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Research Paper

Stability of Quadratic Functional Equation in Random Normed Spaces via Hyers and Fixed Point Methods

Amrit[1],Anil Kumar[2], Manoj Kumar[1*]
[1]*Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak,124021, Haryana, India*
[2]*Department of Mathematics, A.I.J.H.M. College, Rohtak, 124001 Haryana, India*
*(*Corresponding Author)*

Abstract: *The purpose of this study is to provide a new three-variable quadratic functional equation follow as:*

$$\Im(\varsigma_1 + \varsigma_2 - 2\varsigma_3) + \Im(\varsigma_1 - 2\varsigma_2 + \varsigma_3) = \Im(2\varsigma_2 - 2\varsigma_3) + \Im(\varsigma_1 - \varsigma_3) + \Im(\varsigma_1 - \varsigma_2)$$

then use the fixed point approach and direct method in Random Normal Spaces to address the Hyers-Ulam stability of this equation.

Keywords: *fixed point method, Hyers-Ulam stability, Quadratic functional equation, Random Normed Space.*
MSC*: 39B82, 39B52, 39B72, 41A99*

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I. Introduction

The stability of quadratic functional equations in random normed spaces is a fascinating topic in mathematics, particularly in the field of functional analysis. It deals with the behavior of solutions to functional equations when they are perturbed or approximated. The concept originates from a question posed by Stanislaw Ulam [17] in 1940 regarding the stability of group homomorphisms, and it was further developed by Donald Hyers [10], who provided the first affirmative answer in the context of Banach spaces. In random normed spaces, the stability of functional equations like the quadratic functional equation is studied under various conditions and norms. The quadratic functional equation typically takes the form:

$$\Im(\varsigma_1 + \varsigma_2) + \Im(\varsigma_1 - \varsigma_2) = 2\Im(\varsigma) + 2\Im(\varsigma_2)$$

Afterwards, Aoki [2] expanded on Hyers's theorem's result for additive mapping in 1950. For approximately linear mapping, Rassias ([5],[6]) offered a generalized version of Hyers in 1978.

Since then, the researchers gave many new functional equations and discussed their stability in various spaces see ([1], [3], [4] ,[11], [12], [13], [16]).

The fuctional equation

$$\Im(\varsigma_1 + \varsigma_2) + \Im(\varsigma_1 - \varsigma_2) = 2\Im(\varsigma_1) + 2\Im(\varsigma_2) \qquad (1)$$

is referred to as a quadratic functional equation since it has a quadratic function as a solution, $\Im(\varsigma_1) = a\varsigma_1^2$.

To prove our main result, we need some basic notions from literature as follows:

A mapping $\mathrm{F}: R \cup \{-\infty, +\infty\} \to [0,1]$, if it is left-continuous, non-decreasing and satisfies the following condition as : $\mathrm{F}(0) = 0$ and $\mathrm{F}(\infty) = 1$, is called a distribution function. Set A contains all probability distribution functions F with $\mathrm{F}(0) = 0$. A set consisting all function $\mathrm{F} \in A$ for which $\mathrm{F}(\infty) = 1$, where $l^-\mathrm{F}(\varsigma_1) = \lim_{\tau \to \varsigma_1^-} \mathrm{F}(\tau)$ is a subset of A and denoted by D^+. ϵ_a is the element of D^+ for any $a \geq 0$, which is defined as follow:

$$\epsilon_a = \begin{cases} 0, & if \quad \tau \leq a \\ 1, & if \quad \tau > a. \end{cases}$$

Definition 1 [16] Let ζ represent a real linear space, $\varkappa$ represent a function from ζ into D^+ (for any $\varsigma_1 \in \zeta, \varkappa(\varsigma_1)$ is represented by $\varkappa_{\varsigma_1}$) and Υ represent a continuous norm. If $\varkappa$ satisfies the following conditions:

$(RN1) \varkappa_{\varsigma_1}(\tau) = \epsilon_o(\tau)$ for all $\tau > 0$ if and only if $\varsigma_1 = 0$;

$(RN2)\varkappa_{\alpha\varsigma_1}(\tau) = \varkappa_{\varsigma_1}(\frac{\tau}{|\alpha|})$ for all $\varsigma_1 \in \zeta$, $\alpha \neq 0$ and all $\tau \geq 0$;
$(RN3)\varkappa_{\varsigma_1+\varsigma_2}(\tau + s) \geq \Upsilon(\varkappa_{\varsigma_1}(\tau), \varkappa_{\varsigma_2}(s))$for all $\varsigma_1, \varsigma_2 \in \zeta$ and all $\tau, s > 0$.
Then triple $(\zeta, \varkappa, \tau)$ is called a random normed space (briefly RN-space [12]).

