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Research Paper

Approximate analytical solution of a fractional order detritus-based predator-prey model using Homotopy Perturbation Method

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ABSTRACT:*In this work, a detritus-based predator-prey model with fractional order based on the ecosystem of the Sundarban mangrove forest is formulated. Here, Holling type-II function is applied to express the loss of detritus due to micro-organisms, and the food consumption rate of the invertebrate predator is supposed to follow the Ivlev-type response function. Here, we have derived the approximate solution of the fractional order system using Homotopy Perturbation method (HPM) with high accuracy. As HPM is a rapid convergence method, we have done only a few iterations to get approximate analytical series solutions of the system. Numerical simulations have been experimented with different valued fractional orders to better understand our analytical findings.*

2020 Mathematics Subject Classification*: 92B05*
KEYWORDS: *Detritus, Micro-organism pool, Ivlev-type functional response, Invertebrate predator, Caputo fractional derivative, Homotopy perturbation method.*

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I. INTRODUCTION

In recent times, fractional calculus has received much recognition and popularity. It provides some particular descriptions of various linear and non-linear systems. In the past few years, it has been extensively used in various fields of science like mathematics, physics, biology, engineering, etc. [1–11]. In 1695, Leibniz first introduced fractional order derivatives in calculus for differentiation and integration. In most of the cases of real physical problems, the behavior of a non-linear system depends on the instant time as well as the previous time of interval, which may be acquired by a fractional derivative. A differential equation with fractional order is a special case of a differential equation with integer order. Fractional order differential equation can be derived by changing the order of the differential equation from integer to fraction. The key advantage of the system with fractional order is that it permits higher degrees of freedom compared to the differential equations with integer order. In population dynamics, generally, differential equations with integer order are applied to formulate a model, but many authors have already worked on fractional order differential equations in population dynamics and ecology in the past few years [12–15]. In maximum cases, finding the accurate analytical solution of fractional order differential equations is quite challenging [16–19]. As a result, some analytical approximation methods are developed to find the approximate analytical solutions close to the exact analytical solutions. Numerous methods are present for solving fractional differential equations. Among them, some convenient methods are the Adomian Decomposition Method (ADM) [20, 21], Variational Iteration Method (VIM) [22, 23], Homotopy Perturbation Method (HPM) [24, 25], Homotopy Analysis Method (HAM) [26, 27], etc. All these methods are based on some numerical and analytical aspects. The HPM is a very efficacious and appropriate approximation method. This method is not only applicable to linear equations but also suitable for non-linear equations. The HPM was first introduced by HE in 1999 [28–30] for solving both non-linear and linear differential and integral equations. Later, the applications of HPM were widely spread.

Several authors have applied this method in different areas of mathematics like Volterra'sintegrow-differential equation [31, 32], delay-differential equations [33, 34], boundary value problems [35, 36], non-linear wave equations [37, 38], fractional order quadratic Riccati differential equation [39] and so many others. This is a perturbation method by which any differential and integral equations with fractional order can be solved analytically easily by constructing a homotopy. The main benefit of this method is that it has no limitation of having any small parameter for getting an approximate solution, while the other perturbation methods generally require small parameters. This small parameter has a profound impact on the solution of the system. This is a very rapid convergence method requiring few iterations to get an accurate solution.
The main focus of our study is to enhance the implementation of the HPM to our proposed detritus based prey-predator model to get an approximate analytical solution. In this work, a deterministic model is formulated, where detritus is the primary source of energy level, the micro-organism pool acts as the prey, and the predator is the invertebrate predator in the Sundarban mangrove forest in India. The orders of the derivatives used in the model are considered fractions of different values. Here, the uptake rate of the micro-organism pool due to the predation of the invertebrate predator is taken as the Ivlev-type response function.
In this paper, all the sections are arranged in the following manner. Section 2 contains model formulation. In section 3, some preliminaries are discussed, which are used throughout the paper to find the solution to our model. In section 4, different steps of HPM are discussed. In section 4, we have found the approximate solution of our model using the HPM. Numerical simulations are done in 5 to illustrate our analytical solutions.

II. Model formulation:

In this Section, a deterministic detritus-based predator-prey mathematical model is considered as follows:

$$\begin{aligned} \frac{dx}{dt} &= x(b_1 - ax) - \frac{fxy}{k_1+x}. \\ \frac{dy}{dt} &= y\left(b_2 - \frac{dy}{ax}\right) - hz\{1-\exp(-gy)\}. \\ \frac{dz}{dt} &= z[-m + h\{1-\exp(-gy)\}], \end{aligned} \tag{1}$$

where x, y, and z are biomass of detritus, micro-organism pool, and invertebrate predator, respectively at time t, and all are positive at $t = 0$. Here, b_1 is detritus's growth rate, b_2 is the micro-organism pool's growth rate, $\hbar$ is the food conversion efficiency of the invertebrate predator, m is the normal rate of mortality of invertebrate predator, and g represents the hunting success. Here, the uptake function of invertebrate predator is considered as the Ivlev-type response function. For mathematical simplicity, we convert our system into a non-dimensional system using the following transformations:

$x = k_1P, y = \frac{mk_1Q}{f}, z = \frac{k_1m^2R}{fh}, t = \frac{T_1}{m}$.

