delta}\mathbb{H}_t(\xi-m)^2d\xi$$
$$\lesssim_{1+\epsilon,d}\varepsilon^2+\left(2M\|a_m\|_X\right)^2\int_{|\xi|\ge\frac{\delta}{\sqrt{2t}}}e^{-|\xi|^2}d\xi$$

Therefore,

$$\limsup_{t\to 0}\sum_j (4\pi(1+\epsilon)t)^{\frac{d}{4}}\left\|(f_j)_{m,t}\right\|_{L_2(\mathbb{R}^d;Y)}\lesssim_{1+\epsilon,d}C\varepsilon$$

As ε is arbitrary, combining the above estimates, we deduce (14).

Conversely, we can estimate T_{φ_j} in terms of M_{φ_j}.

Lemma 17 (see [37]). Let $f_j:\mathbb{R}^d\to X$ be a C^∞ function with compact support and define the periodization of $(\tilde{f}_j)_t$:

$$(\tilde{f}_j)_t(x)=\sum_{m\in\mathbb{Z}^d}\sum_j (f_j)_t(x+m),\ x\in\mathbb{R}^d$$

Viewing $(\tilde{f}_j)_t$ as a function on $\mathbb{T}^d$, we have

$$\lim_{t\to 0}\sum_j t^{\frac{d}{(1+\epsilon)^1}}\left\|M_{\varphi_j^{(t)}}\left(\left(\tilde{f}_j\right)_t\right)\right\|_{L_{1+\epsilon}(\mathbb{T}^d;Y)}=\sum_j \left\|T_{\varphi_j}(f_j)\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}$$

Proof. The Fourier series of $\left(\tilde{f}_j\right)_t$ is given by

$$\left(\tilde{f}_j\right)_t(x)=\sum_{m\in\mathbb{Z}^d}\sum_j \hat{f}_j(tm)e^{2\pi i m\cdot x}$$

Thus

$$\lim_{t\to 0}\sum_j t^d M_{\varphi_j^{(t)}}\left(\left(\tilde{f}_j\right)_t\right)(tx)=\lim_{t\to 0}t^d\sum_{m\in\mathbb{Z}^d}\sum_j \varphi_j(tm)\hat{f}_j(tm)e^{2\pi i t m\cdot x}$$
$$=\int_{\mathbb{R}^d}\sum_j \varphi_j(\xi)\hat{f}_j(\xi)e^{2\pi i\xi\cdot x}d\xi=\sum_j T_{\varphi_j}(f_j)(x)$$

where we have used $(\mathbf{H}_2)$ to insure that the above integral exists in Riemann's sense. Let η be a nonnegative continuous function with compact support on $\mathbb{R}^d$ such that

$$\eta(0)=1 \text{ and } \sum_{m\in\mathbb{Z}^d}\eta(x+m)^{1+\epsilon}=1$$

(see [29, Lemma VII.3.21]). Then

$$\lim_{t\to 0}\sum_j t^d M_{\varphi_j^{(t)}}\left(\left(\tilde{f}_j\right)_t\right)(tx)\eta(tx)=\sum_j T_{\varphi_j}(f_j)(x)$$

Thus

$$\left\|\sum_j T_{\varphi_j}(f_j)\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}=\lim_{t\to 0}\sum_j t^d\left\|M_{\varphi_j^{(t)}}\left(\left(\tilde{f}_j\right)_t\right)(t\,\cdot)\eta(t\,\cdot)\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}$$

However,

$$\left\|\sum_j M_{\varphi_j^{(t)}}\left(\left(\tilde{f}_j\right)_t\right)(t\,\cdot)\eta(t\,\cdot)\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}^{1+\epsilon}=t^{-d}\int_{\mathbb{R}^d}\sum_j \left\|M_{\varphi_j^{(t)}}\left(\left(\tilde{f}_j\right)_t\right)(x)\right\|_Y^{1+\epsilon}\eta(x)^{1+\epsilon}dx$$
$$=t^{-d}\sum_{m\in\mathbb{Z}^d}\int_{\mathbb{I}^d}\sum_j \left\|M_{\varphi_j^{(t)}}\left(\left(\tilde{f}_j\right)_t\right)(x)\right\|_Y^{1+\epsilon}\eta(x+m)^{1+\epsilon}dx$$
$$=t^{-d}\sum_j \left\|M_{\varphi_j^{(t)}}\left(\left(\tilde{f}_j\right)_t\right)\right\|_{L_{1+\epsilon}(\mathbb{T}^d;Y)}^{1+\epsilon}$$

We then deduce the desired assertion.

Proof of Theorem 14. (i) Let P be a trigonometric polynomial with coefficients in X. By Lemma 16 and Lemma 15

$$\left\|\sum_j M_{\varphi_j}(P)\right\|_{L_{1+\epsilon}(\mathbb{T}^d;Y)}\le\sum_j \left\|T_{\varphi_j}\right\|_{1+\epsilon\to 1+\epsilon}\lim_{t\to 0}(4\pi(1+\epsilon)t)^{\frac{d}{2(1+\epsilon)}}\|P\widehat{\mathbb{H}}_t\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}$$
$$=\sum_j \left\|T_{\varphi_j}\right\|_{1+\epsilon\to 1+\epsilon}\|\ P\ \|_{L_{1+\epsilon}(\mathbb{T}^d;Y)}$$

whence $\left\|\sum_j M_{\varphi_j}\right\|_{1+\epsilon\to 1+\epsilon}\le\sum_j \left\|T_{\varphi_j}\right\|_{1+\epsilon\to 1+\epsilon}$. The second inequality $\left\|\sum_j M_{\varphi_j}^{-1}\right\|_{1+\epsilon\to 1+\epsilon}\le\sum_j \left\|T_{\varphi_j}^{-1}\right\|_{1+\epsilon\to 1+\epsilon}$ is proved in the same way.

(ii) We use Lemma 17 for this part. Let f_j be a compactly supported function with values in a finite dimensional subspace of X. Then for t sufficiently small, $\left(f_j\right)_t$ is supported in the cube $\mathbb{I}^d$, so $\left(\tilde{f}_j\right)_t=\left(f_j\right)_t$. Thus

$$\left\|\sum_j M_{\varphi_j^{(t)}}\left((\tilde{f}_j)_t\right)\right\|_{L_{1+\epsilon}(\mathbb{T}^d;Y)} \leq \sum_j \left\|M_{\varphi_j^{(t)}}\right\|_{1+\epsilon\to 1+\epsilon} \left\|(\tilde{f}_j)_t\right\|_{L_{1+\epsilon}(\mathbb{T}^d;Y)}$$
$$= \sum_j \left\|M_{\varphi_j^{(t)}}\right\|_{1+\epsilon\to 1+\epsilon} \left\|(f_j)_t\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}$$
$$= t^{-\frac{d\epsilon}{1+\epsilon}}\sum_j \left\|M_{\varphi_j^{(t)}}\right\|_{1+\epsilon\to 1+\epsilon} \| f_j \|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}$$

Therefore, by Lemma 17,

$$\left\|\sum_j T_{\varphi_j}(f_j)\right\|_{L_{1+\epsilon}(\mathbb{R}^d;Y)} \leq \liminf_{t\to 0}\sum_j \left\|M_{\varphi_j^{(t)}}\right\|_{1+\epsilon\to 1+\epsilon} \| f_j \|_{L_{1+\epsilon}(\mathbb{R}^d;Y)}$$

whence

$$\left\|\sum_j T_{\varphi_j}\right\|_{1+\epsilon\to 1+\epsilon} \leq \liminf_{t\to 0}\sum_j \left\|M_{\varphi_j^{(t)}}\right\|_{1+\epsilon\to 1+\epsilon}$$

We show similarly the other inequality of part (ii).

Remark 18. Some of the hypotheses $(\mathbf{H}_1) - (\mathbf{H}_4)$ can be weakened for the validity of Theorem 14 We have seen in the proof of Lemma 17 that $(\mathbf{H}_2)$ can be replaced by the Riemann integrability of the function $\xi \mapsto (\varphi_j)\_(\xi)\hat{f}_j(\xi)e^{2\pi i\xi\cdot x}$ for every compactly supported C^∞ function f_j. On the other hand, the proof of Lemma 16 shows that $(\mathbf{H}_3)$ is unnecessary if Y is isomorphic to a Hilbert space.

If the function φ_j is strongly continuous, Theorem 14 can be reformulated as follows: the norms $\|T_{\varphi_j}\|_{1+\epsilon\to 1+\epsilon}$ and $\|T_{\varphi_j}^{-1}\|_{1+\epsilon\to 1+\epsilon}$ coincide with the corresponding ones when $\mathbb{R}^d$ is viewed as a discrete group (see [11] for more details). We then obtain the following corollary as in 11:

Corollary 19. Let $\varphi_j: \mathbb{R}^d \to B(X,Y)$ be a strongly continuous function. Let ψ_j be the restriction of φ_j to $\mathbb{R}^k \subset \mathbb{R}^d$ for some $k < d$. Consider the Fourier multiplier T_{ψ_j} from $L_{1+\epsilon}(\mathbb{R}^k;X)$ to $L_{1+\epsilon}(\mathbb{R}^k;Y)$. Then

$$\left\|\sum_j T_{\psi_j}\right\|_{1+\epsilon\to 1+\epsilon} \leq \sum_j \left\|T_{\varphi_j}\right\|_{1+\epsilon\to 1+\epsilon} \text{ and } \left\|\sum_j T_{\psi_j}^{-1}\right\|_{1+\epsilon\to 1+\epsilon} \leq \sum_j \left\|T_{\varphi_j}^{-1}\right\|_{1+\epsilon\to 1+\epsilon}$$

3. Proofs of Theorem 2 and Corollary 3

As described in the historical comments of subsection 1.1 we need only to show Theorem 2 (ii) for $\epsilon < 0$. We start the proof by some preliminaries. In the sequel, Q will denote a cube of $\mathbb{R}^d$ (with sides parallel to the axes), $|Q|$ and $\ell(Q)$ being respectively its volume and side length. For a locally integrable function f_j on $\mathbb{R}^d$ we let $\langle f_j\rangle_Q$ denote the mean of f_j over Q :

$$\langle f_j\rangle_Q = \frac{1}{|Q|}\int_Q \sum_j f_j(x)dx$$

As mentioned before, part (i) of Theorem 2 for $\epsilon \leq 0$ is proved by singular integrals. We state this result as a lemma for later use.

