Quest Journals
Journal of Research in Applied Mathematics
Volume 10 ~ Issue 2 (2024) pp: 23-27
ISSN (Online): 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Review Paper

Relative Softly Normal Spaces

Hamant kumar
Department of Mathematics
V. A. Govt Degree College, Atrauli-Aligarh, 202280, U. P. (India)

Abstract. *In the present paper, we define new classes of normality and regularity called soft normality and soft regularity in relative sense. Relative soft normality is a generalization of relative π-normality and relative almost normality. Further we have to show that Relative soft normality lies between relative almost normality and relative k-normality and also in relative quasi normality and relative k-normality. Moreover we investigate a relation among some variants of normality such as relative normality, relative π-normality, relative almost normality, relative quasi normality and relative k-normality with relative soft normality and obtain a characterization of relative softly normality in terms of other variants of normality.*
Mathematics Subject Classification 2020 *: 54D10, 54D15*
Keywords *: relative π-normal, relative almost normal, relative softly normal, relative softly regular, relative κ-normal.*

R*eceived 09 Feb., 2024;* ***R****evised 22 Feb., 2024;* ***A****ccepted 24 Feb., 2024*

I. Introduction

Arhangel'skii and Ganedi [**1**] introduced and studied the notion of relative topological properties. Recently, some topologists [**1, 2, 3, 5, 10, 13, 18, 19, 20**] have worked in relative topology and studied some topological properties in relative sense. Normal spaces were first of all investigated in relative sense by Arhangel'skii [**3**]. Zaitsev [**28**] introduced the concept of quasi normality is a generalization of normality and obtained its properties. The concept of almost normality was introduced by Singal and Arya [**23**]. The notion of mild normality was introduced by Shchepin [**21**] and, Singal and Singal [**22**] independently. Arhangel'skii and Ludwig [**4**] introduced the concepts of α-normal and β-normal spaces and obtained their properties. Kalantan [**14, 15**] introduced π-normal spaces and obtained their characterizations. Sharma and Kumar [**25**] introduced a new class of normal spaces called softly normal and obtained a characterization of softly normal space. Das and Bhat [**8**] introduced another class of spaces which lies between densely normal spaces and κ-normal spaces. Kumar and Sharma [**16**] introduced the concepts of softly regular and partly regular spaces and obtained some characterizations of softly regular spaces. Raina and Das [**20**] introduced the versions of normality such as κ-normality, almost normality, quasi normality and π-normality in a relative sense and prove some of their properties and obtain a relation with one another.

II. Preliminaries

Let X be a topological space and let $A \subset X$. Throughout the present paper the **closure** of a set A will be denoted by cl(A) and the **interior** by int(A). A set $U \subset X$ is said to be **regularly open** [**17**] if A = int(cl(A)). The complement of a regularly open set is called **regularly closed**. The finite union of regular open sets is said to be π-**open [28]**. The complement of a π-open set is said to be **π-closed.** A space X is said to be **k-normal** [**21**] (**mildly normal** [**22**]) if for every pair of disjoint regularly closed sets E, F of X there exist disjoint open subsets U and V of X such that $E \subset U$ and $F \subset V$. A space X is said to be **almost normal** [**23**] if for every pair of disjoint closed sets A and B one of which is regularly closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A space X is said to be **π-normal** [**14, 15**] if for every pair of disjoint closed sets A and B, one of which is π-closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A space X is said to be **almost regular** [**24**] if for every regularly closed set A and a point $x \notin A$, there exist disjoint open sets U and V such that $A \subset U$ and $x \in V$. A topological space is said to be **softly regular** [**16**] if for every π-closed set A and a point $x \notin A$, there exist two open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \phi$. A space X is called

almost β-normal [**9**] if for every pair of disjoint closed sets A and B, one of which is regularly closed, there exist disjoint open sets U and V such that $cl(U \cap A) = A$, $cl(V \cap B) = B$, and $cl(U) \cap cl(V) = \phi$.