Then, the model system (1) is reduced to

$$\begin{aligned} \frac{dP}{dT_1} &= P\left\{(\alpha - \eta P) - \frac{Q}{1+P}\right\}. \\ \frac{dQ}{dT_1} &= Q\left(\beta - \frac{\gamma Q}{P}\right) - R\{1-\exp(-\phi Q)\}. \\ \frac{dR}{dT_1} &= R[-1 + \sigma\{1-\exp(-\phi Q)\}] \end{aligned} \tag{2}$$

where, $\alpha = \frac{b_1}{m}, \eta = \frac{ak_1}{m}, \beta = \frac{b_2}{m}, \gamma = \frac{d}{af}, \sigma = \frac{h}{m}, \phi = \frac{gmk_1}{f}$.

Now, we consider the fractional derivatives and considering $0 < m_1 \le 1, 0 < n_1 \le 1, 0 < n_2 \le 1$, we get the following model:

$$D_{T_1}^{m_1} P = P\left\{(\alpha - \eta P) - \frac{Q}{1+P}\right\}.$$
$$D_{T_1}^{n1} Q = Q\left(\beta - \frac{\gamma Q}{P}\right) - R\{1 - \exp(-\phi Q)\}. \quad (3)$$
$$D_{T_1}^{n2} R = R[-1 + \sigma\{1 - \exp(-\phi Q)\}].$$

where the initial conditions of P, Q, and R are assumed as $P_0 = \delta_1 > 0, Q_0 = \delta_2 > 0$, and $R_0 = \delta_3 > 0$. Also, all the parameters α, η, β, γ, σ and ϕ are positive.

III. SOME PRELIMINARIES

In this section, for finding the approximate solution of our system using HPM, some preliminaries of fractional calculus have been provided.

1. Definition:

A function f_1:(0,∞)→R belongs to the space C_α , α∈R if ∃ a number $\beta_1 > \alpha$(α∈R) such that $f_1(t) = t^{\beta_1} f_2(t)$, where f_2:(0,∞)→ R and the function f_2 belongs to the space $C_\alpha^{\beta_2}$ iff $f_3(\beta_2) \in C_\alpha$, β_2∈N, where f_3:(0,∞) →R is a function.

2. Definition:

Let $f_1(t)$: R^+→R be a function.Then, the fractional integral of order $w$of the function $f_1(t)$ is given by

$$J_t^w f_1(t) = \frac{1}{\Gamma(w)} \int_0^t (t-\xi)^{w-1} f_1(\xi) d\xi,$$

where, $\frac{1}{\Gamma(w)} \int_0^t (t-\xi)^{w-1} f_1(\xi) d\xi$ is point wise continuous on $R^+$and w ≥ 0. Also, Γ(w) denotes the gamma function.

3. Definition:

Let g(t) : R^+→R is a continuous function. Then the Caputo derivative of order s > 0 of g(t) is defined as follows:

$$D_t^s g(t) = \frac{1}{\Gamma(z_1 - s)} \int_0^t \frac{g^{(z_1)}(\phi)}{(t-\phi)^{s-z_1+1}} d\phi,$$

where, z_1∈ Z(Z = set of integers), s ∈ R(R = set of real numbers) and , $z_1 - 1 \le s < z_1$. Here, we mention some basic properties of the integral operator J_t^w and the differential operator D_t^s, which are as follows:

$(i) J^\mu J^\eta f(t) = J^{\mu+\eta} f(t) = J^\eta J^\mu f(t),$

$(ii) J^\mu t^\psi = \frac{\Gamma(\psi+1)}{\gamma(\mu+\psi+1)} t^{\mu+\psi},$

$(iii) D^\mu J^\eta f(t) = J^{\eta-\mu} f(t),$

Where, $f(t) \in C_\alpha, \alpha \ge -1, \mu, \eta \ge 0$, and $\psi > -1$.

IV. Analysis of HPM

To understand the method easily, first we will discuss a review of HPM. Then we will come to our problem. For this, a non-linear differential equation is considered as follows:

$$L_1(v) + N_1(v) = f_1(u), u \in \Omega_1 \quad (4)$$

satisfying the following boundary conditions

$$\mathbf{B_1}\left(v, \frac{\nabla v}{\nabla n_3}\right) = 0, u \in \Gamma(5)$$

where L_1, N_1 and B_1 represent the linear, non-linear and boundary operator respectively. Also, Γ represents the boundary of the region Ω_1, and $f_1(u)$ is an analytic function that is known. According to He's HPM [28–30], at first a homotopy is formed as follows:

$$w(u, p_1): \Omega_1 \times [0,1] \longrightarrow \mathbb{R}$$

Satisfying

$$H_1(w,p_1) = (1-p_1)[L_1(w) - L_1(v_0)] + p_1[L_1(w) + N_1(w) - f_1(u)] = 0$$
$$\Rightarrow H_1(w,p_1) = [L_1(w) - L_1(v_0)] + p_1 L_1(v_0) + p_1[N_1(w) - f_1(u)] = 0, \quad (6)$$

where, p_1 represents the embedding parameter and $p_1 \in [0, 1]$, u $\in Q_1$ and v_0 is assumed as the approximation of initial value satisfying the boundary conditions. Using (6) , we obtain

$$H(w,0) = L_1(w) - L_1(v_0) = 0$$
$$H(w,1) = L_1(w) + N_1(w) - f_1(u) = 0 \quad (7)$$