Lemma 20 [37]. Let $\varphi_j \in \mathcal{H}_{\varepsilon,\delta}$ and $f_j \in H_{1+\epsilon}(\mathbb{R}^d)$ with $0 \leq \epsilon \leq 1$. Then

$$\left\|\sum_j G^{\varphi_j}(f_j)\right\|_{1+\epsilon} \lesssim_{d,\varepsilon,\delta} \sum_j \| f_j \|_{H_{1+\epsilon}}$$

Indeed, consider the Hilbert space valued kernel K defined by $K(x) = \{(\varphi_j)_t(x)\}_{t>0}$ for $x \in \mathbb{R}^d$, that is, K is a function from $\mathbb{R}^d$ to $L_2(\mathbb{R}_+)$, where $\mathbb{R}_+$is equipped with the measure $\frac{dt}{t}$. We use K to denote the associated singular integral too:

$$K(f_j) = \int_{\mathbb{R}} \sum_j K(-\epsilon)f_j(x+\epsilon)d(x+\epsilon)$$

Then

$$G^{\varphi_j}(f_j)(x) = \| K(f_j)(x) \|_{L_2(\mathbb{R}_+)},\ x \in \mathbb{R}^d$$

It is easy to show that K satisfies the following regularities (see below for a proof):

$$\|K(x)\|_{L_2(\mathbb{R}^+)} \lesssim_\epsilon \frac{1}{|x|^d} \text{ and } \|K(2(x+\epsilon)) - K(x)\|_{L_2(\mathbb{R}^+)} \lesssim_{\varepsilon,\delta} \frac{|x+2\epsilon|^\delta}{|x|^{d+\delta}},$$
$$x, (x+2\epsilon) \in \mathbb{R}^d, \qquad |x| > 2|x+2\epsilon|. \tag{15}$$

Thus the lemma follows from the L_2-boundedness of K and the Calderón-Zygmund theory.
We will need a reinforcement of the previous lemma for Wilson's intrinsic square functions defined by

$$S_{\varepsilon,\delta}(f_j)(x) = \left(\int_{|\epsilon|<t} \sum_j \sup_{\varphi_j \in \mathcal{H}_{\varepsilon,\delta}} \left|(\varphi_j)_t * f_j(x+\epsilon)\right|^2 \frac{d(x+\epsilon)dt}{t^{d+1}}\right)^{\frac{1}{2}}$$

This square function can also be expressed as a singular integral operator. Let the cone $\Gamma = \{(x+\epsilon, t) \in \mathbb{R}_+^{d+1} : |x+\epsilon| < t\}$ be equipped with the measure $\frac{d(x+\epsilon)dt}{t^{d+1}}$. Let X be the Banach space of square integrable functions on Γ with values in $\ell_\infty(\mathcal{H}_{\varepsilon,\delta})$:

$$X = L_2\left(\Gamma; \ell_\infty(\mathcal{H}_{\varepsilon,\delta})\right)$$

This time, the convolution kernel K is an X-valued kernel: for $x \in \mathbb{R}^d$, $K(x)$ is defined as follows:

$$K(x): \Gamma \to \ell_\infty(\mathcal{H}_{\varepsilon,\delta}),\ (x+\epsilon, t) \mapsto \left\{(\varphi_j)_t(2x+\epsilon)\right\}_{\varphi_j \in \mathcal{H}_{\varepsilon,\delta}}$$

Then

$$S_{\varepsilon,\delta}(f_j)(x) = \| K(f_j)(x) \|_X,\ x \in \mathbb{R}^d$$

Let us show that this new kernel K satisfies (15) too. By (31), we have

$$\begin{aligned} \| K(x) \|_X^2 &= \int_\Gamma \sum_j \sup_{\varphi_j \in \mathcal{H}_{\varepsilon,\delta}} \left|(\varphi_j)_t(2x+\epsilon)\right|^2 \frac{d(x+\epsilon)dt}{t^{d+1}} \\ &\le \int_\Gamma \left[\frac{1}{t^d} \frac{1}{\left(1+\frac{|2x+\epsilon|}{t}\right)^{d+\varepsilon}}\right]^2 \frac{d(x+\epsilon)dt}{t^{d+1}} \\ &\lesssim_{d,\varepsilon} \int_0^\infty \left[\frac{1}{t^d} \frac{1}{\left(1+\frac{|x|}{t}\right)^{d+\varepsilon}}\right]^2 \frac{dt}{t} \lesssim_{d,\varepsilon} \frac{1}{|x|^{2d}} \end{aligned}$$

This gives the first estimate of (15). For the second, let $x, (x+2\epsilon) \in \mathbb{R}^d$ with $|x| > 2|x+2\epsilon|$. Applying (3) once more, we get

$$\| K(2(x+\epsilon)) - K(x) \|_X^2 \lesssim_{\varepsilon,\delta} \int_\Gamma \left[\frac{1}{t^{d+\delta}} \frac{|x+2\epsilon|^\delta}{\left(1+\frac{|2x+\epsilon|}{t}\right)^{d+\varepsilon+\delta}}\right]^2 \frac{d(x+\epsilon)dt}{t^{d+1}} \lesssim_{d,\varepsilon,\delta} \frac{|x+2\epsilon|^{2\delta}}{|x|^{2(d+\delta)}}$$

By [31], $S_{\varepsilon,\delta}$ is bounded on $L_2(\mathbb{R}^d)$. Thus we deduce the following
Lemma 21. Let $0 \le \epsilon \le 1$. Then

$$\left\|\sum_j S_{\varepsilon,\delta}(f_j)\right\|_{1+\epsilon} \lesssim_{d,\varepsilon,\delta} \sum_j \| f_j \|_{H_{1+\epsilon}},\ f_j \in H_{1+\epsilon}(\mathbb{R}^d)$$

Our proof of Theorem 2 (ii) for $\epsilon < 0$ is modelled on Mei's argument [19] (see also the proof of Theorem 1.3 of [33]). We will need a variant of the usual BMO space. For any locally integrable function f_j on $\mathbb{R}^d$ define

$$f_j^\sharp(x) = \sup_{x \in Q} \left(\frac{1}{|Q|}\int_Q \sum_j \left|f_j(x) - \langle f_j \rangle_Q\right|^2 dx\right)^{\frac{1}{2}},$$

and for $0 < \epsilon \le \infty$ let

$$\mathrm{BMO}_{2+\epsilon}(\mathbb{R}^d) = \left\{f_j : f_j^\sharp \in L_{2+\epsilon}(\mathbb{R}^d)\right\} \text{ and } \| f_j \|_{\mathrm{BMO}_{2+\epsilon}} = \left\|f_j^\sharp\right\|_{2+\epsilon}$$

Note that $\mathrm{BMO}_\infty(\mathbb{R}^d)$ coincides with the usual $\mathrm{BMO}(\mathbb{R}^d)$.
The BMO space is closely related to Carleson measures via the following maximal function

$$C^{\varphi_j}(f_j)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|}\int_{T(Q)}\sum_j \left|(\varphi_j)_t * f_j(x+\epsilon)\right|^2 \frac{d(x+\epsilon)dt}{t}\right)^{\frac{1}{2}}$$

where $T(Q) = Q \times (0, \ell(Q)] \subset \mathbb{R}_+^{d+1}$.

The following inequality is known, it can be shown by adapting the proof of [28, Theorem IV.4.3].

Lemma 22. Let $\varphi_j \in \mathcal{H}_{\varepsilon,\delta}$ and f_j be any nice function on $\mathbb{R}^d$. Then $C^{\varphi_j}(f_j) \lesssim_{d,\varepsilon,\delta} f_j^\sharp$.

We now arrive at the key step of our argument.

Lemma 23 (see [37]). Let $\varphi_j, \psi_j \in \mathcal{H}_{\varepsilon,\delta}$ satisfy (4). Let $0 \le \epsilon < 1$. Then

$$\left|\int_{\mathbb{R}^d}\sum_j f_j g_j\right| \lesssim_{d,\varepsilon,\delta} \sum_j \|G^{\varphi_j}(f_j)\|_{1+\epsilon}^{\frac{1+\epsilon}{2}} \| f_j \|_{H_{1+\epsilon}}^{1-\frac{1+\epsilon}{2}} \| g_j \|_{\mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}}$$

for any sufficiently nice functions $f_j \in H_{1+\epsilon}(\mathbb{R}^d)$ and $g_j \in \mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}(\mathbb{R}^d)$.

Proof. Fix (sufficiently nice) functions $f_j \in H_{1+\epsilon}(\mathbb{R}^d)$ and $g_j \in \mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}(\mathbb{R}^d)$. We need to consider a truncated version of $G^{\varphi_j}(f_j)$:

$$G(x,t) = \left(\int_t^\infty \sum_j \left|(\varphi_j)_s * f_j(x)\right|^2 \frac{ds}{s}\right)^{\frac{1}{2}},\ x \in \mathbb{R}^d, t \ge 0 \tag{16}$$

By approximation, we can assume that $G(x,t)$ never vanishes. By (4), we have

$$\int_{\mathbb{R}^d}\sum_j f_j g_j = \int_{\mathbb{R}_+^{d+1}}\sum_j (\varphi_j)_t * f_j(x)\left(\psi_j\right)_t * g_j(x)\frac{dxdt}{t}$$
$$= \int_{\mathbb{R}_+^{d+1}}\sum_j \left[(\varphi_j)_t * f_j(x)G(x,t)^{\frac{\epsilon-1}{2}}\right]\cdot\left[G(x,t)^{\frac{1+\epsilon}{2}}\left(\psi_j\right)_t * g_j(x)\right]\frac{dxdt}{t}$$

Thus by the Cauchy-Schwarz inequality,

$$\left|\int_{\mathbb{R}^d}\sum_j f_j g_j\right| \le \mathrm{A}\cdot\mathrm{B}$$

where

$$\mathrm{A}^2 = \int_{\mathbb{R}^{d+1}}\sum_j G(x,t)^{\epsilon-1}\left|(\varphi_j)_t * f_j(x)\right|^2\frac{dxdt}{t}$$
$$\mathrm{B}^2 = \int_{\mathbb{R}^{d+1}}\sum_j G(x,t)^{1+\epsilon}\left|\left(\psi_j\right)_t * g_j(x)\right|^2\frac{dxdt}{t}$$

The term A is estimated as follows

$$\mathrm{A}^2 = -\int_{\mathbb{R}^d}\int_0^\infty G(x,t)^{\epsilon-1}\frac{\partial}{\partial t}(G(x,t)^2)dtdx = -2\int_{\mathbb{R}^d}\int_0^\infty G(x,t)^{\epsilon}\frac{\partial}{\partial t}G(x,t)dtdx.$$

Since $G(\cdot,t)$ is decreasing in t, $G(\cdot,t)^\epsilon \le G(x,0)^\epsilon = G^{\varphi_j}(f_j)(x)^\epsilon$. Thus

$$\mathrm{A}^2 \le -2\int_{\mathbb{R}^d}\sum_j G^{\varphi_j}(f_j)(x)^\epsilon\int_0^\infty\frac{\partial}{\partial t}G(x,t)dtdx = 2\int_{\mathbb{R}^d}\sum_j G^{\varphi_j}(f_j)(x)^{1+\epsilon}ds = 2\sum_j \|G^{\varphi_j}(f_j)\|_{1+\epsilon}^{1+\epsilon}.$$

The estimate of B is harder. We will need two more variants of $S_{\varepsilon,\delta}(f_j)$. The first one is defined as before for $G(\cdot,t)$:

$$S(x,t)^2 = \int_t^\infty\int_{|\epsilon|<s-\frac{t}{2}}\sum_j \sup_{\varphi_j\in\mathcal{H}_{\varepsilon,\delta}}\left|\left(\varphi_j\right)_s * f_j(x+\epsilon)\right|^2\frac{d(x+\epsilon)ds}{s^{d+1}},\ x \in \mathbb{R}^d, t \ge 0.$$

To introduce the second, let $\mathcal{D}_k$ be the family of dyadic cubes of side length 2^{-k}, and let c_Q denote the center of a cube Q. Define

$$\mathbb{S}(x,k)^2 = \int_{\sqrt{d}2^{-k}}^\infty\int_{|x+\epsilon-c_Q|<s}\sum_j \sup_{\varphi_j\in\mathcal{H}_{\varepsilon,\delta}}\left|\left(\varphi_j\right)_s * f_j(x+\epsilon)\right|^2\frac{d(x+\epsilon)ds}{s^{d+1}},\ \text{ if } x \in Q \in \mathcal{D}_k, k \in \mathbb{Z}.$$

By definition, we have

- $\mathbb{S}(\cdot,k)$ is increasing in k;

- $\mathbb{S}(\cdot,k)$ is constant on every $Q \in \mathcal{D}_k$;
- $\mathbb{S}(\cdot,-\infty) = 0$ and $\mathbb{S}(\cdot,\infty) = S(x,0) = S_{\varepsilon,\delta}(f_j)$.