Example 1 *For any normed space $(\zeta, \|.\|)$, there is a RN-space $(\zeta, \varkappa, \Upsilon_M)$, where Υ_M is the minimal τ-norm and $\varkappa_{\varsigma_1}(\tau) = \frac{\tau}{\tau + \|\varsigma_1\|}$ for all $\tau > 0$. We refer to this space as induced random normed space.*

Definition 2 *[16] Assume that $(\zeta, \varkappa, \Upsilon)$ be a Random Normed space.*

(1) If, for every $\tau > 0$ and $\lambda > 0$, there exists a positive integer N such that $\varkappa_{\varsigma_{1_n}-\varsigma_1}(\tau) > 1 - \lambda$, whenever $n \geq N$, then a sequence $\{\varsigma_{1_n}\}$ in ζ is said to be convergent to a point $\varsigma_1 \in \zeta$. Here, ς_1 is referred to as the limit of the sequence $\{\varsigma_{1_n}\}$, and it is represented by the notation $\lim_{n\to\infty}\varkappa_{\varsigma_{1_n}-\varsigma_1}(\tau) = 1$.

(2) If, for every $\tau > 0$ and $\lambda > 0$, there exists a positive integer N such that $\varkappa_{\varsigma_{1_n}-\varsigma_{1_m}}(\tau) > 1 - \lambda$ whenever $n \geq m \geq M$, then the sequence $\{\varsigma_{1_n}\}$ in ζ is referred to as a Cauchy sequence.

(3) Each Cauchy sequence in ζ that converges to a point in ζ indicates that the RN -space $(\zeta, \varkappa, \Upsilon)$ is complete.

Theorem 1 [15] If $\{\varsigma_{1_n}\}$ is a sequence of ζ and $(\zeta, \varkappa, \Upsilon)$ is a random normed space such that $\varsigma_{1_n} \to \varsigma_1$ then $\lim_{n\to\infty}\varkappa_{\varsigma_{1_n}}(\tau) = \varkappa_{\varsigma_1}(\tau)$ almost everywhere.

Definition 3 [15] If Υ satisfies the following conditions:
(1) Υ is continuous,
(2) Υ is associative and commutative,
(3) $\Upsilon(a, 1) = 1$ for each $a \in [0,1]$,
(4) $\Upsilon(a, b) \leq \Upsilon(c, d)$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$,
then the mapping $\Upsilon: [0,1] \times [0,1] \to [0,1]$ is called a continuous τ-norm (briefly, a triangular norm).
The examples of continuous τ -norm are as follows:

$$\Upsilon_M(a, b) = min\{a, b\}, \Upsilon_P(a, b) = min\{a, b\}, \Upsilon_L(a, b) = max\{a + b - 1, 0\}$$

Recall that , if Υ is a τ-norm and $\{\varsigma_{1_n}\}$ is a sequence of number in $[0,1]$, then $\Upsilon_{i=1}^{n}\varsigma_{1_i}$ is defined recurrently by $\Upsilon_{i=1}^{1}\varsigma_{1_i} = \varsigma_{1_1}$ and $\Upsilon_{i=1}^{n}\varsigma_{1_i} = \Upsilon(\Upsilon_{i=1}^{n-1}\varsigma_{1_i}, \varsigma_{1_n}) = \Upsilon(\varsigma_{1_1}, \varsigma_{1_2}, \varsigma_{1_3}, \dots \varsigma_{1_n})$ for each $n \geq 2$ and $\Upsilon_{i=1}^{\infty}\varsigma_{1_n}$ is defined as $\Upsilon_{i=1}^{\infty}\varsigma_{1_{n+i}}$ [9].

For the sake of this article, let ζ represent a real linear space, $(\Psi, \varkappa', \Upsilon_M)$ an Random Normed space, and $(\psi, \varkappa, \Upsilon_M)$ a complete Random Normed space.

For mapping $\Im: \zeta \to \psi$, we define

$$D\Im(\varsigma_1, \varsigma_2, \varsigma_3) = \Im(\varsigma_1 + \varsigma_2 - 2\varsigma_3) + \Im(\varsigma_1 - 2\varsigma_2 + \varsigma_3) - \Im(2\varsigma_2 - 2\varsigma_3) - \Im(\varsigma_1 - \varsigma_3) - \Im(\varsigma_1 - \varsigma_2) \quad (2)$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$.