Here, the values of p_1 changes from 0 to 1, which means $w(u,p_1)$changes from v_0 to $v(u)$. In topological terms, this is called deformation. Here, $L_1(w) - L_1(v_0)$ and $L_1(w) + N_1(w) - f_1(u)$ are named homotopic. Here, $p_1$acts as a "small embedding parameter." Thus the equation (6) has the solution which is as follows:

$$w = \sum_{n_3=0}^{\infty} p_1^{n_3} w_{n_3}(t) = w_0(t) + p_1 w_1(t) + p_1^2 w_2(t) + p_1^3 w_3(t) + \cdots, \quad (8)$$

which is a power series of the equation (4). Setting $p_1 \to 1$, we get

$$v = \lim_{p_1 \to 1} w = \lim_{p_1 \to 1} \sum_{n_3=0}^{\infty} p_1^{n_3} w_{n_3}(t) = w_0(t) + w_1(t) + w_2(t) + w_3(t) + \cdots, \quad (9)$$

which is the approximate solution of the equation (4). In most of the cases, the series in (9) is convergent, which has been proved in He's works [28-30].

V. Solution of our problem by using HPM

In this portion, for system (3), we will find the approximate solution using HPM. We have already considered the initial conditions as follows:

$$P_0(T_1) = \delta_1, Q_0(T_1) = \delta_2, R_0(T_1) = \delta_3$$

Here, for the system (2), we set up the homotopy, which is as follows:

$$D_{T_1}^{m_1} P = p_1 P \left\{(\alpha - \eta P) - \frac{Q}{1+P}\right\}$$
$$D_{T_1}^{n_1} Q = p_1 \left[Q\left(\beta - \frac{\gamma Q}{P}\right) - R\{1 - \exp(-\phi Q)\}\right] \quad (10)$$
$$D_{T_1}^{n_2} R = p_1 R[-1 + \sigma\{1 - \exp(-\phi Q)\}]$$

where the orders of the derivatives i.e., $m_1, n_1, n_2 \in [0, 1]$, and the homotopy parameter $p_1 \in [0, 1]$. If $p_1 = 0$, then the system (10) will be transformed into a system of homogeneous fractional differential equations. Using fractional approach [6, 7], this transformed system can be solved. The solutions of (10) can be written as:

$$P(T_1) = \sum_{n=0}^{\infty} p_1^n P_n(T_1) = P_0(T_1) + p_1 P_1(T_1) + p_1^2 P_2(T_1) + p_1^3 P_3(T_1) + \cdots,$$
$$Q(T_1) = \sum_{n=0}^{\infty} p_1^n Q_n(T_1) = Q_0(T_1) + p_1 Q_1(T_1) + p_1^2 Q_2(T_1) + p_1^3 Q_3(T_1) + \cdots, \quad (11)$$
$$R(T_1) = \sum_{n=0}^{\infty} p_1^n R_n(T_1) = R_0(T_1) + p_1 R_1(T_1) + p_1^2 R_2(T_1) + p_1^3 R_3(T_1) + \cdots.$$

Setting $p_1 \to 1$, we get the solution of the equation (10) which is close to the accurate solution. The estimated solution is as follows:

$$P(T_1)=\lim_{p_1\to 1}\sum_{n=0}^{\infty} \mathrm{p}_1^{\mathrm{n}} P_n(T_1)=P_0(T_1)+P_1(T_1)+P_2(T_1)+P_3(T_1)+\cdots,$$
$$Q(T_1)=\lim_{p_1\to 1}\sum_{n=0}^{\infty} \mathrm{p}_1^{\mathrm{n}} Q_n(T_1)=Q_0(T_1)+Q_1(T_1)+Q_2(T_1)+Q_3(T_1)+\cdots, \quad (12)$$
$$R(T_1)=\lim_{p_1\to 1}\sum_{n=0}^{\infty} \mathrm{p}_1^{\mathrm{n}} R_n(T_1)=R_0(T_1)+R_1(T_1)+R_2(T_1)+R_3(T_1)+\cdots.$$

Now we have substituted the equations (11) in (10) and then equating the powers of p_1 from both sides, we get