Definition 1.1. Let $Y \subset X$. Y is said to be:
1. relatively T_1 [**1**] in X if for every $y \in Y$, {y} is closed in X.
2. normal in X or **relatively normal** [**1**] in X, if for each pair A, B of disjoint closed subsets of X, there are disjoint open subsets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
3. strongly normal in X or **relatively strongly normal** [**1**] in X, if for each pair A, B of disjoint closed sets in Y, there are disjoint open subsets U and V in X such that $A \subset U$ and $B \subset V$.

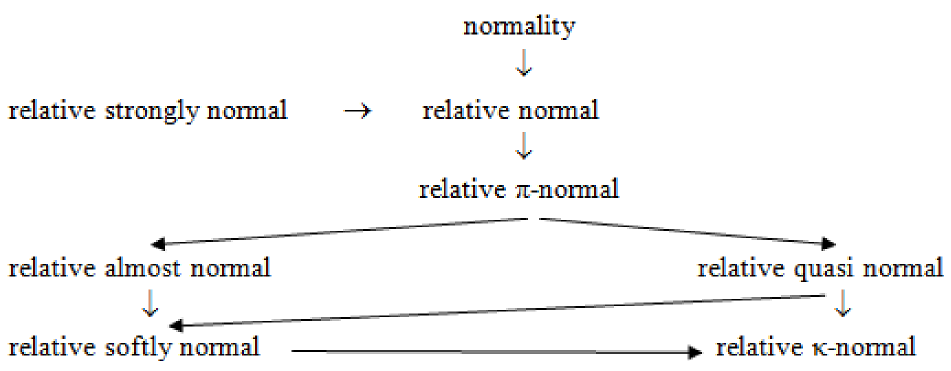
2. Some Variants of Relative Normality

Definition 2.1. Let X be a topological space. Then $Y \subset X$ is said to be:
1. relative softly normal in X if for any two disjoint subsets A and B of X one of which is π-closed and other is regularly closed, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
2. relative κ-normal [**20**] in X if for every pair of disjoint regularly closed sets A and B of X, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
3. relative almost normal [**20**] in X if for any two disjoint closed subsets A and B of X one of which is regularly closed, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
4. relative quasi normal [**20**] in X if for any two disjoint π-closed subsets A and B of X, there exist disjoint open sets U and V such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
5. relative π-normal [**20**] in X if for any two disjoint subsets A and B of X one of which is π-closed and other is closed, there exist disjoint open sets U and V such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
It is clear from the definitions that if X is softly normal (κ-normal, almost normal, quasi normal, π-normal), then $Y \subset X$ is relative softly normal (relative κ-normal, relative almost normal, relative quasi normal, relative π-normal respectively) in X.
The following example shows that none of the converses are true.
Example 2.2. [**20, 26**] Let X be the set of integers. Define a topology $\Im$ on X, where every odd integer is open and a set U is open if for every even integer $p \in U$, the successor and the predecessor of p also belong to U. Let Y be the set of all odd integers. Then Y is relative κ-normal, relative almost normal, relative quasi normal, relative softly normal as well as relative π-normal in X. But X is none of the absolute forms of these properties because A = {2, 3, 4} and B = {6, 7, 8} are disjoint regularly closed sets in X which are π-closed as well and there do not exist disjoint open sets in X separating them.