On the other hand, if $s \geq t \geq \sqrt{d}2^{-k}$ and $x \in Q \in \mathcal{D}_k$, then $B\left(x, s-\frac{t}{2}\right) \subset B(c_Q, s)$. Here $B(x,r)$ denotes the ball of center x and radius r. It then follows that

$$S(\cdot,t) \leq \mathbb{S}(\cdot,k) \text{ on every } Q \in \mathcal{D}_k \text{ whenever } t \geq \sqrt{d}2^{-k}$$

Here, the crucial observation is the elementary pointwise inequality: $G(x,t) \lesssim_{d,\varepsilon,\delta} S(x,t)$. This inequality is easily proved by the arguments of [31]. Indeed, by a lemma due to Uchiyama [30] (see also Lemma 3 of [31]), we can assume that the function φ_j defining $G(x,t)$ in (16) is supported in the unit ball of $\mathbb{R}^d$. Then we get $G(x,t) \lesssim_{d,\varepsilon,\delta} S(x,t)$ exactly as Wilson did on page 784 of [31].

After these preparations, we are ready to estimate the term B. We have

$$\begin{aligned}
\mathrm{B}^2 &\lesssim_{d,\varepsilon,\delta} \int_{\mathbb{R}^{d+1}} \sum_j S(x,t)^{1+\epsilon} \left|(\psi_j)_t * g_j(x)\right|^2 \frac{dxdt}{t} \\
&= \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_k} \int_Q \int_{\sqrt{d}2^{-k}}^{\sqrt{d}2^{-k+1}} \sum_j S(x,t)^{1+\epsilon} \left|(\psi_j)_t * g_j(x)\right|^2 \frac{dt}{t} dx \\
&\leq \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_k} \int_Q \int_{\sqrt{d}2^{-k}}^{\sqrt{d}2^{-k+1}} \sum_j \mathbb{S}(x,k)^{1+\epsilon} \left|(\psi_j)_t * g_j(x)\right|^2 \frac{dt}{t} dx \\
&= \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}} \sum_{j\leq k} D(x,j) \int_{\sqrt{d}2^{-k}}^{\sqrt{d}2^{-k+1}} \sum_j \left|(\psi_j)_t * g_j(x)\right|^2 \frac{dt}{t} dx
\end{aligned}$$

where $D(x,k) = \mathbb{S}(x,k)^{1+\epsilon} - \mathbb{S}(x,k-1)^{1+\epsilon}$. Thus

$$\begin{aligned}
\mathrm{B}^2 &\lesssim d,\varepsilon,\delta \\
&\int_{\mathbb{R}^d} \sum_j D(x,j_0) \sum_{k\geq j_0} \int_{\sqrt{d}2^{-k}}^{\sqrt{d}2^{-k+1}} \sum_j \left|(\psi_j)_t * g_j(x)\right|^2 \frac{dt}{t} dx \\
&= \sum_j \sum_{Q\in\mathcal{D}_j} \int_Q D(x,j_0) \int_0^{\sqrt{d}2^{-j+1}} \sum_j \left|(\psi_j)_t * g_j(x)\right|^2 \frac{dt}{t} dx
\end{aligned}$$

Since $D(\cdot,j)$ is constant on every $Q \in \mathcal{D}_j$, we have

$$\begin{aligned}
\mathrm{B}^2 &\lesssim d,\varepsilon,\delta \\
&\sum_j \sum_{Q\in\mathcal{D}_j} D(x,j_0)\mathbb{1}_Q(x) \int_Q \int_0^{2\sqrt{d}\ell(Q)} \sum_j \left|(\psi_j)_t * g_j(x)\right|^2 \frac{dt}{t} dx \\
&\leq \sum_j \sum_{Q\in\mathcal{D}_j} \sum_j D(x,j_0)\mathbb{1}_Q(x) \min_{(x+\epsilon)\in Q} C^{\psi_j}(g_j)(x+\epsilon)^2 |Q| \\
&\leq \int_{\mathbb{R}^d} \sum_j D(x,j_0) C^{\psi_j}(g_j)(x)^2 dx \\
&= \int_{\mathbb{R}^d} \sum_j S_{\varepsilon,\delta}(f_j)(x)^{1+\epsilon} C^{\psi_j}(g_j)(x)^2 dx \\
&\leq \sum_j \|S_{\varepsilon,\delta}(f_j)\|_{1+\epsilon}^{1+\epsilon} \|C^{\psi_j}(g_j)\|_{\frac{1+\epsilon}{\epsilon}}^2
\end{aligned}$$

By Lemma 21 $\left\|\sum_j S_{\varepsilon,\delta}(f_j)\right\|_{1+\epsilon} \lesssim_{d,\varepsilon,\delta} \| f_j \|_{H_{1+\epsilon}}$. Hence,

$$\mathrm{B} \lesssim_{d,\varepsilon,\delta} \sum_j \| f_j \|_{H_{1+\epsilon}}^{1-\frac{1+\epsilon}{2}} \|C^{\psi_j}(g_j)\|_{\frac{1+\epsilon}{\epsilon}}.$$

Combining the estimates of A and B together with Lemma 22, we get the desired assertion.

The preceding lemma implies the following

$$\left|\int_{\mathbb{R}^d} \sum_j f_j g_j\right| \lesssim_d \sum_j \| f_j \|_{H_{1+\epsilon}} \| g_j \|_{\mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}}$$

This shows that every function $g_j \in \mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}(\mathbb{R}^d)$ induces a continuous linear functional on $H_{1+\epsilon}(\mathbb{R}^d)$. Like the H_1-BMO duality theorem, the converse is true too. The following lemma is known, its noncommutative analogue is [19, Theorem 4.4]. We include a proof by adapting Mei's argument.

Lemma 24 (see [37]). Let $0 \le \epsilon < 1$. Then every continuous functional ℓ on $H_{1+\epsilon}(\mathbb{R}^d)$ is represented by a function $g_j \in \mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}(\mathbb{R}^d)$:

$$\ell(f_j) = \int_{\mathbb{R}^d} \sum_j f_j g_j,\ \forall f_j \in H_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$$

Moreover,

$$(1+\epsilon) \parallel \sum_j g_j \parallel_{\mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}} \lesssim_d \parallel \ell \parallel_{H_{1+\epsilon}(\mathbb{R}^d)^*} \lesssim_d \sum_j \parallel g_j \parallel_{\mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}}$$

Proof. We will use the characterization of $H_{1+\epsilon}(\mathbb{R}^d)$ by the g-function defined by (99), namely

$$\parallel \sum_j f_j \parallel_{H_{1+\epsilon}} \approx_d \sum_j \left\| G_\nabla^{\mathbb{P}}(f_j) \right\|_{1+\epsilon}$$

Let $\ell \in H_{1+\epsilon}(\mathbb{R}^d)^*$. Then by the Hahn-Banach theorem, there exist $d+1$ functions h_i on the upper half space $\mathbb{R}_+^{d+1}$ such that

$$\left(\int_{\mathbb{R}^d} \left(\sum_{i=1}^{d+1} \int_0^\infty |h_i(x+\epsilon,t)|^2 \frac{dt}{t} \right)^{\frac{\epsilon}{2(1+\epsilon)}} d(x+\epsilon) \right)^{\frac{\epsilon}{1+\epsilon}} \approx_d \parallel \ell \parallel_{H_{1+\epsilon}(\mathbb{R}^d)^*}$$

and (with $x_{d+1} = t$)

$$\ell(f_j) = \int_{\mathbb{R}^d} \int_0^\infty \sum_j \left(\sum_{i=1}^{d+1} t \frac{\partial}{\partial x_i} \mathbb{P}_t(f_j)(x+\epsilon) h_i(x+\epsilon,t) \right) \frac{dt}{t} d(x+\epsilon) = \int_{\mathbb{R}^d} \sum_j f_j(x) g_j(x) dx$$

where

$$g_j(x) = \int_{\mathbb{R}^d} \int_0^\infty \left(\sum_{i=1}^{d+1} t \frac{\partial}{\partial x_i} \mathbb{P}_t(\epsilon)(x+\epsilon) h_i(x+\epsilon,t) \right) \frac{dt}{t} d(x+\epsilon)$$

It remains to show that $g_j \in \mathrm{BMO}_{\frac{1+\epsilon}{\epsilon}}(\mathbb{R}^d)$. All the $d+1$ terms on the right hand side are treated in the same way, so we need only to deal with one of them, say the i-th term. For notational simplicity, let

$$(\varphi_j)_t(x) = t \frac{\partial}{\partial x_i} \mathbb{P}_t(-x),\ h = h_i$$

and (with some abuse of notation)

$$g_j = \int_0^\infty \sum_j (\varphi_j)_t * h(\cdot,t) \frac{dt}{t},\ H(x+\epsilon) = \left(\int_0^\infty |h(x+\epsilon,t)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

Given $x \in \mathbb{R}^d$ and a cube Q containing x, let

$$a_Q = \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^d} \int_0^\infty \sum_j (\varphi_j)_t(-\epsilon) h(x+\epsilon,t) \mathbb{1}_{(2Q)^c}(x+\epsilon) \frac{dt}{t} d(x+\epsilon) d(x+2\epsilon)$$

Then

$$\begin{aligned} g_j(u) - a_Q &= \int_{\mathbb{R}^d} \int_0^\infty \sum_j (\varphi_j)_t(x+\epsilon-u) h(x+\epsilon,t) \mathbb{1}_{2Q}(x+\epsilon) \frac{dt}{t} d(x+\epsilon) \\ &+ \int_{\mathbb{R}^d} \int_0^\infty \left[\frac{1}{|Q|} \int_Q \sum_j \left((\varphi_j)_t(x+\epsilon-u) - (\varphi_j)_t(-\epsilon) \right) d(x+2\epsilon) \right] h(x+\epsilon,t) \mathbb{1}_{(2Q)^c}(x+\epsilon) \frac{dt}{t} d(x+\epsilon) d(x+2\epsilon) \\ &\overset{\text{def}}{=} A(u) + B(u). \end{aligned}$$

By the Plancherel identity and the Cauchy-Schwarz inequality, letting $\tilde{h}_t(x+\epsilon) = h(x+\epsilon,t) \mathbb{1}_{2Q}(x+\epsilon)$, we have

$$\int_Q |A(u)|^2 du \leq \int_{\mathbb{R}^d} |A(u)|^2 du = \int_{\mathbb{R}^d} \left| \int_0^\infty \sum_j \hat{\varphi}_j(t\xi) \hat{\bar{h}}_t(\xi) \frac{dt}{t} \right|^2 d\xi$$
$$\leq \int_{\mathbb{R}^d} \int_0^\infty \sum_j |\hat{\varphi}_j(t\xi)|^2 \frac{dt}{t} \int_0^\infty \left| \hat{\bar{h}}_t(\xi) \right|^2 \frac{dt}{t} d\xi .$$
$$\lesssim_d \int_{2Q} \int_0^\infty |h(x+\epsilon,t)|^2 \frac{dt}{t} d(x+\epsilon)$$