$$p_1^0: D^{m_1}P_0(T_1)=0,$$
$$D^{\mathrm{n}_1}Q_0(T_1)=0,$$
$$D^{\mathrm{n}_2}R_0(T_1)=0,$$
$$p_1^1: D^{m_1}P_1(T_1)=P_0(\alpha-\eta P_0-Q_0+P_0Q_0-P_0^2Q_0+P_0^3Q_0),$$
$$D^{\mathrm{n}_1}Q_1(T_1)=Q_0\left(\beta-\frac{\gamma Q_0}{F_0}\right)-R_0\left(\phi Q_0-\frac{\phi^2Q_0^2}{2!}+\frac{\phi^3Q_0^3}{3!}\right),$$
$$D^{n_2}R_1(T_1)=R_0\left(-1+\sigma\phi Q_0-\frac{\sigma\phi^2Q_0^2}{2!}+\frac{\sigma\phi^2Q_0^3}{3!}\right).$$
$$p_1^2: D^{m_1}P_2(T_1)=P_0(-\eta P_1-Q_1+P_1Q_0+P_0Q_1-2P_0P_1Q_0-P_0^2Q_1+3P_0^2P_1Q_0$$
$$+P_0^3Q_1)+P_1(\alpha-\eta P_0-Q_0+P_0Q_0-P_0^2Q_0+P_0^3Q_0),$$
$$D^{n_1}Q_2(T_1)=Q_0\left\{-\frac{P_1}{P_0}\left(\beta-\frac{\gamma Q_0}{P_0}\right)-\frac{\gamma Q_1}{P_0}\right\}+Q_1\left(\beta-\frac{\gamma Q_0}{P_0}\right)-R_0(\phi Q_1-\phi^2Q_0Q_1$$
$$+\frac{\phi^3Q_0^2Q_1}{2}\Big)-R_1\left(\phi Q_0-\frac{\phi^2Q_0^2}{2!}+\frac{\phi^3Q_0^3}{3!}\right).$$
$$D^{n_2}R_2(T_1)=R_0\left(\sigma\phi Q_1-\phi^2\sigma Q_0Q_1+\frac{\phi^3\sigma Q_0^2Q_1}{2}\right)+R_1\left(-1+\sigma\phi Q_0-\frac{\phi^2\sigma Q_0^2}{2!}+\frac{\phi^3\sigma Q_0^3}{3!}\right)$$
$$p_1^3: D^{m_1}P_3(T_1)=P_0(-\eta P_2-Q_2+P_2Q_0+P_1Q_1+P_0Q_2-P_1^2Q_0-2P_0P_2Q_0 \quad (13)$$
$$-2P_0P_1Q_1-P_0^2Q_2+3P_0P_1^2Q_0+3P_0^2P_2Q_0+3P_0^2P_1Q_1+P_0^3Q_2)$$
$$+P_1(-\eta P_1-Q_1+P_1Q_0+P_0Q_1-2P_0P_1Q_0-P_0^2Q_1+3P_0^2P_1Q_0$$
$$+P_0^3Q_1)+P_2(\alpha-\eta P_0-Q_0+P_0Q_0-P_0^2Q_0+P_0^3Q_0),$$
$$D^{n_1}Q_3(T_1)=Q_0\left\{\left(\beta-\frac{\gamma Q_0}{P_0}\right)\left(-\frac{P_2}{P_0}+\frac{P_1^2}{P_0^2}\right)+\frac{\gamma P_1Q_1}{P_0^2}-\frac{\gamma Q_2}{P_0}\right\}+Q_1\left\{-\frac{P_1}{P_0}\right.$$
$$\left.\left(\beta-\frac{\gamma Q_0}{P_0}\right)-\frac{\gamma Q_1}{P_0}\right\}+Q_2\left(\beta-\frac{\gamma Q_0}{P_0}\right)-R_0\left\{\phi Q_2-\frac{\phi^2}{2!}(Q_1^2+2Q_0Q_2)\right.$$
$$\left.+\frac{\phi^3}{3!}(3Q_0Q_1^2+3Q_0^2Q_2)\right\}-R_1\left(\phi Q_1-\phi^2Q_0Q_1+\frac{\phi^3Q_0^2Q_1}{2}\right)$$
$$-R_2\left(\phi Q_0-\frac{\phi^2Q_0^2}{2!}+\frac{\phi^3Q_0^3}{3!}\right).$$
$$D^{n_2}R_3(T_1)=R_0\left\{\sigma\phi Q_2-\frac{\phi^2\sigma}{2!}(Q_1^2+2Q_0Q_2)+\frac{\phi^3\sigma}{3!}(3Q_0Q_1^2+3Q_0^2Q_2)\right\}+R_1(\sigma\phi Q_1$$
$$-\phi^2\sigma Q_0Q_1+\frac{\phi^3\sigma Q_0^2Q_1}{2}\Big)+R_2\left(-1+\sigma\phi Q_0-\frac{\phi^2\sigma Q_0^2}{2!}+\frac{\phi^3\sigma Q_0^3}{3!}\right),$$

and so on. Now, we have applied $J_{T_1}^{m_1}, J_{T_1}^{n_1}, J_{T_1}^{n_2}$ on the set of equations (13), and we get