Note. If Y is κ-normal (almost normal, quasi normal, π-normal, softly normal) in itself that is with respect to the subspace topology, then Y need not be relative κ-normal (relative almost normal, relative quasi normal, relative π-normal, relative softly normal) in X respectively. See the following example.
Example 2.3. Let X be the set of integers with the topology defined in Example 2.2. Let Y be the set of all even integers. Then Y has discrete topology. So Y is a normal space and hence Y is κ-normal as well as quasi normal, softly normal and π-normal in itself but Y is none of the relative forms of any of these variants in X because A = {2, 3, 4} and B = {6, 7, 8} are disjoint regularly closed sets which are π-closed as well in X and there do not exist disjoint open sets in X separating $A \cap Y = \{2, 4\}$ and $B \cap Y = \{6, 8\}$.
Example 2.4.[20] Let X = {1, 2, 3, 4} and $\Im_X = \{ \phi, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3\}, X\}$. Let Y = {1, 3, 4}. It is clear that Y is almost normal in itself, i.e. with respect to the subspace topology but it is not relative almost normal in X because {3, 4} is regularly closed set in X and {1} is closed in X such that $\{3, 4\} \cap Y = \{3, 4\}$ and $\{1\} \cap Y = \{1\}$ cannot be separated by disjoint open sets in X.
Theorem 2.5. Let $Y \subset X$. If Y is relative softly normal in X, then:
(i) for every π-closed subset A and every regularly open subset B of X such that $A \subset B$, there exists an open set U of X such that $A \cap Y \subset U \subset cl(U) \subset B \cup (X - Y)$.
(ii) for every regularly closed subset A and every π-open subset B of X such that $A \subset B$, there exists an open set U of X such that $A \cap Y \subset U \subset cl(U) \subset B \cup (X - Y)$.
Proof. (i). Let A be a π-closed subset and B be a regularly open subset of X such that $A \subset B$. Then $X - B$ is regularly closed subset of X and $A \cap (X - B) = \phi$. Since Y is softly normal in X, there exist disjoint open sets U and V of X such that $A \cap Y \subset U$ and $(X - B) \cap Y \subset V$. Thus, $(X - V) \subset B \cup (X - Y)$. So, $A \cap Y \subset U \subset (X - V) \subset B \cup (X - Y)$. Since $X - V$ is a closed set containing U and cl(U) is a smallest closed set containing U, $A \cap Y \subset U \subset cl(U) \subset B \cup (X - Y)$.
To prove **(ii).**, let A be a regularly closed subset of X and B be a π-open subset of X such that $A \subset B$. Then $X - A$ is regularly open set containing closed set $X - B$. By (i), there is an open set U of X such that $(X - B) \cap Y \subset U \subset cl(U) \subset (X - A) \cup (X - Y)$. Thus $A \cap Y \subset X - cl(U) \subset X - U \subset B \cup (X - Y)$. Let $X - cl(U) = V$. Then V is open in X and $A \cap Y \subset V \subset cl(V) \subset B \cup (X - Y)$.