It then follows that

$$\sup_{x\in Q} \frac{1}{|Q|} \int_Q |A(u)|^2 du \lesssim {}_d M(H^2)(x).$$

We turn to the term B. Let c be the center of Q and $u \in Q$, then

$$|B(u)| \lesssim_d \int_{(2Q)^c} \int_0^\infty \frac{\ell(Q)}{(|x+\epsilon-c|+t)^{d+1}} |h(x+\epsilon,t)| \frac{dt}{t} d(x+\epsilon)$$
$$\lesssim_d \int_{(2Q)^c} \frac{\ell(Q)}{|x+\epsilon-c|^{d+1}} \left(\int_0^\infty |h(x+\epsilon,t)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} d(x+\epsilon)$$
$$\lesssim_d \sum_{k=1}^\infty 2^{-k} \frac{1}{(2^k \ell(Q))^d} \int_{2^{k-1}\ell(Q) \leq |x+\epsilon-c| < 2^k \ell(Q)} H(x+\epsilon) d(x+\epsilon)$$
$$\lesssim_d M(H)(x).$$

Thus

$$\sup_{x\in Q} \left(\frac{1}{|Q|} \int_Q |B(u)|^2 du \right)^{\frac{1}{2}} \lesssim_d M(H)(x).$$

Combining the preceding estimates, we get

$$g_j^\sharp(x) \lesssim \sup_{x\in Q} \left(\frac{1}{|Q|} \int_Q \sum_j |g_j(u) - a_Q|^2 du \right)^{\frac{1}{2}}$$
$$\lesssim \sup_{x\in Q} \left(\frac{1}{|Q|} \int_Q |A(u)|^2 du \right)^{\frac{1}{2}} + \sup_{x\in Q} \left(\frac{1}{|Q|} \int_Q |B(u)|^2 du \right)^{\frac{1}{2}}$$
$$\lesssim d(M(H^2)(x))^{\frac{1}{2}} + M(H)(x) \lesssim_d (M(H^2)(x))^{\frac{1}{2}}.$$

Hence,

$$\|g_j^\sharp\|_{\frac{1+\epsilon}{\epsilon}} \lesssim_d \left\| \left(M(H^2)\right)^{\frac{1}{2}} \right\|_{\frac{1+\epsilon}{\epsilon}} \lesssim_d \frac{1}{1+\epsilon} \| H \|_{\frac{1+\epsilon}{\epsilon}} \lesssim_d \frac{1}{1+\epsilon} \| \ell \|_{H_{1+\epsilon}(\mathbb{R}^d)^*}.$$

This is the desired inequality.

It is now easy to show part (ii) of Theorem 2 for $0 \leq \epsilon \leq 1$.

Proof of Theorem 2. (ii) for $\epsilon < 0$. Using Lemma 24 and taking the supremum in the inequality of Lemma 23 over all g_j with $\| g_j \|_{\mathrm{BMO}_{2(1+\epsilon)}} \leq 1$, we obtain

$$\| \sum_j f_j \|_{H_{2+\epsilon}} \lesssim_{d,\varepsilon,\delta} \frac{1}{\epsilon} \sum_j \|G^{\varphi_j}(f_j)\|_{2+\epsilon}^{\frac{2+\epsilon}{2}} \| f_j \|_{H_{2+\epsilon}}^{1-\frac{2+\epsilon}{2}}$$

whence

$$\| \sum_j f_j \|_{H_{2+\epsilon}} \lesssim_{d,\varepsilon,\delta} (\epsilon)^{-\frac{2}{2+\epsilon}} \sum_j \|G^{\varphi_j}(f_j)\|_{2+\epsilon}.$$

Since $\| \sum_j f_j \|_{2+\epsilon} \leq \sum_j \| f_j \|_{H_{2+\epsilon}}$, we deduce

$$\| \sum_j f_j \|_{2+\epsilon} \lesssim_{d,\varepsilon,\delta} (\epsilon)^{-\frac{2}{2+\epsilon}} \sum_j \|G^{\varphi_j}(f_j)\|_{2+\epsilon}.$$

This implies $\mathrm{L}^{\varphi_j}_{t,2+\epsilon} \lesssim_{d,\varepsilon,\delta} 1$ for $\epsilon \leq 0$. For $0 < \epsilon < \frac{1}{2}$ we use duality and $\mathrm{L}^{\psi_j}_{c,(\frac{3}{2}+\epsilon)'} \lesssim_{d,\varepsilon,\delta} \sqrt{(\frac{3}{2}+\epsilon)'}$ to conclude that $\mathrm{L}^{\varphi_j}_{t,\frac{3}{2}+\epsilon} \lesssim_{d,\varepsilon,\delta} \sqrt{(\frac{3}{2}+\epsilon)'} \lesssim_{d,\varepsilon,\delta} 1$ too.

Proof of Corollary 3. In the previous proof we have obtained $\| f_j \|_{H_{1+\epsilon}} \lesssim d,\varepsilon,\delta \|G^{\varphi_j}(f_j)\|_{1+\epsilon}$ for $0 \leq \epsilon \leq 1$. The converse inequality is contained in Lemma 20. On the other hand, by Lemma 21 we get

$$\left\|\sum_j S^{\psi_j}(f_j)\right\|_{1+\epsilon} \lesssim_{d,\varepsilon,\delta} \sum_j \|f_j\|_{H_{1+\epsilon}} \lesssim_{d,\varepsilon,\delta} \sum_j \|G^{\varphi_j}(f_j)\|_{1+\epsilon}.$$

Hence the corollary is proved.

4. Proof of Theorem 1 (see [37])

We have seen in the previous section that the g-function $G^{\mathbb{P}}$ can be expressed as a singular integral operator. Equivalently, $G^{\mathbb{P}}$ can be also written as a Fourier multiplier with values in $L_2(\mathbb{R}_+)$(recalling that $\mathbb{R}_+$is equipped with the measure $\frac{dt}{t}$). Let $\varphi_j\colon \mathbb{R}^d \to L_2(\mathbb{R}_+)$be the function defined by $\varphi_j(\xi)(t) = -2\pi t|\xi|e^{-2\pi t|\xi|}$ for $\xi \in \mathbb{R}^d$ and $t > 0$. Let T_{φ_j} be the Fourier multiplier introduced in section 2 (with $X = \mathbb{C}$ and $Y = L_2(\mathbb{R}_+)$). Then

$$G^{\mathbb{P}}(f_j)(x) = \|T_{\varphi_j}(f_j)(x)\|_{L_2(\mathbb{R}_+)},\ x \in \mathbb{R}^d, f_j \in L_{1+\epsilon}(\mathbb{R}^d).$$

It is clear that this symbol φ_j satisfies the assumption of Theorem 14 (see also Remark 18). Thus the results in that section apply to φ_j.

The corresponding periodic Fourier multiplier M_{φ_j} gives rise to the g-function defined by the circular Poisson semigroup on $\mathbb{T}^d$:

$$\mathrm{P}_r(f_j) = \sum_{m\in\mathbb{Z}^d} \sum_j \hat{f}_j(m) r^{|m|} z^m,\ f_j \in L_{1+\epsilon}(\mathbb{T}^d).$$

The associated g-function is defined by

$$G^{\mathrm{P}}(f_j) = \left(\int_0^1 \sum_j (1-r)\left|\frac{d}{dr}\mathrm{P}_r(f_j)\right|^2 dr\right)^{\frac{1}{2}}. \tag{17}$$

By the change of variables $r = e^{-2\pi t}$, elementary computations show that for any $0 \leq \epsilon \leq \infty$

$$\left\|\sum_j G^{\mathrm{P}}(f_j)\right\|_{L_{1+\epsilon}(\mathbb{T}^d)} \approx_d \sum_j \|M_{\varphi_j}(f_j)\|_{L_{1+\epsilon}(\mathbb{T}^d;L_2(\mathbb{R}_+))},\ f_j \in L_{1+\epsilon}(\mathbb{T}^d).$$

We refer to [10, Section 8] for more details.

Now we proceed to the proof of Theorem 1. For clarity, we will divide this proof into two subsections. Let us first make an elementary observation as a prelude. $\{\mathbb{P}_t\}_{t>0}$ is the Poisson semigroup subordinated to $\{\mathbb{H}_t\}_{t>0}$ in Bochner's sense:

$$\mathbb{P}_t(f_j) = \frac{1}{\sqrt{\pi}}\int_0^\infty \sum_j \frac{e^{-s}}{\sqrt{s}} \mathbb{H}_{\frac{t^2}{4s}}(f_j)ds.$$

This formula immediately implies

$$\mathrm{L}^{\mathbb{H}}_{c,1+\epsilon} \gtrsim \mathrm{L}^{\mathbb{P}}_{c,1+\epsilon} \text{ and } \mathrm{L}^{\mathbb{H}}_{t,1+\epsilon} \lesssim \mathrm{L}^{\mathbb{P}}_{t,1+\epsilon}.$$

Combining this with the historical comments at the end of subsection 1.1, it remains for us to show that $\mathcal{L}^{\mathbb{P}}_{c,1+\epsilon} \gtrsim \max\left(\sqrt{1+\epsilon}, \frac{1+\epsilon}{\epsilon}\right)$, and $\mathcal{L}^{\mathbb{H}}_{t,1+\epsilon} \gtrsim \sqrt{1+\epsilon}$ for $\epsilon > 0$. The former is already contained in [35], but we will reproduce the proof there for the convenience of the reader.

4.1. Proof of $\mathcal{L}^{\mathbb{P}}_{c,1+\epsilon} \gtrsim \max\left(\sqrt{1+\epsilon}, \frac{1+\epsilon}{\epsilon}\right)$ (see [37]). By the discussion at the beginning of this section and Corollary 19., the constant $\mathcal{L}^{\mathbb{P}}_{c,2+\epsilon}$ increases in the dimension d. So it suffices to consider the case $d = 1$. This inequality for $\epsilon \leq 0$ is well known. It can be easily proved as follows. Fix $s > 0$ and let $f_j = \mathbb{P}_s$. Then

$$t\frac{\partial}{\partial t}\sum_j \mathbb{P}_t(f_j)(x) = \frac{t}{\pi}\frac{x^2 - (t+s)^2}{(x^2 + (t+s)^2)^2},\ x \in \mathbb{R}.$$

For $x \geq 6s$, we have

$$G^{\mathbb{P}}(f_j)(x) \geq \left(\int_{\frac{x}{3}-s}^{\frac{x}{2}-s} \sum_j \left|t\frac{\partial}{\partial t}\mathbb{P}_t(f_j)(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \gtrsim \frac{1}{x}.$$

Thus

$$\left\|\sum_j G^{\mathbb{P}}(f_j)\right\|_{2+\epsilon} \gtrsim \left(\int_{6s}^{\infty} \frac{1}{x^{2+\epsilon}} dx\right)^{\frac{1}{2+\epsilon}} \gtrsim \frac{s^{-\frac{1}{2(2+\epsilon)}}}{1+\epsilon}.$$

On the other hand,

$$\| f_j \|_{2+\epsilon} \approx s^{-\frac{1}{2(1+\epsilon)}}.$$

Hence, $L^{\mathbb{P}}_{c,2+\epsilon} \gtrsim 2(1+\epsilon)$.