In this work, we apply the fixed-point and direct methods to investigate the generalized Hyers-Ulam stability of the quadratic functional equation (1) under the minimum τ-norm in random normed spaces.

II. Results

Here, the new quadratic functional equation will be introduced and its stability in random normed space will be discussed using both the direct and fixed point methods.

Proposition 2 *The functional equation*

$$\Im(\varsigma_1 + \varsigma_2 - 2\varsigma_3) + \Im(\varsigma_1 - 2\varsigma_2 + \varsigma_3) = \Im(2\varsigma_2 - 2\varsigma_3) + \Im(\varsigma_1 - \varsigma_3) + \Im(\varsigma_1 - \varsigma_2) \quad (3)$$

is a quadratic functional equation.

Proof: Putting $\varsigma_1 = \varsigma_2$ and $\varsigma_3 = 0$ in equation (3), we get

$$\Im(2\varsigma_2) + \Im(-\varsigma_2) = \Im(2\varsigma_2) + \Im(\varsigma_2) + \Im(0)$$

$$\Im(-\varsigma_2) = \Im(\varsigma_2) + \Im(0). \quad (4)$$

Taking $\varsigma_1 = \varsigma_2 = \varsigma_3$ in equation (3) it will be $\Im(0) = 0$.

Then equation (4) becomes

$$\Im(-\varsigma_2) = \Im(\varsigma_2). (5)$$

Taking $\varsigma_3 = 0$ in equation (3), we get

$$\Im(\varsigma_1 + \varsigma_2) + \Im(\varsigma_1 - 2\varsigma_2) = \Im(2\varsigma_2) + \Im(\varsigma_1) + \Im(\varsigma_1 - \varsigma_2)$$

$$\Im(\varsigma_1 + \varsigma_2) = \Im(2\varsigma_2) + \Im(\varsigma_1) + \Im(\varsigma_1 - \varsigma_2) - \Im(\varsigma_1 - 2\varsigma_2). \quad (6)$$

Similarly, taking $\varsigma_3 = 0$ and $\varsigma_2 = -\varsigma_2$ in equation (3), we obtain

$$\Im(\varsigma_1 - \varsigma_2) = \Im(-2\varsigma_2) + \Im(\varsigma_1) + \Im(\varsigma_1 + \varsigma_2) - \Im(\varsigma_1 + 2\varsigma_2). (7)$$

Adding equation (6) and equation (7), and using $\Im(-\varsigma_2) = \Im(\varsigma_2)$ we have

$$2\Im(2\varsigma_2) + 2\Im(\varsigma_1) = \Im(\varsigma_1 - 2\varsigma_2) + \Im(\varsigma_1 + 2\varsigma_2).$$

Now putting $\varsigma_2 = \frac{\varsigma_2}{2}$, we obtain

$$2\Im(\varsigma_1) + 2\Im(\varsigma_2) = \Im(\varsigma_1 + \varsigma_2) + \Im(\varsigma_1 - \varsigma_2), \qquad (8)$$

taking $\varsigma_1 = \varsigma_2$, in above equation, we get

$$\Im(2\varsigma_1) = 2^2 f(\varsigma_1), \qquad (9)$$

clearly, this equation become a quadratic equation.

Theorem 3 (Direct Method) Let $\phi: \zeta^3 \to \Psi$ be a function such that for some $0 < \alpha < 3^2$,

$$\varkappa'_{\phi(3\varsigma_1, 3\varsigma_2, 3\varsigma_3)}(\tau) \geq \varkappa'_{\alpha\phi(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau), \qquad (10)$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$ and

$$\lim_{n\to\infty} \varkappa'_{\phi(3^n\varsigma_1, 3^n\varsigma_2, 3^n\varsigma_3)}(3^{2n}\tau) = 1, \quad (11)$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$.

If $\Im: \zeta \to \psi$ is a mapping with $\Im(0) = 0$ such that

$$\varkappa_{D\Im(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau), \quad (12)$$

for all $\varsigma_1 \in \zeta$ and $\tau > 0$, then there exists a unique quadratic mapping $\Theta: \zeta \to \psi$, which satisfies equation (2) such that

$$\varkappa_{(\Im(\varsigma_1) - \Theta(\varsigma_1))}(\tau) \geq \varkappa'_{\phi(\varsigma_1, 2\varsigma_1, 3\varsigma_1)}((3^2 - \alpha)\tau) \qquad (13)$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$.