Definition 2.6. A subset A of X is said to be **concentrated** [**2**] on Y if A is contained in the closure in X of the trace $A \cap Y$ of the set A on Y. A space X is normal on Y if every two disjoint closed subsets of X concentrated on Y can be separated by disjoint open neighborhoods in X.
Definition 2.7. A space X is called **densely normal** [**2**] if there exists a dense subspace Y of X such that X is normal on Y.
Definition 2.8. A subset A of X is said to be **strongly concentrated** [**8**] on Y if $A \subset cl(int(A \cap Y))$. Let Y be a subspace of X. Then X is said to be **weakly normal** on Y if for every disjoint closed subsets A and B of X strongly concentrated on Y, there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.
Definition 2.9. A space is said to be **weakly densely normal** [**8**] if there exists a proper dense subspace Y of X such that X is weakly normal on Y.
Recall that a space X is called **extremally disconnected** if it is T_1 and the closure of any open set in X is open. Any π-open (π-closed) subset of an extremally disconnected space is an open domain (closed domain). Any extremally disconnected space is π-normal space [**14**]. Also a space X is called **weakly extremally disconnected** [**15**] if the closure of any open set is open. In a weakly extremally disconnected space any regularly closed set is clopen. Every weakly extremally disconnected space is almost normal [**15**].
Theorem 2.10. Every subset of an extremally disconnected space is relative softly normal.
Theorem 2.11. Every subset of a weakly extremally disconnected space is relative softly normal.
Theorem 2.12. $Y \subset X$ is relative softly normal in X if for every pair of disjoint closed sets A and B of X one of which is regularly closed and other is π-closed in X, there exists a continuous function f on X into closed interval [0, 1] such that $f(A \cap Y) = \{0\}$ and $f(A \cap Y) = \{1\}$.
Definition 2.13. $Y \subset X$ is said to be **relative almost regular** [**19**] in X if for every regularly closed set A in X and a point $y \in Y$ such that $y \notin A$, there exist disjoint open sets U and V such that $A \cap Y \subset U$ and $y \in V$.
Definition 2.14. $Y \subset X$ is said to be **relative softly regular** in X if for every π-closed set A in X and a point $y \in Y$ such that $y \notin A$, there exist disjoint open sets U and V such that $A \cap Y \subset U$ and $y \in V$.
Theorem 2.15. If Y is relative π-normal and relative T_1 in X, then Y is relative softly regular.
In general relative almost normality does not necessarily imply relative almost regularity. See the following example.
Example 2.16. Let X = {p, q, r} and $\Im = \{\phi, \{p\}, \{q\}, \{p, q\}, X\}$. Let Y = {p, r}. Here, the set {q, r} is regularly closed in X and $p \in Y$ such that $p \notin \{q, r\}$. But p and $\{q, r\} \cap Y = \{r\}$ can not be separated by two disjoint open sets in X. Hence Y is not relative almost regular in X. But Y is relative almost normal in X as there is no pair of disjoint closed sets in X.

III. Interrelations

From the above definitions and examples, the interrelations shown in the following diagram follows immediately.



Where none of the implications is reversible:

IV. Normal Spaces where all these Variants are Equivalent

Definition 4.1. A space X is said to be **β-normal** [**4**] if for any two disjoint closed subsets A and B of X there exist open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B, and $cl(U) \cap cl(V) = \phi$.
Theorem 4.2. Let X be a β-normal space. Then following statements are equivalent:
(i) Y is relative normal in X.
(ii) Y is relative π-normal in X.
(iii) Y is relative almost normal in X.