$$P_0(T_1) = \delta_1$$
$$Q_0(T_1) = \delta_2,$$
$$R_0(T_1) = \delta_3,$$
$$P_1(T_1) = \{\alpha\delta_1 - \eta\delta_1^2 - \delta_1\delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\frac{T_1^{m_1}}{\Gamma(m_1 + 1)},$$
$$Q_1(T_1) = \left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) - \delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{n_1}}{\Gamma(n_1 + 1)},$$
$$R_1(T_1) = \left\{-\delta_3 + \sigma\delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{n_2}}{\Gamma(n_2 + 1)},$$
$$P_2(T_1) = \{(\alpha - 2\eta\delta_1) - \delta_1\delta_2(-1 + 2\delta_1 - 3\delta_1^2) - \delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}$$
$$\{\alpha\delta_1 - \eta\delta_1^2 - \delta_1\delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\frac{T_1^{2m_1}}{\Gamma(2m_1 + 1)} - \delta_1(1 - \delta_1 + \delta_1^2 - \delta_1^3)$$
$$\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) - \delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{m_1+n_1}}{\Gamma(m_1 + n_1 + 1)},$$
$$Q_2(T_1) = -\frac{\delta_2}{\delta_1}\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right)\{\alpha\delta_1 - \eta\delta_1^2 - \delta_1\delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\frac{T_1^{m_1+n_1}}{\Gamma(m_1 + n_1 + 1)}$$
$$+\left\{-\frac{2\gamma\delta_2}{\delta_1} + \beta - \delta_3\left(\phi - \phi^2\delta_2 + \frac{\phi^3\delta_2^2}{2}\right)\right\}\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) - \delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!}\right.\right.$$
$$\left.\left.+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{2n_1}}{\Gamma(2n_1 + 1)} - \left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\left\{-\delta_3 + \sigma\delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!}\right.\right.$$
$$\left.\left.+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{n_1+n_2}}{\Gamma(n_1 + n_2 + 1)},$$
$$R_2(T_1) = \sigma\delta_3\left(\phi - \phi^2\delta_2 + \frac{\phi^3\delta_2^2}{2}\right)\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) - \delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}$$
$$\frac{T_1^{n_1+n_2}}{\Gamma(n_1 + n_2 + 1)} + \left\{-1 + \sigma\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\{-\delta_3 + \sigma\delta_3$$
$$\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\}\frac{T_1^{2n_2}}{\Gamma(2n_2 + 1)},$$

$$P_3(T_1) = \{\alpha - 2\eta\delta_1 - \delta_1\delta_2(-1 + 2\delta_1 - 3\delta_1^2) - \delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}[\{(\alpha - 2\eta\delta_1) -\delta_1\delta_2(-1 + 2\delta_1 - 3\delta_1^2) - \delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\{\alpha\delta_1 - \eta\delta_1^2 - \delta_1\delta_2 (1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\frac{T_1^{3m_1}}{\Gamma(3m_1 + 1)} - \delta_1(1 - \delta_1 + \delta_1^2 - \delta_1^3)\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) -\delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{2m_1+n_1}}{\Gamma(2m_1 + n_1 + 1)}\Bigg] + \{-\eta - \delta_1\delta_2(1 - 3\delta_1) -\delta_2(-1 + 2\delta_1 - 3\delta_1^2)\}\{\alpha\delta_1 - \eta\delta_1^2 - \delta_1\delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}^2\frac{\Gamma(2m_1 + 1)}{\Gamma(m_1 + 1)^2} \frac{T_1^{3m_1}}{\Gamma(3m_1 + 1)} + \{-\delta_1(-1 + 2\delta_1 - 3\delta_1^2) - (1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\{\alpha\delta_1 - \eta\delta_1^2 -\delta_1\delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) - \delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\} \frac{1}{\Gamma(m_1 + 1)}\frac{1}{\Gamma(n_1 + 1)}\frac{\Gamma(m_1 + n_1 + 1)}{\Gamma(2m_1 + n_1 + 1)}T_1^{2m_1+n_1} - \delta_1(1 - \delta_1 + \delta_1^2 - \delta_1^3) \Bigg[-\frac{\delta_2}{\delta_1}\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right)\{\alpha\delta_1 - \eta\delta_1^2 - \delta_1\delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\frac{T_1^{2m_1+n_1}}{\Gamma(2m_1 + n_1 + 1)} +\left\{-\frac{2\gamma\delta_2}{\delta_1} + \beta - \delta_3\left(\phi - \phi^2\delta_2 + \frac{\phi^3\delta_2^2}{2}\right)\right\}\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) - \delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} +\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{m_1+2n_1}}{\Gamma(m_1 + 2n_1 + 1)} - \left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\{-\delta_3 + \sigma\delta_3(\phi\delta_2 -\frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\Big)\Big\}\frac{T_1^{m_1+n_1+n_2}}{\Gamma(m_1 + n_1 + n_2 + 1)}\Bigg],$$