Proof: Replacing $(\varsigma_1, \varsigma_2, \varsigma_3)$ by $(\varsigma_3, 2\varsigma_3, 3\varsigma_3)$ in equation (12), we get

$$\varkappa_{\frac{1}{3^2}\Im(3\varsigma_3) - \Im(\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_3, 2\varsigma_3, 3\varsigma_3)}(9t). \qquad (14)$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$. Replacing ς_3 by $3^n\varsigma_3$ in equation (14), we get

$$\varkappa_{\left(\frac{\Im(3^{n+1}\varsigma_3)}{3^2} - \Im(3^n\varsigma_3)\right)}(\tau) \geq \varkappa'_{\phi(3^n\varsigma_3, 2.3^n\varsigma_3, 3.3^n\varsigma_3)}(9\tau) \qquad (15)$$

$$\varkappa_{\left(\frac{\Im(3^{n+1}\varsigma_3}{3^{2(n+1)}} - \frac{\Im(3^n\varsigma_3)}{3^{2n}}\right)}(\tau) \geq \varkappa'_{\phi(\varsigma_3, 2\varsigma_3, 3\varsigma_3)}\left(\left(\frac{3^2}{\alpha}\right)^n (9\tau)\right), \quad (16)$$

for all $\varsigma_3 \in \zeta$ and $\tau > 0$, since

$$\frac{\Im(3^n\varsigma_3)}{3^{2n}} - \Im(\varsigma_3) = \sum_{j=0}^{n-1} \left(\frac{\Im(3^{j+1}\varsigma_3)}{3^{2(j+1)}} - \frac{\Im(3^j\varsigma_3)}{3^{2j}}\right)$$

$$\varkappa_{\left(\frac{\Im(3^n\varsigma_3)}{3^{2n}} - \Im(\varsigma_3)\right)}\left(\sum_{j=0}^{n-1} \frac{1}{9}\left(\frac{\alpha}{3^2}\right)^j \tau\right) \geq \Upsilon_M((\varkappa'_{\phi(\varsigma_3, 2\varsigma_3, 3\varsigma_3)}(\tau))$$

$$= \varkappa'_{\phi(\varsigma_3, 2\varsigma_3, 3\varsigma_3)}(\tau), \qquad (17)$$

for all $\varsigma_3 \in \zeta$ and $\tau > 0$. Replacing ς_3 by $3^m\varsigma_3$ in equation (17), we get

$$\varkappa_{\left(\frac{\Im(3^{n+m}\varsigma_3)}{3^{2(n+m)}} - \frac{(3^m\varsigma_3)}{3^{2m}}\right)}(\tau) \geq \varkappa'_{\phi(\varsigma_3, 2\varsigma_3, 3\varsigma_3)}\left(\frac{9\tau}{\sum_{j=m}^{n+m-1}\left(\frac{\alpha}{3^2}\right)^j}\right). \ (18)$$

This implies that $\{\frac{\Im(3^n\varsigma_3)}{3^{2n}}\}$ is a Cauchy sequence in complete RN-space, so it converges to some point $\Theta(\varsigma_3) \in \psi$, for all $\varsigma_3 \in \zeta$ and $\tau > 0$.

Letting $m = 0$ in equation (18) we get

$$\varkappa_{\left(\frac{\Im(3^n\varsigma_3)}{3^{2n}} - \Im(\varsigma_3)\right)}(\tau) \geq \varkappa'_{\phi(\varsigma_3, 2\varsigma_3, 3\varsigma_3)}\left(\frac{9\tau}{\sum_{j=0}^{n-1}\left(\frac{\alpha}{3^2}\right)^j}\right) \qquad (19)$$

Let $\Theta(\varsigma_3) = \lim_{n\to\infty} \frac{\Im(3^n\varsigma_3)}{3^n}$, and for any $\delta > 0$ we have

$$\varkappa_{\Theta(\varsigma_3) - \Im(\varsigma_3)}(\delta + \tau) \geq \Upsilon_M\left(\varkappa_{\Theta(\varsigma_3) - \frac{\Im(3^n\varsigma_3)}{3^{2n}}}(\delta), \varkappa_{\frac{\Im(3^n\varsigma_3)}{3^{2n}} - \Im(\varsigma_3)}(\tau)\right)$$

$$\geq \left(\varkappa_{\Theta(\varsigma_3) - \frac{(3^n\varsigma_3)}{3^n}}(\delta), \varkappa'_{\phi(\varsigma_3, 2z, 3z)}\left(\frac{9\tau}{\sum_{j=0}^{n-1}\left(\frac{\alpha}{3^2}\right)^j}\right)\right), \qquad (20)$$

for all $\varsigma_3 \in \zeta$ and $\tau > 0$.