(iv) Y is relative quasi normal in X.
(v) Y is relative softly normal in X.
(vi) Y is relative κ-normal in X.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (v) ⇒ (vi) and (i) ⇒ (ii) ⇒ (iv) ⇒ (v) ⇒ (vi) are obvious from the interrelations.
To prove (vi) ⇒ (i), let A and B be two closed sets in X. Since X is β-normal, there exist open subsets U and V of X such that cl(U) ∩ cl(V) = ϕ, cl(U ∩ A) = A, and cl(V ∩ B) = B. So, cl(U) and cl(V) are disjoint regularly closed sets such that A ⊂ cl(U) and B ⊂ cl(V). Which implies A ∩ Y ⊂ cl(U) ∩ Y and B ∩ Y ⊂ cl(V) ∩ Y. Since Y is relative κ-normal in X, there exist disjoint open subsets U_1 and V_2 of X such that A ∩ Y ⊂ cl(U) ∩ Y ∩ U_1 and B ∩ Y ⊂ cl(V) ∩ Y ⊂ V_1. Hence Y is relative normal in X.
Definition 4.3. A space is said to be **seminormal** [**27**] if for every closed set F and each open set U containing F, there exists a regular open set V such that F ⊂ V ⊂ U.
Theorem 4.4. Let X be a seminormal space. Then following statements are equivalent:
(i) Y is relative normal in X.
(ii) Y is relative π-normal in X.
(iii) Y is relative almost normal in X.
(iv) Y is relative quasi normal in X.
(v) Y is relative softly normal in X.
(vi) Y is relative κ-normal in X.
Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (v) ⇒ (vi) and (i) ⇒ (ii) ⇒ (iv) ⇒ (v) ⇒ (vi) are obvious.
To prove (vi) ⇒ (i), let A and B be two disjoint closed sets in X. Then X − B is an open set containing A. Since X is seminormal, there exists a regularly open set U in X such that A ⊂ U ⊂ X − B. Now X − U is a regularly closed set contained in the open set X − A. Again by seminormality of X, there exists a regularly open set V in X such that X − U ⊂ V ⊂ X − A. Here, X −V and X − U are disjoint regularly closet sets in X such that A ⊂ X − V and B ⊂ X − U. Thus, A ∩ Y ⊂ (X − V) ∩Y and B ∩Y ⊂ (X − U) ∩ Y. Since Y is relative κ-normal in X, there exist disjoint open sets P and Q in X such that A ∩ Y ⊂ (X − V) ∩ Y ⊂ P and B ∩ Y ⊂ (X − U) ∩ Y ⊂ Q. Hence Y is relative normal in X.
Definition 4.5. A space is said to be **almost β-normal** [**9**] if for any two disjoint closed subsets A and B of X, one of which is regularly closed, there exist disjoint open subsets U and V of X such that A ∩ U is dense in A, B ∩ V is dense in B, and cl(U) ∩ cl(V) = ϕ.
Definition 4.6. Y ⊂ X is said to be **relative β-normal** [**10**] in X or β-normal in X if for any two disjoint closed subsets A and B of X, there exist open subsets U and V of X such that (A ∩ Y) ∩ U is dense in A ∩ Y and (B ∩ Y) ∩ V is dense in B ∩ Y and cl(U) ∩ cl(V) = ϕ.
Theorem 4.7. Let Y be a relative β-normal space in X. Then following statements are equivalent:
(i) Y is relative normal in X.
(ii) Y is relative π-normal in X.
(iii) Y is relative almost normal in X.
(iv) Y is relative quasi normal in X.
(v) Y is relative softly normal in X.
(vi) Y is relative κ-normal in X.
Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (v) ⇒ (vi) and (i) ⇒ (ii) ⇒ (iv) ⇒ (v) ⇒ (vi) are obvious.
Definition 4.8. Y ⊂ X is said to be **strong β-normal** [**10**] in X or **relative strong β-normal** in X if for any two disjoint closed subsets A and B of Y, there exist open subsets U and V of X such that A ∩ U is dense in A and B ∩ V is dense in B and cl(U) ∩ cl(V) = ϕ.

Theorem 4.9. Let Y be a relative strong β-normal space in X. Then following statements are equivalent:
(i) Y is relative strong normal in X.
(ii) Y is relative normal in X.
(iii) Y is relative π-normal in X.
(iv) Y is relative almost normal in X.
(v) Y is relative quasi normal in X.
(vi) Y is relative softly normal in X.
(vii) Y is relative κ-normal in X.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (vi) ⇒ (vi) and (i) ⇒ (ii) ⇒ (iii) ⇒ (v) ⇒ (vi) ⇒ (vii) are obvious.

V. Conclusion

In the present paper, we define new classes of normality and regularity called soft normality and soft regularity in relative sense. Relative soft normality is a generalization of relative π-normality and relative almost normality. Further we have to show that Relative soft normality lies between relative almost normality and relative k-normality and also in relative quasi normality and relative k-normality. Moreover we investigate a relation among some variants of normality with relative soft normality and obtain a characterization of relative softly normality in terms of other variants of normality. This idea can be extended to bitopology, ordered topological, ordered bitopological and fuzzy topological spaces etc.

Acknowledgement. I am heartily thankful to Dr M. C. Sharma, Department of Mathematics, N. R. E. C. College, Khurja, Uttar Pradesh (India) for his encouragement and support during the preparation of this paper.