$$
\begin{aligned}
Q_3(T_1) = &-\frac{\delta_2}{\delta_1}\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)[\{(\alpha-2\eta\delta_1)-\delta_1\delta_2(-1+2\delta_1-3\delta_1^2)-\delta_2(1-\delta_1+\delta_1^2-\delta_1^3)\} \\
&\{\alpha\delta_1-\eta\delta_1^2-\delta_1\delta_2(1-\delta_1+\delta_1^2-\delta_1^3)\}\frac{T_1^{2m_1+n_1}}{\Gamma(2m_1+n_1+1)}-\delta_1(1-\delta_1+\delta_1^2 \\
&-\delta_1^3)\left\{\delta_2\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)-\delta_3\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{m_1+2n_1}}{\Gamma(m_1+2n_1+1)}\Bigg]+\frac{\delta_2}{\delta_1^2} \\
&\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)\{\alpha\delta_1-\eta\delta_1^2-\delta_1\delta_2(1-\delta_1+\delta_1^2-\delta_1^3)\}^2\frac{1}{\Gamma(m_1+1)^2} \\
&\frac{\Gamma(2m_1+1)}{\Gamma(2m_1+n_1+1)}T_1^{2m_1+n_1}+\left\{\frac{\gamma\delta_2}{\delta_1^2}-\frac{1}{\delta_1}\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)\right\}\{\alpha\delta_1-\eta\delta_1^2-\delta_1\delta_2 \\
&(1-\delta_1+\delta_1^2-\delta_1^3)\}\left\{\delta_2\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)-\delta_3\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{1}{\Gamma(m_1+1)} \\
&\frac{1}{\Gamma(n_1+1)}\frac{\Gamma(m_1+n_1+1)}{\Gamma(m_1+2n_1+1)}T_1^{m_1+2n_1}+\left\{\beta-\frac{2\gamma\delta_2}{\delta_1}-\delta_3\left(\phi-\phi^2\delta_2+\frac{\phi^3\delta_2^2}{2}\right)\right\} \\
&\Bigg[-\frac{\delta_2}{\delta_1}\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)\{\alpha\delta_1-\eta\delta_1^2-\delta_1\delta_2(1-\delta_1+\delta_1^2-\delta_1^3)\}\frac{T_1^{m_1+2n_1}}{\Gamma(m_1+2n_1+1)} \\
&+\left\{-\frac{2\gamma\delta_2}{\delta_1}+\beta-\delta_3\left(\phi-\phi^2\delta_2+\frac{\phi^3\delta_2^2}{2}\right)\right\}\left\{\delta_2\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)-\delta_3\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}\right.\right. \\
&\left.\left.+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{3n_1}}{\Gamma(3n_1+1)}+\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\right)\left\{-\delta_3+\sigma\delta_3\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}\right.\right. \\
&\left.\left.+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{2n_1+n_2}}{\Gamma(2n_1+n_2+1)}\Bigg]+\left\{-\frac{\gamma}{\delta_1}-\delta_3\left(-\frac{\phi^2}{2}+\frac{\phi^3\delta_2}{2}\right)\right\}\left\{\delta_2\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)\right. \\
&\left.-\delta_3\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\right)\right\}^2\frac{1}{\Gamma(n_1+1)^2}\frac{\Gamma(2n_1+1)}{\Gamma(3n_1+1)}T_1^{3n_1}-(\phi-\phi^2\delta_2 \\
&+\frac{\phi^3\delta_2^2}{2}\Bigg)\left\{\delta_2\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)-\delta_3\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\{-\delta_3+\sigma\delta_3(\phi\delta_2 \\
&-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\Bigg)\Bigg\}\frac{1}{\Gamma(n_1+1)}\frac{1}{\Gamma(n_2+1)}\frac{\Gamma(n_1+n_2+1)}{\Gamma(2n_1+n_2+1)}T_1^{2n_1+n_2}-(\phi\delta_2 \\
&-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\Bigg)\Bigg[\sigma\delta_3(\phi-\phi^2\delta_2+\frac{\phi^3\delta_2^2}{2})\left\{\delta_2\left(\beta-\frac{\gamma\delta_2}{\delta_1}\right)-\delta_3\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}\right.\right. \\
&\left.\left.+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{2n_1+n_2}}{\Gamma(2n_1+n_2+1)}+\left\{-1+\sigma\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\right)\right\} \\
&\left\{-\delta_3+\sigma\delta_3\left(\phi\delta_2-\frac{\phi^2\delta_2^2}{2!}+\frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{n_1+2n_2}}{\Gamma(n_1+2n_2+1)}\Bigg],
\end{aligned}
$$

$$R_3(T_1) = \left\{-1 + \sigma\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\left[\sigma\delta_3(\phi - \phi^2\delta_2 + \frac{\phi^3\delta_2^2}{2})\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right)\right.\right.$$
$$\left.-\delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{n_1+2n_2}}{\Gamma(n_1+2n_2+1)} + \left\{-1 + \sigma\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!}\right.\right.$$
$$\left.\left.\left. + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\left\{-\delta_3 + \sigma\delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{3n_2}}{\Gamma(3n_2+1)}\right] + \sigma\delta_3$$
$$\left(\phi - \phi^2\delta_2 + \frac{\phi^3\delta_2^2}{2}\right)\left[-\frac{\delta_2}{\delta_1}\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right)\{\alpha\delta_1 - \eta\delta_1^2 - \delta_1\delta_2(1 - \delta_1 + \delta_1^2 - \delta_1^3)\}\right.$$
$$\frac{T_1^{m_1+n_1+n_2}}{\Gamma(m_1+n_1+n_2+1)} + \left\{-\frac{2\gamma\delta_2}{\delta_1} + \beta - \delta_3\left(\phi - \phi^2\delta_2 + \frac{\phi^3\delta_2^2}{2}\right)\right\}\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right)\right.$$
$$\left.-\delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{2n_1+n_2}}{\Gamma(2n_1+n_2+1)} - \left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right) \tag{14}$$
$$\left.\left\{-\delta_3 + \sigma\delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{T_1^{n_1+2n_2}}{\Gamma(n_1+2n_2+1)}\right] + \sigma\delta_3\left(-\frac{\phi^2}{2!} + \frac{\phi^3\delta_2}{2}\right)$$
$$\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) - \delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}^2\frac{1}{\Gamma(n_1+1)^2}\frac{\Gamma(2n_1+1)}{\Gamma(2n_1+n_2+1)}$$
$$T_1^{2n_1+n_2} + \sigma\left(\phi - \phi^2\delta_2 + \frac{\phi^3\delta_2^2}{2}\right)\left\{\delta_2\left(\beta - \frac{\gamma\delta_2}{\delta_1}\right) - \delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}$$
$$\left\{-\delta_3 + \sigma\delta_3\left(\phi\delta_2 - \frac{\phi^2\delta_2^2}{2!} + \frac{\phi^3\delta_2^3}{3!}\right)\right\}\frac{1}{\Gamma(n_1+1)}\frac{1}{\Gamma(n_2+1)}$$
$$\frac{\Gamma(n_1+n_2+1)}{\Gamma(n_1+2n_2+1)}T_1^{n_1+2n_2},$$