Unfortunately, the above simple argument does not apply to the case $\epsilon > 0$. Our proof for the latter is much harder. By the discussion at the beginning of this section, it is equivalent to considering the torus $\mathbb{T}$ and the g-function defined by (17). Recall that

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

It is shown in [34] that the inequality

$$\left\|\sum_j G^{P}(f_j)\right\|_{L_{2+\epsilon}(\mathbb{T})} \leq L^{\mathbb{P}}_{c,2+\epsilon} \sum_j \| f_j \|_{L_{2+\epsilon}(\mathbb{T})}$$

is equivalent to the corresponding dyadic martingale inequality on $\Omega = \{-1,1\}^{\mathbb{N}}$. It is well known that the relevant constant in the latter martingale inequality is of order $\sqrt{2+\epsilon}$ as $\epsilon \to \infty$. To reduce the determination of optimal order of $\mathcal{L}^{P}_{c,2+\epsilon}$ to the martingale case, we need to refine an argument in the proof of [34. Theorem 3.1] whose idea originated from [4].

Keeping the notation there, let $M = (M_k)_{0\leq k\leq K}$ be a finite dyadic martingale and

$$M_k - M_{k-1} = d_k(\varepsilon_1, \cdots, \varepsilon_{k-1})\varepsilon_k$$

where (ε_k) are the coordinate functions of Ω. The transformation $\varepsilon_k = \operatorname{sgn}(\cos\theta_k)$ establishes a measure preserving embedding of Ω into $\mathbb{T}^{\mathbb{N}}$. Accordingly, define

$$a_k(e^{i\theta_1}, \cdots, e^{i\theta_{k-1}}) = d_k(\operatorname{sgn}(\cos\theta_1), \cdots, \operatorname{sgn}(\cos\theta_{k-1}))$$
$$b_k(e^{i\theta_k}) = \operatorname{sgn}(\cos\theta_k).$$

Note that to enlighten notation, we write an element $(x+2\epsilon) \in \mathbb{T}$ as $(x+2\epsilon) = e^{-i\theta}$, so identify $\mathbb{T}$ with $[-\pi,\pi)$, a slightly different convention from the one of section 2

Given (n_k) a rapidly increasing sequence of positive integers, put

$$a_{k,(n)}(e^{i\theta}) = a_{k,(n)}(e^{i\theta}; e^{i\theta_1}, \cdots, e^{i\theta_{k-1}}) = a_k(e^{i(\theta_1+n_1\theta)}, \cdots, e^{i(\theta_{k-1}+n_{k-1}\theta)})$$
$$b_{k,(n)}(e^{i\theta}) = b_{k,(n)}(e^{i\theta}; e^{i\theta_k}) = b_k(e^{i(\theta_k+n_k\theta)})$$
$$(f_j)_{(n)}(e^{i\theta}) = (f_j)_{(n)}(e^{i\theta}; e^{i\theta_1}, \cdots, e^{i\theta_K}) = \sum_{k=1}^{K} a_{k,(n)}(e^{i\theta}) b_{k,(n)}(e^{i\theta}).$$

The functions $(f_j)_{(n)}, a_{k,(n)}$ and $b_{k,(n)}$ are viewed as functions on $\mathbb{T}$ for each $(\theta_1, \cdots, \theta_K)$ arbitrarily fixed. Furthermore, by approximation, we can assume that all a_k and b_k are polynomials. Then, if the sequence (n_k) rapidly increases, Lemmas 3.4 and 3.5 of [34] imply

$$\frac{1}{2}G^{P}((f_j)_{(n)}) \leq \left(\sum_{k=1}^{K} |a_{k,(n)}|^2 G^{P}(b_{k,(n)})^2\right)^{\frac{1}{2}} \leq 2G^{P}\left((f_j)_{(n)}\right).$$

Therefore,

$$\left\|\left(\sum_{k=1}^{K} |a_{k,(n)}|^2 G^{P}(b_{k,(n)})^2\right)^{\frac{1}{2}}\right\|_{L_{2+\epsilon}(\mathbb{T})} \leq 2\, L^{P}_{c,2+\epsilon} \| f_j \|_{L_{2+\epsilon}(\mathbb{T})}. \tag{18}$$

The discussion so far comes from [34]. Now we require a finer analysis of the g-function $G^{\mathbb{P}}(b_{k,(n)})$. To this end we write the Fourier series of the function $b = \operatorname{sgn}(\cos\theta)$:

$$b(e^{\mathrm{i}\theta}) = \frac{2}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\left[e^{\mathrm{i}(2j+1)\theta} + e^{-\mathrm{i}(2j+1)\theta}\right].$$

Then

$$\frac{d}{dr}\mathrm{P}_r\left(b_{k,(n)}\right)(e^{\mathrm{i}\theta}) = \frac{4}{\pi}n_k r^{n_k-1}\,\mathrm{Re}\left(\sum_{j=0}^{\infty} (-1)^j r^{2n_k j} e^{\mathrm{i}(2j+1)(\theta_k+n_k\theta)}\right).$$

The real part on the right side is easy to compute. Indeed, letting $\rho = r^{2n_k}$ and $\eta = \theta_k + n_k\theta$, we have

$$\mathrm{Re}\left(e^{\mathrm{i}\eta}\sum_{j=0}^{\infty} (-1)^j \rho^j e^{\mathrm{i}2j\eta}\right) = \mathrm{Re}\left(\frac{e^{\mathrm{i}\eta}}{1+\rho e^{\mathrm{i}2\eta}}\right) = \frac{1+\rho}{1+2\rho\cos\eta+\rho^2}\cos\eta.$$

As

$$\frac{1+\rho}{1+2\rho\cos\eta+\rho^2} \geq \frac{1+\rho}{(1+\rho)^2} \geq \frac{1}{2}$$

it then follows that

$$\left|\mathrm{Re}\left(e^{\mathrm{i}\eta}\sum_{j=0}^{\infty} (-1)^j \rho^j e^{\mathrm{i}2j\eta}\right)\right| \geq \frac{|\cos\eta|}{2}.$$

Therefore, we deduce

$$\left|\frac{d}{dr}\mathrm{P}_r\left(b_{k,(n)}\right)(e^{\mathrm{i}\theta})\right|^2 \gtrsim n_k^2 r^{2(n_k-1)}\cos^2(\theta_k+n_k\theta).$$

Thus

$$\begin{aligned} G^{\mathrm{P}}\left(b_{k,(n)}\right)^2 &\gtrsim \cos^2(\theta_k+n_k\theta)n_k^2\int_0^1 (1-r)^{2-1}r^{2(n_k-1)}dr \\ &\approx \left[1+\mathrm{O}\left(\frac{1}{n_k}\right)\right]\cos^2(\theta_k+n_k\theta). \end{aligned}$$

Now lifting both sides of (18) to power $(2+\epsilon)$, then integrating the resulting inequality over $\mathbb{T}^K$ with respect to $(\theta_1,\cdots,\theta_K)$, we get

$$\begin{aligned} &\int_{\mathbb{T}}\int_{\mathbb{T}^K}\left(\sum_{k=1}^{K} \left|a_{k,(n)}\left(e^{\mathrm{i}(\theta_1+n_1\theta)},\cdots,e^{\mathrm{i}(\theta_{k-1}+n_{k-1}\theta)}\right)\right|^2\left[1+\mathrm{O}\left(\frac{1}{n_k}\right)\right]\cos^2(\theta_k+n_k\theta)\right)^{\frac{2+\epsilon}{2}} d\theta_1\cdots d\theta_K d\theta \\ &\leq \left(C\,\mathrm{L}^{\mathrm{P}}_{c,2+\epsilon}\right)^{2+\epsilon}\int_{\mathbb{T}}\int_{\mathbb{T}^K}\sum_j \left|(f_j)_{(n)}\left(e^{\mathrm{i}(\theta_1+n_1\theta)},\cdots,e^{\mathrm{i}(\theta_K+n_K\theta)}\right)\right|^{2+\epsilon} d\theta_1\cdots d\theta_K d\theta. \end{aligned}$$

For each fixed θ, the change of variables $(\theta_1,\cdots,\theta_K)\mapsto(\theta_1-n_1\theta,\cdots,\theta_K-n_K\theta)$ being a measure preserving transformation of $\mathbb{T}^K$, we deduce

$$\begin{aligned} &\int_{\mathbb{T}^K}\left(\sum_{k=1}^{K} \left|a_{k,(n)}\left(e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_{k-1}}\right)\right|^2\left[1+\mathrm{O}\left(\frac{1}{n_k}\right)\right]\cos^2\theta_k\right)^{\frac{2+\epsilon}{2}} d\theta_1\cdots d\theta_K \\ &\leq \left(C\,\mathrm{L}^{\mathrm{P}}_{c,2+\epsilon}\right)^{2+\epsilon}\int_{\mathbb{T}^K}\sum_j \left|(f_j)_{(n)}\left(e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_K}\right)\right|^{2+\epsilon} d\theta_1\cdots d\theta_K. \end{aligned}$$

Letting $n_1\to\infty$, we get

$$\int_{\mathbb{T}^K}\left(\sum_{k=1}^{K} \left|d_k\left(\mathrm{sgn}(\cos\theta_1),\cdots,\mathrm{sgn}(\cos\theta_{k-1})\right)\right|^2\cos^2\theta_k\right)^{\frac{2+\epsilon}{2}} d\theta_1\cdots d\theta_K \leq \left(C\,\mathrm{L}^{\mathrm{P}}_{c,2+\epsilon}\right)^{2+\epsilon}\|M_K\|^{2+\epsilon}_{L_{2+\epsilon}(\Omega)}.$$

Now we consider an elementary example where M is simple random walk stopped at ± 2, namely

$$d_k = \mathbb{1}_{\{\tau\geq k\}} \text{ with } \tau = \inf\left\{k: \left|\sum_{j=1}^{k} \varepsilon_j\right| = 2\right\}.$$

Note that the probability of the event $\{\tau = j\}$ is zero for odd j and $2^{-\frac{j}{2}}$ for even j. On the other hand, recalling $\varepsilon_k = \mathrm{sgn}(\cos\theta_k)$ and letting

$$A_j = \left\{\tau = j, |\cos\theta_k| \geq \frac{1}{\sqrt{2}}, 1 \leq k \leq j\right\}$$

we easily check that the probability of A_j is $8^{-\frac{j}{2}}$ for even j. Thus

$$\sum_{k=1}^{K} |d_k(\varepsilon_1, \cdots, \varepsilon_{k-1})|^2 \cos^2\theta_k \geq \mathbb{1}_{A_j} \sum_{k=1}^{j} \mathbb{1}_{\{\tau \geq k\}} \cos^2\theta_k \geq \frac{j}{2}\mathbb{1}_{A_j}$$

consequently, for $K = 2j$

$$\int_{\mathbb{T}^K} \left(\sum_{k=1}^{K} |d_k(\mathrm{sgn}(\cos\theta_1), \cdots, \mathrm{sgn}(\cos\theta_{k-1}))|^2 \cos^2\theta_k\right)^{\frac{2+\epsilon}{2}} d\theta_1 \cdots d\theta_K$$

$$\gtrsim \sum_{m=1}^{J} j^{\frac{2+\epsilon}{2}} 8^{-j} \geq c^{2+\epsilon}(2+\epsilon)^{\frac{2+\epsilon}{2}}.$$

Noting that $|M_K| \leq 2$ and combining all the previous inequalities together, we finally obtain

$$\mathrm{L}^{\mathrm{P}}_{c,2+\epsilon} \gtrsim \sqrt{2+\epsilon}$$

4.2. Proof of $L^{\mathbb{H}}_{t,2+\epsilon} \gtrsim \sqrt{2+\epsilon}$ for $\epsilon > 0$. Again, it suffices to consider the torus case. The g-function relative to the heat semigroup on $\mathbb{T}$ is defined by

$$G^{\mathrm{H}}(f_j) = \left(\int_0^1 \sum_j (1-r)\left|\frac{d}{dr}\mathrm{H}_r(f_j)\right|^2 dr\right)^{\frac{1}{2}}$$

where

$$\mathrm{H}_r(f_j)(\theta) = \sum_{n\in\mathbb{Z}} \sum_j \hat{f}_j(n) r^{n^2} e^{in\theta}$$