Letting $n \to \infty$, in equation (20) , we get

$$\varkappa_{\Theta(\varsigma_3) - \Im(\varsigma_3)}(\delta + \tau) \geq \varkappa'_{\phi(\varsigma_3, 2\varsigma_3, 3\varsigma_3)}((3^2 - \alpha)\tau) \qquad (21)$$

for all $\varsigma_3 \in \zeta$ and $\tau > 0$.

Letting $\delta \to 0$, we obtain

$$\varkappa_{\Theta(\varsigma_3) - \Im(\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_3, 2\varsigma_3, 3\varsigma_3)}((3^2 - \alpha)\tau) \ (22)$$

So, condition of equation (12) holds.

Replacing ς_1 by $3^n\varsigma_1$, ς_2 by $3^n\varsigma_2$ and ς_3 by $3^n\varsigma_3$ in equation (12) respectively, we get

$$\varkappa_{\frac{Df(3^n\varsigma_1, 3^n\varsigma_2, 3^n\varsigma_3)}{3^{2n}}}(\tau) \geq \varkappa'_{\phi(3^n\varsigma_1, 3^n\varsigma_2, 3^n\varsigma_3)}(3^{2n}\tau) \qquad (23)$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$. Letting $n \to \infty$, in equation (23), we get, Θ satisfy the equation (2).

To prove the uniqueness: let, if possible, there exists $W: \zeta \to \psi$ which satisfing equation (2) and (6). Hence

$$\Theta(3^n z) = 3^{2n}\Theta(\varsigma_3)$$
$$W(3^n\varsigma_3) = 3^{2n}W(\varsigma_3)$$

Thus

$$\varkappa_{\Theta(\varsigma_3)-W(\varsigma_3)}(\tau) = \varkappa_{\frac{\Theta(3^n\varsigma_3)}{3^{2n}}-\frac{W(3^n\varsigma_3)}{3^{2n}}}(\tau) \qquad (24)$$

$$\geq \Upsilon_M\left(\varkappa_{(\frac{\Theta(3^n\varsigma_3)}{3^{2n}}-\frac{\Im(3^n\varsigma_3)}{3^{2n}}}\tau), \varkappa_{\frac{\Im(2^n\varsigma_3)}{2^{2n}}-\frac{W(2^n\varsigma_3)}{2^{2n}}}\left(\frac{\tau}{2}\right)^2\right) \qquad (25)$$

$$\geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}\left((3^2-\alpha)\left(\frac{3^2}{\alpha}\right)^n t\right), (26)$$

for all $\varsigma_3 \in \zeta$ and all $\tau > 0$. Since,

$$\lim_{n\to\infty}\left((3^2-\alpha)\left(\frac{3^2}{\alpha}\right)^n t\right) = \infty$$

we have

$$\varkappa_{\Theta(\varsigma_1)-W(\varsigma_1)}(\tau) = 1$$

for all $\tau > 0$.

Thus, the Quadratic mapping Θ is unique.

Theorem 4 *Let $\phi: \zeta^3 \to \Psi$ be a function such that, for some $3^2 < \alpha$,*

$$\varkappa'_{\phi(\frac{\varsigma_1}{3},\frac{\varsigma_2}{3},\frac{\varsigma_3}{3})}(\tau) \geq \varkappa'_{\phi(\varsigma_1,\varsigma_2,\varsigma_3)}(\alpha\tau) \qquad (27)$$

and $\lim_{n\to\infty}\varkappa'_{3^{2n}\phi(\frac{\varsigma_1}{3^n},\frac{\varsigma_2}{3^n},\frac{\varsigma_3}{3^n})}(\tau) = 1$ for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and all $\tau > 0$. If $\Im: \zeta \to \psi$ is mapping such that $\Im(0) = 0$ and satisfies equation (11), then there exists a unique quadratic mapping $\Theta: \zeta \to \psi$ such that

$$\varkappa_{\Im(\varsigma_3)-\Theta(\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}(\alpha - 3^2)\tau), \quad (28)$$

for all $\varsigma_3 \in \zeta$ and $\tau > 0$.