Therefore, we have got the approximate solution of order 3, which is as follows:

$$\begin{aligned} P(T_1) &= \textstyle\sum_{n=0}^{3} P_n(T_1) = P_0(T_1) + P_1(T_1) + P_2(T_1) + P_3(T_1) + \cdots, \\ Q(T_1) &= \textstyle\sum_{n=0}^{3} Q_n(T_1) = Q_0(T_1) + Q_1(T_1) + Q_2(T_1) + Q_3(T_1) + \cdots, \quad (15) \\ R(T_1) &= \textstyle\sum_{n=0}^{3} R_n(T_1) = R_0(T_1) + R_1(T_1) + R_2(T_1) + R_3(T_1) + \cdots. \end{aligned}$$

One may also take more terms in the same manner to get a more suitable solution close to the exact solution.

VII. Numerical simulation results

In this section, we have executed a numerical simulation to find the graphs of the approximate solution of the system (3). In this simulation process, we used the series' first four terms to get the approximate solution. Throughout the numerical illustration, we have used a set of parameter values as follows: α = 0.8, η = 0.3, β = 1.2, γ = 0.94, ϕ = 1.1, σ = 2.006, and the initial values of three populations are assumed as P_0 = 0.3, Q_0 = 0.3, R_0 = 0.3. We have carried out the simulation for different valued fractional orders as well as for standard order 1. Figure 1 shows the graph of solution of $P(T_1)$ with respect to time T_1 for different values of m_1 i.e. for m_1 = 1/3, 1/2, 2/3, 1, n_1 = 1 and n_2 = 1. In this figure, it has been shown that initially, the population density of detritus increases more rapidly with decreasing fractional order m_1. But, after a certain period of time, the population density increases more rapidly with increasing fractional order m1. But for the standard order 1, the population density of detritus initially increases with increasing time, reaches its highest value, and then decreases. Figure 2 describe the solution graph of $P(T_1)$ with respect to time T_1 for $m_1 = n_1 = n_2$= 1/3, 1/2, 2/3, 1. Figures 1 and 2 show the same kind of solution graphs.

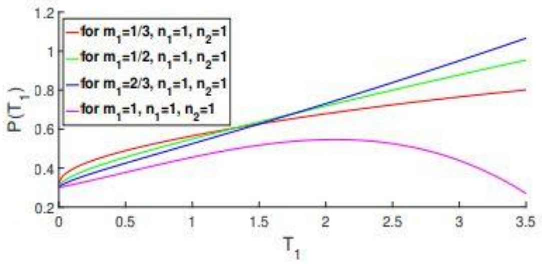

Figure 1: Approximate solutions of $P(T_1)$of the system (3) for fractional orders:
m_1 = 1/3, 1/2, 2/3, 1, n_1 = 1 and n_2 = 1

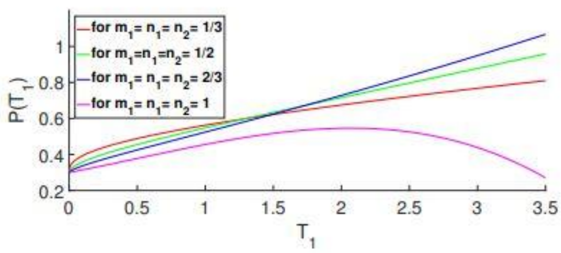

Figure 2: Approximate solutions of $P(T_1)$ of the system (3) for fractional orders:
m_1= n_1 = n_2 = 1/3, 1/2, 2/3, 1

Figure 3 represents the solution graph of the population $Q(T_1)$ with respect to time T_1 for n_1 = 1/3, 1/2, 2/3, 1, and keeping the values of m_1 and n_2 fixed to 1. In Figure 3, it has been shown that when the value of order n_1 decreases, initially, the population density of the micro organism pool increases more rapidly with increasing time, but after a certain period of time, the population density decreases more rapidly with increasing time. A similar type of picture is observed in Figure 4, when we plot the solution graph of the population $Q(T_1)$ with respect to time T_1 for m_1 = n_1 = n_2 = 1/3, 1/2, 2/3, 1.