We will need the following elementary inequality that is known to experts. Let $a = (a_k)$ be a finite complex sequence and $f_j = \sum_k a_k e^{i^k\theta}$. Then

$$\left\|G^{\mathrm{H}}(f_j)\right\|_{L_{2+\epsilon}(\mathbb{T})} \approx \| a \|_{\ell_2} \tag{19}$$

See [8] for related results in a more general setting. The proof is easy:

$$G^{\mathrm{H}}(f_j)(e^{i\theta})^2 = \int_0^1 (1-r)\left|\sum_k a_k 4^k r^{4^k-1} e^{i2^k\theta}\right|^2 dr$$

$$\leq \sum_{j,k} a_j \bar{a}_k \frac{4^{j+k}}{(4^j + 4^k - 1)^2}$$

$$\leq \sum_{j,k} |a_j|^2 \frac{4^{j+k}}{(4^j + 4^k - 1)^2}$$

$$\lesssim \sum_j |a_j|^2$$

This implies

$$\left\|G^{\mathrm{H}}(f_j)\right\|_{L_{2+\epsilon}(\mathbb{T})} \lesssim \| a \|_{\ell_2}$$

However, for $0 \leq \epsilon \leq \infty$,

$$\left\|\sum_j G^{\mathrm{H}}(f_j)\right\|_{L_{2+\epsilon}(\mathbb{T})} \geq \sum_j \left\|G^{\mathrm{H}}(f_j)\right\|_{L_2(\mathbb{T})} \approx \| a \|_{\ell_2}$$

Therefore, for $0 \leq \epsilon \leq \infty$,

$$\left\|G^{\mathrm{H}}(f_j)\right\|_{L_{2+\epsilon}(\mathbb{T})} \approx \| a \|_{\ell_2}$$

The remaining case $0 \leq \epsilon < 1$ then follows from the Hölder inequality.

Now it is easy to show $\mathrm{L}^{\mathrm{H}}_{t,1+\epsilon} \gtrsim \sqrt{1+\epsilon}$ for $\epsilon > 0$. Indeed, 19) shows that $\mathrm{L}^{\mathrm{H}}_{t,2+\epsilon}$ dominates the best constant $C_{2+\epsilon}$ in the following inequality

$$\left\|\sum_{k\geq 1} a_k e^{i2^k\theta}\right\|_{L_{2+\epsilon}(\mathbb{T})} \leq C_{2+\epsilon} \| a \|_{\ell_2}$$

for any finite sequence $a = (a_k)$. It is well known that the best constant $C_{2+\epsilon}$ in the above inequality is of order $\sqrt{2+\epsilon}$ as $\epsilon \to \infty$ (see [26]); this fact can be also seen from the equivalence, up to universal constants, between this inequality and the classical Khintchine inequality (cf. [24]). Thus $L^{\mathbb{H}}_{t,2+\epsilon} \gtrsim \sqrt{2+\epsilon}$. This completes the proof of Theorem 1

5. Proofs of Theorem 9, Theorem 10 and Corollary 11 (see [37])

We begin with the proof of Theorem 9

Proof of Theorem 9. Let $H = \ell_2(\Delta)$ be the Hilbert space indexed by the family Δ of dyadic rectangles and $\{e_R\}_{R\in\Delta}$ be its canonical basis. Let $\varphi_j\colon \mathbb{R}^d \to H$ be the function given by

$$\varphi_j = \sum_{R\in\Delta} \mathbb{1}_R e_R$$

We will apply Theorem 14 to the case where $X = \mathbb{C}$ and $Y = H$. With the notation introduced in section 2 for any $f_j \in L_{2+\epsilon}(\mathbb{R}^d)$ we have

$$S^{\Delta}(f_j) = \left\|T_{\varphi_j}(f_j)\right\|_H$$

so

$$\mathrm{L}^{\Delta}_{c,2+\epsilon,d} = \left\|T_{\varphi_j}\right\|_{2+\epsilon\to 2+\epsilon} \text{ and } \mathrm{L}^{\Delta}_{t,2+\epsilon,d} = \left\|T^{-1}_{\varphi_j}\right\|_{2+\epsilon\to 2+\epsilon}$$

Note that

$$\left\|T_{\varphi_j^{(t)}}\right\|_{2+\epsilon\to 2+\epsilon} = \left\|T_{\varphi_j}\right\|_{2+\epsilon\to 2+\epsilon} \text{ and } \left\|T^{-1}_{\varphi_j^{(t)}}\right\|_{2+\epsilon\to 2+\epsilon} = \left\|T^{-1}_{\varphi_j}\right\|_{2+\epsilon\to 2+\epsilon}, \ \forall t > 0$$

Choose t irrational. Then $\varphi_j^{(t)}$ satisfies the assumption of Theorem 14 (i), so

$$\left\|\sum_j M_{\varphi_j^{(t)}}\right\|_{2+\epsilon\to 2+\epsilon} \leq \left\|\sum_j T_{\varphi_j^{(t)}}\right\|_{2+\epsilon\to 2+\epsilon} \text{ and } \left\|\sum_j M^{-1}_{\varphi_j^{(t)}}\right\|_{2+\epsilon\to 2+\epsilon} \leq \sum_j \left\|T^{-1}_{\varphi_j^{(t)}}\right\|_{2+\epsilon\to 2+\epsilon}$$

Letting $t \to 1$, we deduce

$$\sum_j \left\|M_{\varphi_j}\right\|_{2+\epsilon\to 2+\epsilon} \leq \sum_j \left\|T_{\varphi_j}\right\|_{2+\epsilon\to 2+\epsilon} \text{ and } \left\|\sum_j M^{-1}_{\varphi_j}\right\|_{2+\epsilon\to 2+\epsilon} \leq \sum_j \left\|T^{-1}_{\varphi_j}\right\|_{2+\epsilon\to 2+\epsilon}$$

This means

$$\mathrm{L}^{\tilde{\Delta}}_{c,2+\epsilon,d} \leq \mathrm{L}^{\Delta}_{c,2+\epsilon,d} \text{ and } \mathrm{L}^{\tilde{\Delta}}_{t,2+\epsilon,d} \leq \mathrm{L}^{\Delta}_{t,2+\epsilon,d}$$

To show the converse inequalities, we note that for any integer j and any $f_j \in L_{2+\epsilon}(\mathbb{T}^d)$ we have

$$\left\|M_{\varphi_j^{(2^j)}}(f_j)\right\|_H = S^{\tilde{\Delta}}(f_j)$$

This implies

$$\mathrm{L}^{\tilde{\Delta}}_{c,2+\epsilon,d} = \left\|M_{\varphi_j^{(2^j)}}\right\|_{2+\epsilon\to 2+\epsilon} \text{ and } \mathrm{L}^{\tilde{\Delta}}_{t,2+\epsilon,d} = \left\|M^{-1}_{\varphi_j^{(2^j)}}\right\|_{2+\epsilon\to 2+\epsilon}$$

Thus by Theorem 14 (ii), we get

$$\mathrm{L}^{\Delta}_{c,2+\epsilon,d} \leq \liminf_{j\to-\infty} \sum_j \left\|M_{\varphi_j^{(2^j)}}\right\|_{2+\epsilon\to 2+\epsilon} = \mathrm{L}^{\tilde{\Delta}}_{c,2+\epsilon,d} \text{ and } \mathrm{L}^{\Delta}_{t,2+\epsilon,d} \leq \liminf_{j\to-\infty} \sum_j \left\|M^{-1}_{\varphi_j^{(2^j)}}\right\|_{2+\epsilon\to 2+\epsilon} = \mathrm{L}^{\tilde{\Delta}}_{t,2+\epsilon,d}$$

The proof is finished.

Remark 25 [37]. The above proof is also applicable to the one-sided Littlewood-Paley-Rubio Francia inequality in [25, 17]: This inequality and its dual form in [5, 21] do not make difference for $\mathbb{R}^d$ and $\mathbb{T}^d$.

We will need the following lemma for the proof of Theorem 10. This lemma is of interest for its own right and is the Hilbert space valued extension of a classical theorem of Marcinkiewicz and Zygmund (cf. 14, Theorem V.2.7]).

Let (Ω, μ) be a measure space, H a Hilbert space and $0 \leq \epsilon < \infty$. Consider a linear operator T from $L_{1+\epsilon}(\Omega)$ to $L_{1+\epsilon}(\Omega; H)$. We are interested in the following inequalities

$$(1+\epsilon)^{-1} \| \sum_j f_j \|_{L_{1+\epsilon}(\Omega)} \leq \sum_j \| T(f_j) \|_{L_{1+\epsilon}(\Omega;H)} \leq \sum_j (1+2\epsilon) \| f_j \|_{L_{1+\epsilon}(\Omega)}, \ f_j \in L_{1+\epsilon}(\Omega).$$

Like for (12), if the first inequality holds for some $\epsilon \leq \infty$, the least $(1+\epsilon)$ is denoted by $\|T^{-1}\|_{1+\epsilon\to 1+\epsilon}$ while the least $(1+2\epsilon)$, if exists, is equal to $\| T \|_{1+\epsilon\to 1+\epsilon}$. If $(1+\epsilon)$ or $(1+2\epsilon)$ does not exist, $\|T^{-1}\|_{1+\epsilon\to 1+\epsilon}$ or $\| T \|_{1+\epsilon\to 1+\epsilon}$ is interpreted as infinite.
Given another Hilbert space K, consider the tensor $\mathrm{I}_K \otimes T: L_{1+\epsilon}(\Omega; K) \to L_{1+\epsilon}(\Omega; K \otimes H)$, where $K \otimes H$ denotes the tensor Hilbert space. Let $\|\mathrm{I}_K \otimes T\|_{1+\epsilon\to 1+\epsilon}$ and $\|(\mathrm{I}_K \otimes T)^{-1}\|_{1+\epsilon\to 1+\epsilon}$ be the best constants, if exist, in the following inequalities

$$(1+\epsilon)^{-1} \| \sum_j f_j \|_{L_{1+\epsilon}(\Omega;K)} \leq \sum_j \left\|\mathrm{I}_K \otimes T(f_j)\right\|_{L_{1+\epsilon}(\Omega;K\otimes H)}$$
$$\leq (1+2\epsilon) \sum_j \| f_j \|_{L_{1+\epsilon}(\Omega;K)},\ f_j \in L_{1+\epsilon}(\Omega; K).$$

Lemma 26 (see [37]). Let $0 \leq \epsilon < \infty$. Then

$$\|\mathrm{I}_K \otimes T\|_{1+\epsilon\to 1+\epsilon} \lesssim \| T \|_{1+\epsilon\to 1+\epsilon} \quad \text{and} \quad \|(\mathrm{I}_K \otimes T)^{-1}\|_{1+\epsilon\to 1+\epsilon} \lesssim \|T^{-1}\|_{1+\epsilon\to 1+\epsilon}.$$