Proof: It follows from equation (11) that

$$\varkappa_{(\Im(\varsigma_3)-3^2\Im(\frac{\varsigma_3}{3}))}(\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}(\alpha\tau), \qquad (29)$$

for all $\varsigma_3 \in \zeta$. Using the triangular inequality and equation (28), we get

$$\varkappa_{(\Im(\varsigma_3)-3^{2n}\Im(\frac{\varsigma_3}{3^n}))}(\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}\left(\frac{\alpha\tau}{\sum_{j=m}^{m+n-1}(\frac{3^2}{\alpha})^j}\right), \quad (30)$$

for all $\varsigma_3 \in \zeta$ and $m, n \in \Psi$ with $n > m \geq 0$. In the complete random normed spces, then the sequence $\{3^{2n}\Im(\frac{\varsigma_3}{3^n})\}$ is a Cauchy sequence, so it converges to some point $\Theta(\varsigma_1) \in \psi$. We can define a mapping $\Theta: \zeta \to \psi$ by

$$\Theta(\varsigma_3) = \lim_{n\to\infty} 3^{2n}\Im(\frac{\varsigma_3}{3^n}),$$

for all $\varsigma_3 \in \zeta$. Then the above mapping satisfies the equation (1) and (28) . The remaining proof is same as in Theorem 2, one can easily deduce it.

Corollary 1. Let θ be a non negative real number and $(\varsigma_3)_0$ be a unique fixed point of Ψ. If $\Im: \zeta \to \psi$ is a mapping with $\Im(0) = 0$ which satisfies

$$\varkappa_{D(\Im(\varsigma_1,\varsigma_2,\varsigma_3))}(\tau) \geq \varkappa'_{\theta{\varsigma_3}_0}(\tau), \qquad (31)$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$, then there exists a unique quadratic mapping $C: \zeta \to \psi$ such that

$$\varkappa_{\Im(\varsigma_3)-\Theta(\varsigma_3)}(\tau) \geq \varkappa'_{\theta{\varsigma_3}_0}(8\tau), \qquad (32)$$

for all $\varsigma_3 \in \zeta$ and $\tau > 0$.

Proof: Let $\phi: \zeta^3 \to \Psi$ be defined by $\phi(\varsigma_1, \varsigma_2, \varsigma_3) = \theta(\varsigma_3)_0$. Then, the proof follow from Theorem 1 by taking $\alpha = 1$. This complete the proof.

Corollary 2. Let $p, q, r \in R$ be a positive real number with $p, q, r < 3$ and $(\varsigma_3)_0$ be a fixed unit point of Ψ. If $\Im: \zeta \to \psi$ is a mapping with $\Im(0) = 0$ which satisfies

$$\varkappa_{D\Im(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau) \geq \varkappa'_{(\|\varsigma_1\|^p+\|\varsigma_2\|^q+\|\varsigma_3\|^r)(\varsigma_3)_0}(\tau), \qquad (33)$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$, then there exists a unique quadratic function $\Theta: \zeta \to \psi$ such that

$$\varkappa_{\Im(\varsigma_3)-\Theta(\varsigma_3)}(\tau) \geq \varkappa'_{\|\varsigma_1\|^p{\varsigma_3}_0}((3^2 - 3^p)\tau), \quad (34)$$

for all $\varsigma_1, \varsigma_3 \in \zeta$ and $\tau > 0$.

Proof: Let $\phi: \zeta^3 \to \Psi$ be defined by $\phi(\varsigma_1, \varsigma_2, \varsigma_3) = (\| \varsigma_1 \|^p + \| \varsigma_2 \|^q + \| \varsigma_3 \|^r)(\varsigma_3)_0$. Then the proof follow from Theorem 1 by $\alpha = 3^p$. This complete the proof.

Theorem 5 [7]. Suppose that $\mathrm{J}: \Omega \to \Omega$ is a strictly contractive mapping and (Ω, d) is a complete generalized metric space with Lipschitz constant $\mathrm{L} < 1$. Then, for each $\varsigma_1 \in \Omega$, either $\mathrm{d}(\mathrm{J}^n\varsigma_1, \mathrm{J}^{n+1}) = \infty$ for all non-negative integers $\mathrm{n} \geq 0$ or there exists a natural number n_0 such that

(1) $d(J^n\varsigma_1, J^{n+1}\varsigma_1) < \infty$ for all $n \geq n_0$;
(2) The sequence $\{J^n\varsigma_1\}$ is convergent to a fixed point ς_2^* of J;
(3) ς_2^* is the unique fixed point of J in the set $A = \{\varsigma_2 \in \Omega: d(J^{n_0}\varsigma_1, \varsigma_2) < \infty\}$;
(4) $d(\varsigma_2, \varsigma_2^*) \leq \frac{1}{1-L} d(\varsigma_2, J\varsigma_2)$ for all $\varsigma_2 \in A$.