References

[1]. A. V. Arhangel'skii, H. M. M. Genedi, Beginnings of the theory of relative topological properties, General Topology, Spaces and Mappings, MGU Moscow (1989), 3–48.
[2]. A. V. Arhangel'skii, A relative topological properties and relative topological spaces, Topology Appl. 70 (1996), 87–99.
[3]. A. V. Arhangel'kii, Relative normality and dense subspaces, Topology Appl. 123 (2002), 27–36.
[4]. A. V. Arhangel'skii and L. Ludwig, On α-normal and β-normal spaces, Comment. Math. Univ. Carolin. 42 (2001), 507–519.
[5]. A. Chigogidze, On relative dimensions, in: General Topology, Spaces of Functions and Dimension, MGU, Moscow (1985), 67– 117 (in Russian).
[6]. A. K. Das, Δ-normal spaces and decomposition of normality, Appl. Gen. Topology 10 (2009), 197–206.
[7]. A. K. Das, A note on spaces between normal and κ-normal spaces, Filomat 27 (2013), 85–88.
[8]. A. K. Das and P. Bhat, A class of spaces containing all densely normal space, Indian J. Math. 57 (2015), 217–224.
[9]. A. K. Das, P. Bhat and J. K. Tartir, On a simultaneous generalization of β-normality and almost normality, Filomat 31 (2017), 425–430.
[10]. A. K. Das and S. S. Raina, On relative β-normality, Acta Math. Hungar. 160 (2020), 468–477.
[11]. A. Dow and J. Vermeer, An example concerning the property of a space being Lindel¨of in another, Topology Appl. 51(1993), 255–260.
[12]. E Grabner, G. Grabner, K. Miyazaki and J. Tartir, Relative star normal type, Topology Appl. 153 (2005), 874–885.
[13]. Lj. D. Koˇcinac, Some relative topological properties, Mat. Vesnik 44 (1992), 33-44.
[14]. L. Kalantan, π-normal topological spaces, Filomat 22 (2008), 173–181.
[15]. L. Kalantan, F. Allahabi, On almost normality, Demonstr. Math. XLI (2008) 962–968.
[16]. H. Kumar and M. C. Sharma, Softly regular spaces in topological spaces, Journal of Emerging Technologies and Innovative Research, Vol. 6, Issue 11, (2018), 183-190.
[17]. K. Kuratowski, Topologie I, Hafner, New York, 1958.
[18]. M. V. Matveev, O. I. Pavlov and J. K. Tartir, On relatively normal spaces, relatively regular spaces, and on relative property (a), Topology Appl. 93 (1999), 121–129.
[19]. S. S. Raina and A. K. Das, Some new variants of relative regularity via regularly closed sets, J. Math. (2021).
[20]. S. S. Raina and A. K. Das, Some variants of normality in relative topological spaces, Filomat 36:12 (2022), 4241-4249.
[21]. E. V. Shchepin, Real functions, and spaces that are nearly normal. (Russian), Sibirsk. Mat. ˇZ. 13 (1972), 1182–1196.
[22]. M. K. Singal and A. R. Singal, Mildly normal spaces, Kyungpook Math. J. 13 (1973), 27–31.
[23]. M. K. Singal and S. P. Arya, Almost normal and almost completely regular spaces, Kyungpook Math. J. 25 (1970), 141–152.
[24]. M. K. Singal and S. P. Arya, On almost-regular spaces, Glas. Mat. Ser. III 4(24) (1969) 89–99.
[25]. M. C. Sharma and Hamant Kumar, Softly normal topological spaces, Acta Ciencia Indica, Vol. XLI, M. No. 2, (2015), 81-84.
[26]. L. A. Steen, J. A. Seebach, jr., Counter Examples in Topology, Springer Verlag, New york, 1978.
[27]. G. Viglino, Seminormal and C-compact spaces, Duke J. Math. 38 (1971) 57–61.
[28]. V. Zaitsev, Some classes of topological spaces and their bicompact extensions, Dokl. Akad. Nauk SSSR 178 (1968), 778–779.