Proof. It suffices to consider a finite dimensional K, say, $K = \ell_2^n$. Let $\left((g_j)_1, \cdots, (g_j)_n\right)$ be a standard complex Gaussian system, the corresponding expectation denoted by $\mathbb{E}$. Then for any $(\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n$ we have

$$\mathbb{E}\left|\sum_{k=1}^{n} \sum_j \alpha_k (g_j)_k\right|^{1+\epsilon} = \gamma_{1+\epsilon}^{1+\epsilon}\left(\sum_{k=1}^{n} |\alpha_k|^2\right)^{\frac{1+\epsilon}{2}} \quad \text{with } \gamma_{1+\epsilon}^{1+\epsilon} = \mathbb{E}\left|(g_j)_1\right|^{1+\epsilon}.$$

Now let $f_j = \left((f_j)_1, \cdots, (f_j)_n\right) \in L_{1+\epsilon}(\Omega; \ell_2^n)$. Then

$$\| \sum_j f_j \|_{L_{1+\epsilon}(\Omega;\ell_2^n)}^{1+\epsilon} = \int_\Omega \sum_j \left(\sum_{k=1}^{n} \left|(f_j)_k\right|^2\right)^{\frac{1+\epsilon}{2}} = \gamma_{1+\epsilon}^{-(1+\epsilon)} \int_\Omega \sum_j \mathbb{E}\left|\sum_{k=1}^{n} (g_j)_k (f_j)_k\right|^{1+\epsilon}.$$

On the other hand,

$$\left\|\sum_j \mathrm{I}_K \otimes T(f_j)\right\|_{L_{1+\epsilon}(\Omega;\ell_2^n\otimes H)}^{1+\epsilon} = \int_\Omega \left(\sum_{k=1}^{n} \sum_j \left\|T\left((f_j)_k\right)\right\|_H^2\right)^{\frac{1+\epsilon}{2}}.$$

Recall the following well known fact, the Hilbert-valued Khintchine inequality:

$$A_{1+\epsilon}^{-1}\left(\sum_{k=1}^{n} \|a_k\|_H^2\right)^{\frac{1}{2}} \leq \left(\mathbb{E}\left\|\sum_{k=1}^{n} \sum_j (g_j)_k a_k\right\|_H^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \leq B_{1+\epsilon}\left(\sum_{k=1}^{n} \|a_k\|_H^2\right)^{\frac{1}{2}},\ a_k \in H$$

moreover, $1 \lesssim A_{1+\epsilon} \leq 1$ for $0 \leq \epsilon \leq 1$ and $1 \leq B_{1+\epsilon} \lesssim \sqrt{1+\epsilon} \approx \gamma_{1+\epsilon}$ for $0 \leq \epsilon < \infty$. Using this inequality, we can write (recalling $K = \ell_2^n$)

$$\left\|\sum_j \mathrm{I}_K \otimes T(f_j)\right\|_{L_{2+\epsilon}(\Omega;\ell_2^n\otimes H)}^{2+\epsilon} \approx \int_\Omega \sum_j \mathbb{E}\left\|\sum_{k=1}^{n} (g_j)_k T\left((f_j)_k\right)\right\|_H^{2+\epsilon},\ \epsilon \leq 0$$

$$\left\|\sum_j \mathrm{I}_K \otimes T(f_j)\right\|_{L_{2+\epsilon}(\Omega;\ell_2^n\otimes H)}^{2+\epsilon} \gtrsim \gamma_{2+\epsilon}^{-(2+\epsilon)} \int_\Omega \sum_j \mathbb{E}\left\|\sum_{k=1}^{n} (g_j)_k T\left((f_j)_k\right)\right\|_H^{2+\epsilon},\ \epsilon > 0.$$

Thus for $\epsilon \leq 0$,

$$\left\|\sum_j \mathrm{I}_K \otimes T(f_j)\right\|_{L_{2+\epsilon}(\Omega;\ell_2^n\otimes H)}^{2+\epsilon} \approx \mathbb{E}\int_\Omega \sum_j \left\|\sum_{k=1}^n (g_j)_k T((f_j)_k)\right\|_H^{2+\epsilon}$$
$$\lesssim \| T \|_{2+\epsilon\to 2+\epsilon}^{2+\epsilon}\, \mathbb{E}\int_\Omega \sum_j \left|\sum_{k=1}^n (g_j)_k (f_j)_k\right|^{2+\epsilon}$$
$$= \| T \|_{2+\epsilon\to 2+\epsilon}^{2+\epsilon}\, \gamma_{2+\epsilon}^{2+\epsilon} \sum_j \| f_j \|_{L_{2+\epsilon}(\Omega;\ell_2^n)}^{2+\epsilon}$$
$$\approx \| T \|_{2+\epsilon\to 2+\epsilon}^{2+\epsilon} \sum_j \| f_j \|_{L_{2+\epsilon}(\Omega;\ell_2^n)}^{2+\epsilon}.$$

This yields $\|\mathrm{I}_K \otimes T\|_{2+\epsilon\to 2+\epsilon} \lesssim \| T \|_{2+\epsilon\to 2+\epsilon}$.

On the other hand, for $\epsilon \geq 0$, we have

$$\left\|\sum_j \mathrm{I}_K \otimes T(f_j)\right\|_{L_{1+\epsilon}(\Omega;\ell_2^n\otimes H)}^{1+\epsilon} \gtrsim \gamma_{1+\epsilon}^{-(1+\epsilon)}\mathbb{E}\int_\Omega \sum_j \left\|\sum_{k=1}^n (g_j)_k T((f_j)_k)\right\|_H^{1+\epsilon}$$
$$\gtrsim \gamma_{1+\epsilon}^{-(1+\epsilon)}\|T^{-1}\|_{1+\epsilon\to 1+\epsilon}^{-(1+\epsilon)}\mathbb{E}\int_\Omega \sum_j \left|\sum_{k=1}^n (g_j)_k (f_j)_k\right|^{1+\epsilon}$$
$$= \gamma_{1+\epsilon}^{-(1+\epsilon)}\|T^{-1}\|_{1+\epsilon\to 1+\epsilon}^{-(1+\epsilon)}\gamma_{1+\epsilon}^{1+\epsilon} \sum_j \| f_j \|_{L_{1+\epsilon}(\Omega;\ell_2^n)}^{1+\epsilon}$$
$$= \|T^{-1}\|_{1+\epsilon\to 1+\epsilon}^{-(1+\epsilon)} \sum_j \| f_j \|_{L_{1+\epsilon}(\Omega;\ell_2^n)}^{1+\epsilon}$$

whence $\|(\mathrm{I}_K \otimes T)^{-1}\|_{1+\epsilon\to 1+\epsilon} \lesssim \|T^{-1}\|_{1+\epsilon\to 1+\epsilon}$.

It remains to show $\|\mathrm{I}_K \otimes T\|_{1+\epsilon\to 1+\epsilon} \lesssim \| T \|_{1+\epsilon\to 1+\epsilon}$ for $\epsilon > 0$. To this end, we use duality. If $\| T \|_{2+\epsilon\to 2+\epsilon}$ is finite, then the adjoint $T^*: L_{2+2\epsilon}(\Omega; H) \to L_{2+2\epsilon}(\Omega)$ is a bounded operator with norm equal to $\| T \|_{2+\epsilon\to 2+\epsilon}$. In the same way, we have

$$\|\mathrm{I}_K \otimes T\|_{2+\epsilon\to 2+\epsilon} = \|\mathrm{I}_K \otimes T^*: L_{2+2\epsilon}(\Omega; K \otimes H) \to L_{2+2\epsilon}(\Omega; K)\|.$$

Then arguing as above with T^* and $(2 + 2\epsilon)$ instead of T and $(2 + \epsilon)$, respectively, we get

$$\|\mathrm{I}_K \otimes T^*: L_{2+2\epsilon}(\Omega; K \otimes H) \to L_{2+2\epsilon}(\Omega; K)\| \lesssim \|T^*: L_{2+2\epsilon}(\Omega; H) \to L_{2+2\epsilon}(\Omega)\|$$

which implies $\|\mathrm{I}_K \otimes T\|_{2+\epsilon\to 2+\epsilon} \lesssim \| T \|_{2+\epsilon\to 2+\epsilon}$ for $\epsilon > 0$.

Remark 27. Except the last part, the above proof works for T defined on a closed linear subspace S of $L_{1+\epsilon}(\Omega)$. Namely, letting $S \otimes_{1+\epsilon} H$ be the closure of $S \otimes H$ in $L_{1+\epsilon}(\Omega; H)$, then

$$\|\mathrm{I}_K \otimes T: S \otimes_{1+\epsilon} K \to S \otimes_{1+\epsilon} (K \otimes H)\| \lesssim \|T: S \to S \otimes_{1+\epsilon} H\|, 0 \leq \epsilon \leq 1$$
$$\|(\mathrm{I}_K \otimes T)^{-1}: S \otimes_{1+\epsilon} (K \otimes H) \to S \otimes_{1+\epsilon} K\| \lesssim \|| T^{-1}: S \otimes_{1+\epsilon} H \to S\|, 0 \leq \epsilon < \infty.$$

Proof of Theorem 10. By considering d-fold tensor products of functions on $\mathbb{T}$, i.e., functions on $\mathbb{T}^d$ of the form $f_j(x + 2\epsilon) = (f_j)_1(z_1) \cdots (f_j)_d(z_d)$, we easily check

$$\left(\mathrm{L}_{c,1+\epsilon,1}^{\widetilde{\Delta}}\right)^d \leq \mathrm{L}_{c,1+\epsilon,d}^{\widetilde{\Delta}} \text{ and } \left(\mathrm{L}_{t,1+\epsilon,1}^{\widetilde{\Delta}}\right)^d \leq \mathrm{L}_{t,1+\epsilon,d}^{\widetilde{\Delta}}.$$

Lemma 26 allows us to prove the converse inequalities. We do this for the second, the first being similarly treated; we consider only the case $d = 2$, an iteration argument will then give the general case. Let $\widetilde{\Delta}^{(1)}$ be the family of dyadic intervals of $\mathbb{Z}$ and $\widetilde{\Delta}^{(2)}$ the same family but with $\mathbb{Z}$ viewed as the second factor of $\mathbb{Z}^2$. Then (with $d = 2$)

$$\widetilde{\Delta} = \{R^{(1)} \times R^{(2)}: R^{(i)} \in \widetilde{\Delta}^{(i)}, i = 1,2\}.$$

Now let f_j be a polynomial on $\mathbb{T}^2$. By the Fubini theorem and (11) applied to the first variable $z_1 \in \mathbb{T}$, we get

$$\| f_j \|_{L_{1+\epsilon}(\mathbb{T}^2)}^{1+\epsilon} = \int_{\mathbb{T}} dz_2 \int_{\mathbb{T}} \sum_j |f_j(z_1, z_2)|^{1+\epsilon} dz_1$$
$$\leq \left(\mathrm{L}_{t,1+\epsilon,1}^{\widetilde{\Delta}}\right)^{1+\epsilon} \int_{\mathbb{T}} dz_2 \int_{\mathbb{T}} \left(\sum_{R^{(1)}\in\widetilde{\Delta}^{(1)}} \sum_j \left|S_{R^{(1)}}\left(f_j(\cdot, z_2)\right)(z_1)\right|^2\right)^{\frac{1+\epsilon}{2}} dz_1.$$