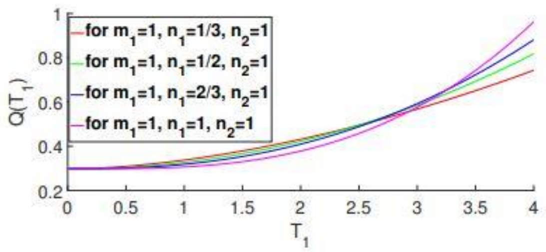

Figure 3: Approximate solutions of $Q(T_1)$, of the system (3) for fractional orders:
m_1 = 1, n_1 = 1/3, 1/2, 2/3, 1 and n_2 =1

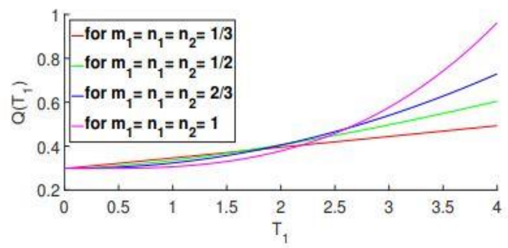

Figure 4: Approximate solutions of $Q(T_1)$, of the system (3) for fractional orders:
m_1 = n_1 = n_2 = 1/3, 1/2, 2/3, 1

Figure 5 presents the solution graph of $R(T_1)$ with respect to time T_1 for m_1 = n_1 = 1 and n_2 = 1/3, 1/2, 2/3, 1. This graph shows the rapid decrement of population density with decreasing order n_2 initially. But after a certain period of time, a rapid increment of population density of $R(T_1)$ is observed with decreasing order n_2. Figures 6 describe the solution graph of $R(T_1)$ with respect to time T_1 for m_1 = n_1 = n_2 = 1/3, 1/2, 2/3, 1. Figures 5 and 6 show the same type of solution graph.

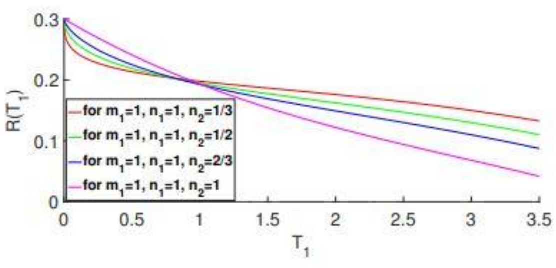

Figure 5: Approximate solutions of $R(T_1)$, of the system (3) for fractional orders: m_1= 1, n_1= 1 and n_2 = 1/3, 1/2, 2/3, 1

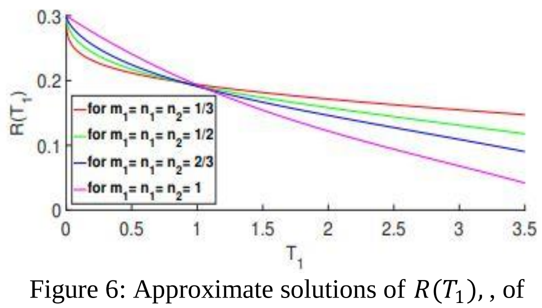

Figure 6: Approximate solutions of $R(T_1)$, , of the system (3) for fractional orders:
m_1= n_1 = n_2 = 1/3, 1/2, 2/3, 1

Lastly, Figure 7 shows the 3-dimensional phase portrait of solutions of $P(T_1)$, $Q(T_1)$, and $R(T_1)$ for m_1 = n_1 = n_1 = 1/3, 1/2, 2/3, 1.

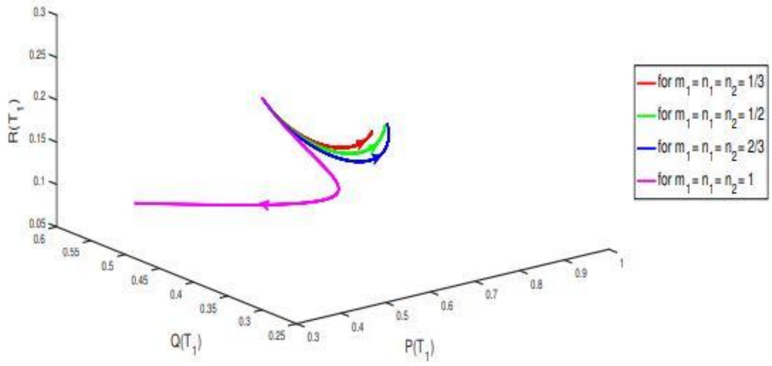

Figure 7: 3-dimensional phase portrait of solutions of $P(T_1)$, $Q(T_1)$ and $R(T_1)$, for m_1 = n_1 = n_1 = 1/3, 1/2, 2/3, 1.

VIII. Conclusions

Integer-order differential equations are commonly used to describe the prey-predator model. Nowadays, fractional order differential equations attract researchers a lot, so they have applied it in different fields of science, including biology, ecology, etc. Many authors have used the HPM to get an approximate solution of the fractional order prey-predator model. But in most of the cases, the models they considered in the papers are two-dimensional. In our study, we have extended the HPM to a three-dimensional prey-predator model, where the three components are detritus, micro-organism pool, and invertebrate predator. Here, we have derived the approximate analytical solution of our model applying the HPM, which is better than other perturbation methods generally used in fractional calculus. In our model, the detritus grows logistically, and loss of detritus due to the micro-organism pool follows Holling type-II response function. Here, we have considered the Ivlev-type response function as the uptake function of the invertebrate predator. This Ivlev-type response function is rarely used in the fractional-order prey-predator models by other researchers. This can be a motivation for solving a much more complicated form of the prey-predator model in the future.

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