Theorem 6 (Fixed Point Method) Let $\phi: \zeta^3 \to D^+$ be a function such that, for some $0 < \alpha < 3^2$,
$\varkappa'_{\phi(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau) \leq \varkappa'_{\phi(3\varsigma_1,3\varsigma_2,3\varsigma_3)}(\alpha\tau)$, (35)
for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$.If $\Im: \zeta \to \psi$ is a mapping with $\Im(0) = 0$ such that
$\varkappa_{Df(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau)$, (36)
for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \psi$ and $\tau > 0$, then there exists a unique quadratic mapping $\Theta: \zeta \to \psi$ such that
$\varkappa_{\Im(\varsigma_1)-\Theta(\varsigma_1)}(\tau) \geq \varkappa'_{\phi(\varsigma_1,2\varsigma_1,3\varsigma_1)}((3^2-\alpha)\tau)$, (37)
for all $\varsigma_1, \varsigma_3 \in \psi$ and $\tau > 0$.
Proof: Taking $\varsigma_1 = \varsigma_3$, $\varsigma_2 = 2\varsigma_3$ and $\varsigma_3 = 3\varsigma_3$ in equation (35), we get
$\varkappa_{\frac{1}{3^2}(3\varsigma_3)-\Im(\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}(9\tau)$, (38)
for all $\varsigma_3 \in \zeta$ and $\tau > 0$. Let $\Omega = \{ð: \zeta \to \psi, ð(\varsigma_1) = 0\}$ and the mapping d defined on Ω by
$d(ð, \hbar) = \inf\{c \in [0,\infty): \varkappa_{ð(\varsigma_3)-\hbar(\varsigma_3)}(c\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}(\tau), \forall\varsigma_3 \in \zeta\}$.
In this case, (Ω, d) is a generalized complete metric space, where $\inf\phi = -\infty$, as usual. Let's now examine the mapping $J: \Omega \to \Omega$, which is defined as
$Jð(\varsigma_3) = \frac{1}{3^2} ð(3\varsigma_3)$
for each $ð \in \Omega$ and $\varsigma_3 \in \zeta$.
Let $ð, \hbar \in \Omega$ and $c \in [0,\infty)$ be an arbitrary constant $d(ð, \hbar) < c$.Then
$\varkappa_{ð(\varsigma_3)-\hbar(\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}(\tau)$
for all $\varsigma_3 \in \zeta$ and $\tau > 0$ and so
$\varkappa_{Jð(\varsigma_3)-J\hbar(\varsigma_3)}(\frac{\alpha c\tau}{3^2}) = \varkappa_{ð(3\varsigma_3)-\hbar(3\varsigma_3)}(\alpha c\tau)$
$\geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}(\tau)$, (39)
for all $\varsigma_3 \in \zeta$ and $\tau > 0$. Hence, we have

$$d(Jð, J\hbar) \leq \frac{ac}{3^2} \leq \frac{\alpha}{3^2} d(ð, \hbar),$$

for all $ð, \hbar \in \Omega$. Then J is a contractive mapping on Ω with Lipchitz constant $L = \frac{\alpha}{3^2} < 1$.
So, by using Theorem 3 there exists a unique fixed point of J in the set of function $\Omega_1 = \{ð \in \Omega: d(ð, \hbar) < \infty\}$, which is $\Theta: \zeta \to \psi$, such that
$\Theta(\varsigma_1) = \lim_{n\to\infty} \frac{\Im(3^2\varsigma_1)}{3^{2n}}$,
for each $\varsigma_1 \in \zeta$ since $\lim_{n\to\infty} d(J^n f, \Theta) = 0$ also from
$\varkappa_{\Im(\frac{3\varsigma_3}{3^2})-\Im(\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}(9\tau)$,
it follows that $d(\Im, J\Im) \leq \frac{1}{3^2}$ therefore using Theorem 3 again, we get
$d(\Im, \Theta) \leq \frac{1}{1-L} d(\Im, J\Im) \leq \frac{1}{3^2-\alpha}$
This means that
$\varkappa_{\Im(\varsigma_3)-\Theta(\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}((3^2-\alpha)\tau)$,
for all $\varsigma_3 \in \zeta$ and $\tau > 0$.
Also, by substituting $3^n\varsigma_1$ for ς_1, $3^n\varsigma_2$ for ς_2 and $3^n\varsigma_3$ for ς_3 in equation (35), respectively, we get
$\varkappa_{DQ(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau) \geq \lim_{n\to\infty} \varkappa'_{\phi(3^n\varsigma_1,3^n\varsigma_2,3^n\varsigma_3)}(3^{2n}\tau)$
$= \lim_{n\to\infty} \varkappa'_{\phi(\varsigma_1,\varsigma_2,\varsigma_3)}\left(\left(\frac{3^2}{\alpha}\right)^n \tau\right) = 1$, (40)
for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$. By using $(RN1)$, the mapping Θ is quadratic
Assume that there is a quadratic mapping $\Theta': \zeta \to \psi$ that fulfills (36) in order to demonstrate the uniqueness. As a result, Θ is a fixed point in Ω_1 for J. Even so, J has just one fixed point in Ω_1, as Theorem 3 indicates. Thus, $\Theta = \Theta'$. This concludes the proof.
Theorem 7 Let $\phi: \zeta^3 \to D^+$ be a function such that, for some $0 < 3^2 < \alpha$,
$\varkappa'_{\phi(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\frac{\varsigma_1}{3},\frac{\varsigma_2}{3},\frac{\varsigma_3}{3})}(\alpha\tau)$, (41)
for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$. If $\Im: \zeta \to \psi$ is a mapping with $\Im(0) = 0$ which satisfies (30) then there exists a unique quadratic mapping $\Theta: \zeta \to \psi$ such that
$\varkappa_{\Im(\varsigma_3)-\Theta(\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_3,2\varsigma_3,3\varsigma_3)}((\alpha-3^2)\tau)$,
for all $\varsigma_3 \in \zeta$ *and* $\tau > 0$.