Let $K = \ell_2(\widetilde{\Delta}^{(1)})$ equipped with the canonical basis $\{e_{R^{(1)}}\}_{R^{(1)}\in\widetilde{\Delta}^{(1)}}$. Then

$$\left(\sum_{R^{(1)}\in\widetilde{\Delta}^{(1)}}\sum_j \left|S_{R^{(1)}}\left(f_j(\cdot,z_2)\right)(z_1)\right|^2\right)^{\frac{1}{2}} = \left\|\sum_{R^{(1)}\in\widetilde{\Delta}^{(1)}}\sum_j S_{R^{(1)}}\left(f_j(\cdot,z_2)\right)(z_1)e_{R^{(1)}}\right\|_K.$$

For each fixed z_1, we apply Lemma 26 to the K-valued function on the right hand side in the variable z_2 in order to infer

$$\int_{\mathbb{T}}\left\|\sum_{R^{(1)}\in\widetilde{\Delta}^{(1)}}\sum_j S_{R^{(1)}}\left(f_j(\cdot,z_2)\right)(z_1)e_{R^{(1)}}\right\|_K^{1+\epsilon} dz_2$$

$$\lesssim \left(\mathrm{L}^{\triangle}_{t,1+\epsilon,1}\right)^{1+\epsilon}\int_{\mathbb{T}}\left(\sum_{R^{(2)}\in\widetilde{\Delta}^{(2)}}\left\|S_{R^{(2)}}\left[\sum_{R^{(1)}\in\widetilde{\Delta}^{(1)}}\sum_j S_{R^{(1)}}\left(f_j(\cdot,z_2)\right)(z_1)e_{R^{(1)}}\right]\right\|_K^2\right)^{\frac{1+\epsilon}{2}} dz_2$$

$$= \left(\mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,1}\right)^{1+\epsilon}\int_{\mathbb{T}}\sum_j S^{\widetilde{\Delta}}(f_j)(z_1,z_2)^{1+\epsilon}dz_2.$$

Combining the previous inequalities, we get

$$\| f_j \|_{L_{1+\epsilon}(\mathbb{T}^2)}\lesssim \left(\mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,1}\right)^2\sum_j \left\|S^{\widetilde{\Delta}}(f_j)\right\|_{L_{1+\epsilon}(\mathbb{T}^2)}$$

whence $\mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,2} \lesssim \left(\mathrm{L}^{\widetilde{\Delta}}_{t,1+\epsilon,1}\right)^2$.

Remark 28. Pichorides [23 studied the first inequality of (11) for $d = 1$ restricted to functions in the Hardy space, and proved that the corresponding constant is of $\frac{1+\epsilon}{\epsilon}$ as $\epsilon \to 0$. Combined with Remark 27. the above proof shows that Pichorides' result extends to higher dimensions, we thus recover a result of [3] (see also [2] for related results).

We conclude the paper with the proof of Corollary 11.

Proof of Corollary 11. By Theorem 9. Theorem 10 and the known results mentioned in the historical comments at the end of section 1 we need only to show $(2+\epsilon)^{\frac{d}{2}} \lesssim \mathrm{L}^{\Delta}_{t,2+\epsilon,1} \lesssim 2+\epsilon$ for $0 \le \epsilon < \infty$. The first inequality is proved by using lacunary series as in the last part of the proof of Theorem 1. It remains to show the second, that is, we must prove

$$\| f_j \|_{2+\epsilon}\lesssim \sum_j (2+\epsilon)\left\|S^{\Delta}(f_j)\right\|_{2+\epsilon},\ f_j \in L_{2+\epsilon}(\mathbb{R}).$$

To this end, it suffices to consider a (nice) function f_j whose Fourier transform is supported in $\mathbb{R}_+$. Fixing such an f_j, let $S_k(f_j) = S_R(f_j)$ for $R = [2^{k-1}, 2^k)$, i.e., $\hat{S}_k(\hat{f}_j) = \mathbb{1}_{[2^{k-1},2^k)}\hat{f}_j$.

We will use the smooth version of S^{Δ}. Let φ_j be a C^∞ function on $\mathbb{R}$ whose Fourier transform is supported in $\left\{\xi: \frac{1}{2} < |\xi| < 4\right\}$ and satisfies

$$\sum_{k\mathbb{Z}}\sum_j \hat{\varphi}_j(2^{-k}\xi) = 1,\ \xi \in \mathbb{R}\setminus\{0\}.$$

Then the smooth version of S^{Δ} is the discretization of the g-function in (5):

$$G^{\varphi_j}_{\mathrm{dis}}(g_j)(x) = \left(\sum_{k\in\mathbb{Z}}\left|(\varphi_j)_k * g_j(x)\right|^2\right)^{\frac{1}{2}},\ x \in \mathbb{R}$$

for any (nice) function g_j on $\mathbb{R}$. Then

$$\int_{\mathbb{R}}\sum_j f_j(x)\overline{g_j(x)}dx = \sum_{k\in\mathbb{Z}}\sum_{j_0=k-1}^{k+1}\sum_j \int_{\mathbb{R}} S_k(f_j)(x)\overline{(\varphi_j)_{j_0} * g_j(x)}dx.$$

Thus by the Hölder inequality,

$$\left|\int_{\mathbb{R}}\sum_j f_j(x)\overline{g_j(x)}dx\right| \le 3\sum_j \left\|S^{\Delta}(f_j)\right\|_{2+\epsilon}\left\|G^{\varphi_j}_{\mathrm{dis}}(g_j)\right\|_{2+2\epsilon}.$$

However, it is well known that

$$\left\| \sum_j G_{\text{dis}}^{\varphi_j}(g_j) \right\|_{2+2\epsilon} \lesssim \sum_j (2+\epsilon) \| g_j \|_{2+2\epsilon}.$$

This is also the discrete analogue of Theorem 2 (i). It then follows that

$$\left| \int_{\mathbb{R}} \sum_j f_j(x)\overline{g_j(x)}dx \right| \lesssim (2+\epsilon) \sum_j \|S^{\Delta}(f_j)\|_{2+\epsilon} \| g_j \|_{2+2\epsilon}.$$

Taking the supremum over all g_j with $\| g_j \|_{2+2\epsilon} \leq 1$ yields the desired inequality on f_j.

References

[1] O. Bakas. Endpoint mapping properties of the Littlewood-Paley square function. Colloq. Math. 157 (2019), $1-15$.

[2] O. Bakas. On a problem of Pichorides. J. Geom. Anal. 31 (2021), 12637-12639.

[3] O. Bakas, S. Rodríguez-López, and A. A. Sola. Multi-parameter extensions of a theorem of Pichorides. Proc. Amer. Math. Soc. 147 (2019),1081-1095

[4] J. Bourgain. Some remarks on Banach spaces in which martingale differences are unconditional. Ark. Mat. 21 (1983), 163-168.

[5] J. Bourgain. On square functions on the trigonometric system. Bull. Soc. Math. Belg. Sér. B 37 (1985), 20-26.

[6] J. Bourgain. On the behavior of the constant in the Littlewood-Paley inequality. Lecture Notes in Math. 1376 (1989), 202-208.

[7] S-Y. A. Chang, J. M. Wilson, and T. Wolff. Some weighted norm inequalities concerning the Schrödinger operator. Comm. Math. Helv. 60 (1985), 217-246

[8] C. Y. Chuah, Y. Han, Z. Liu, and T. Mei. Paley's inequality for nonabelian groups. Preprint 2021 (arXiv).

[9] P. Chen, X. T. Duong, L. Wu, and L. Yan. Exponential-square integrability, weighted inequalities for the square functions associated to operators, and applications. Int. Math. Res. Not. 23 (2021), 18057-18117.

[10] Z. Chen, Q. Xu and Z. Yin. Harmonic Analysis on Quantum Tori. Commun. Math. Phys. 322 (2013), 755-805.

[11] K. de Leeuw. On L^p multipliers. Ann. Math. 91 (1965), 364-379.

[12] C. Fefferman, and E.M. Stein. H^p spaces of several variables. Acta Math. 129 (1972), 137-193.

[13] R. Fefferman, and J. Pipher. Multiparameter operators and sharp weighted inequalities. Amer. J. Math. 119 (1997), 337-69.

[14] J. García-Cuerva, and J.L. Rubio de Francia. Weighted norm inequalities and related topics. North-Holland Publishing Co., Amsterdam, 1985.

[15] R. Gong, and L. Yan. Weighted L_p estimates for the area integral associated to selfadjoint operators. Manuscripta Math. 144 (2014), 25-49.

[16] L. Grafakos. Classical Fourier analysis. Second edition. Springer, New York, 2008.

[17] J-L. Journé. Calderón-Zygmund operators on product spaces. Rev. Mat. Iberoam. 1 (1985), 55-91.

[18] A. Lerner. Quantitative weighted estimates for the Littlewood-Paley square function and Marcinkiewicz multipliers. Math. Res. Lett. 26 (2019), 537-556.

[19] T. Mei. Operator valued Hardy spaces. Memoirs Amer. Math. Soc. 881 (2007), vi+64 pp.

[20] P-A. Meyer. Démonstration probabiliste de certaines inégalités de Littlewood-Paley I: les inégalités classiques Sém. Probab. de Strasbourg 10 (1976), 125-141.

[21] N. N. Osipov. A one-sided Littlewood-Paley inequality in $\mathbb{R}^n$ for $0 < p \leq 2$. J. Math. Sci. 172 (2011), 229-242.

[22] S. Pichorides. On the Littlewood-Paley square inequality. Colloq. Math. 60/61 (1990), 687-691.

[23] S. Pichorides. A remark on the constants of the Littlewood-Paley square inequality. Proc. Amer. Math. Soc. 114 (1992), 787-789.

[24] G. Pisier. Les inégaliés de Khintchine-Kahane, d'après C. Borell. Séminaire d'Analyse Fonctionnelle, Ecole Polytechnique, 1977-1978, Exp. 7.

[25] J. L. Rubio de Francia. A Littlewood-Paley inequality for arbitrary intervals. Rev. Mat. Iberoam. 1 (1985), $1-14$.

[26] W. Rudin. Trigonometric series with gaps. J. Math. Mech. 2 (1960), 203-227.

[27] E.M. Stein. The development of square functions in the work of A. Zygmund. Bull. Amer. Math. Soc. 7 (1982), 359-376.

[28] E. M. Stein. Harmonic analysis. Princeton University Press, Princeton, 1993.

[29] E. M. Stein, and G. Weiss. Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton, 1975.
[30] A. Uchiyama. A constructive proof of the Fefferman-Stein decomposition of BMO ($\mathbf{R}^n$). Acta Math. 148 1(1982), $215-241$.
[31] J. M. Wilson. The intrinsic square function. Rev. Mat. Iberoam. 23 (2007), 771-791.
[32] J. M. Wilson. Weighted Littlewood-Paley theory and exponential-square integrability. Lecture Notes in Math. Springer, Berlin, 1924 (2008), pp. xiv +224.
[33] R. Xia, X. Xiong, and Q. Xu. Characterizations of operator-valued Hardy spaces and applications to harmonic analysis on quantum tori. Adv. Math. 291 (2016), 183-227.
[34] Q. Xu. Littlewood-Paley theory for functions with values in uniformly convex spaces. J. Reine Angew. Math. 504 (1998), 195-226
[35] Q. Xu. Holomorphic functional calculus and vector-valued Littlewood-Paley-Stein theory for semigroups. Preprint 2021 (arXiv).
[36] Z. Xu, and H. Zhang. From the Littlewood-Paley-Stein inequality to the Burkholder-Gundy inequality. Preprint 2021 (arXiv).
[37] Quanhua Xu, Optimal Orders of the Best Constants in The Littlewood-Paley Inequalities, Journal of Functional Analysis, 283 (2022), 109570.