Proof: The intended result can be easily obtained by making a tweak to the proof of Theorems 4 and 6. The proof is now complete.

Corollary 3. Let ζ be a Banach space , ϵ and p be a positive real number with $p \neq 2$. Assume that $\Im: \zeta \to \psi$ is a function with $\Im(0) = 0$ which satisfies

$$\| D\Im(\varsigma_1, \varsigma_2, \varsigma_3) \| \leq \epsilon(\| \varsigma_1 \|^p + \| \varsigma_2 \|^p + \| \varsigma_3 \|^p),$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$.Then there exists a unique quadratic function $\Theta: \zeta \to \psi$ such that

$$\| \Theta(\varsigma_1) - \Im(\varsigma_1) \| \leq \frac{\epsilon(1+2^p+3^p)\|\varsigma_1\|^p}{|3^2-3^p|},$$

for all $\varsigma_1 \in \zeta$ and $\tau > 0$.

Proof: Define $\varkappa: \zeta \times \mathbb{R} \to \mathbb{R}$ by

$$\varkappa_{\varsigma_1}(\tau) = \begin{cases} \frac{\tau}{\tau + \| \varsigma_1 \|}, & if \quad \tau > 0 \\ 0, & if \quad \tau \leq 0. \end{cases}$$

for all $\varsigma_1 \in \zeta$ and $\tau \in \mathbb{R}$.Then $(\zeta, \varkappa, \Upsilon_M)$ is a complete RN-space. Denote $\phi: \zeta \times \zeta \to R$ by

$$\phi(\varsigma_1, \varsigma_2, \varsigma_3) = \epsilon(\| \varsigma_1 \|^p + \| \varsigma_2 \|^p + \| \varsigma_3 \|^p),$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$. It follows from $\| D\Im(\varsigma_1, \varsigma_2, \varsigma_3) \| \leq \theta(\| \varsigma_1 \|^p + \| \varsigma_2 \|^p + \| \varsigma_3 \|^p)$ that

$$\varkappa_{D\Im(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau) \geq \varkappa'_{\phi(\varsigma_1,\varsigma_2,\varsigma_3)}(\tau),$$

for all $\varsigma_1, \varsigma_2, \varsigma_3 \in \zeta$ and $\tau > 0$, where $\varkappa': \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\varkappa'_{\varsigma_1}(\tau) = \begin{cases} \frac{\tau}{\tau + \| \varsigma_1 \|}, & if \quad \tau > 0 \\ 0, & if \quad \tau \leq 0. \end{cases}$$

is a RN on $\mathbb{R}$.Then all the condition of Theorem 6 and 7 hold and so there exists a unique quadratic mapping $\Theta: \zeta \to \zeta$ such that

$$\frac{\tau}{\tau + \|\Theta(\varsigma_1) - \Im(\varsigma_1)\|} = \varkappa_{\Theta(\varsigma_1) - \Im(\varsigma_1)}(\tau)$$

$$\geq \varkappa'_{\phi(\varsigma_1, 2\varsigma_1, 3\varsigma_1)}(|3^2 - \alpha|\tau)$$

$$= \frac{|3^2-\alpha|\tau}{|3^2-\alpha|\tau + \epsilon(1+2^p+3^p)\|\varsigma_1\|^p},$$

So, we can obtain the required result after taking $\alpha = 3^p$.

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