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Review Paper

An Extension of Generalized Wright Function and its Properties Pertaining to Integral Transforms

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ABSTRACT: *This paper is divided in two sections: Section A &Section B.*
Section A we have obtained integral representations and differential formulae for our function $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$.
and Section B we have derived different types of integral transform namely
Euler Transform, Laplace Transform, Whittaker Transform, K-Transform for this function $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$.
We aim to introduce the extended Wright Function via extended Beta function and obtained certain integral & differential representations of them. Further we drive different types of integral transforms, including Euler Transform, Laplace Transform, Whittaker Transform & K – Transform.
All main results are obtained in terms of generalized extended Wright Function and given in the form of theorems corollaries of the theorems are also discussed Interesting special cases are also discussed.
KEY WORDS: *Beta Function, Gamma Function, Wright Hypergeometric Function, Euler Transform, Laplace Transform, Whittaker Transform, K – Transform.*

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I. INTRODUCTION

The Special functions of mathematical physics are found to be very useful for finding solutions of initial and boundary value problems governed by partial differential equations and fractional differential equations. Special functions have widespread applications in other areas of mathematics and often new perspectives in special functions are motivated by such connections. Several Special functions, called recently Special functions of fractional calculus, play a very important and interesting role as solutions of fractional order differential equations, such as the Mittag-Leffler function, Wright function with its auxiliary functions, and Fox's H -function.

The Wright function is one of the Special functions which plays an important role in the solution of linear partial fractional differential equations. It was introduced for the first time in the year 1940, [1,2] in connection with a problem in the number theory regarding the asymptotic of the number of some special partitions of the natural numbers. Recently this function has appeared in papers related to partial differential equations of fractional order. Considering the boundary value problems for the fractional diffusion-wave equation, that is, the linear partial integrodifferential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order α with $0 < \alpha < 2$, it was found that the corresponding Green Function can be represented in terms of the Wright Function.

The wright function is defined by the series representation [3]:

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\alpha n+\beta)} \qquad \alpha > -1,\ \beta \in C \tag{1}$$

A number of researchers like Gajic & Stankovic [4], Detsenko [5], Luchko [6 & 7], Kilbas, Saigo & Trujillo [8], Mainardi & Pagnini [9], Rudlf, Luchko & Mainardi [10], and again Luchko [11], Shadeb & Salim [3], have studied the main properties of the Wright Function including its integral representations, asymptotes, representations in terms of special functions of the hypergeometric and integral transforms related to Wright Function.
E. Ata. [12] has given generalized Beta Function defined by Wright Function and Mohsen & Salem [13] have given further extended Gamma and Beta functions in terms of generalized Wright Function.

Further, Shahed and Salim [3] introduced a new generalized Wright function $W_{\alpha,\beta}^{\gamma,\delta}(z)$ defined as:

$$W_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{2}$$

$\alpha, \beta, \gamma, \delta \in C$; $\alpha > -1, \delta \neq 0, -1, -2, \ . \ . \ .$ with $z \in C$ and $|z| < 1$ with $\alpha = -1$

where $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1 & (n = 0; a \neq 0) \\ \gamma(\gamma+1)(\gamma+2) \ . \ . \ . \ (\gamma+n-1) & (n \in N; a \in C) \end{cases}$
$N = \{1,2,3 \ . \ . \ . \}$ and $Z_0^- = \{0,-1,-2, \ . \ . \ . \}$
is a Pochhammer symbol and $\Gamma(.)$ is gamma function. [14]

In view of the above recent works, we in the present paper have introduced and defined extended generalized Wright Function $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$ by the following equation:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n, \delta-\gamma; p)}{B(\gamma, \delta-\gamma)} \frac{z^n}{n!\,\Gamma(\alpha n+\beta)} \quad , \tag{3}$$

$\alpha, \beta, \gamma, \delta \in C$; $\alpha > -1, \delta \neq 0, -1, -2, \ . \ . \ .$ with $z \in C$; $p \geq 0$

where $B^{((k_l)_{l\in N_0})}(a, b; p)$ is extended beta function defined by Srivastava in the year (2012) as:

$$B^{((k_l)_{l\in N_0})}(a, b; p) = \int_0^1 t^{a-1} (1-t)^{b-1} \mathcal{J}\left((k_l)_{l\in N_0}; \frac{-p}{t(1-t)}\right) dt \quad , \tag{4}$$
$$\min \{\mathfrak{R}(a), \mathfrak{R}(b)\} > 0; \mathfrak{R}(p) \geq 0$$

and $\mathcal{J}\left((k_l)_{l\in N_0}; z\right)$ is a function of an appropriately bounded sequence $(k_l)_{l\in N_0}$ of arbitrary real or complex numbers defined as follows:

$$\mathcal{J}\left((k_l)_{l\in N_0}; \frac{-p}{t(1-t)}\right) = \begin{cases} \sum_{n=0}^{\infty} (k_l)_{l\in N_0} \frac{z^l}{l!}, \\ M_0 z^w e^z \left[1 + O\left(\frac{1}{2}\right)\right], \end{cases} \tag{5}$$

$|z| < \mathfrak{R}$; $0 < \mathfrak{R} < \infty$; $k_0=1$, $\mathfrak{R}(z) \rightarrow \infty$; $M_0 > 0$; $w \in C$

We have obtained integral representation, differential formula for this function $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$ and also derived deferent types of integral transforms for this function.

Special cases

Some important special cases of extended generalized Wright function defined in eq[n](3) are described below, which will be useful in the present paper while deriving certain main results.
Case I. If we put $k_l = \frac{(\rho)_l}{(\sigma)_l}$ $(l \in N_0)$ in eq[n] (3), we obtain another form of extended generalized Wright function as:

$$W_{\alpha,\beta}^{((\rho,\sigma);\gamma,\delta)}(z;p) = \sum_{n=0}^{\infty} \frac{B^{(\rho,\sigma)}(\gamma+n, \delta-\gamma; p)}{B(\gamma, \delta-\gamma)} \frac{z^n}{n!\,\Gamma(\alpha n+\beta)} \tag{6}$$

$\alpha, \beta, \gamma, \delta \in C$; $\mathfrak{R}(\rho) > 0$, $\mathfrak{R}(\sigma) > 0$; $\alpha > -1, \delta \neq 0, -1, -2, \ . \ . \ .$ with $z \in C$; $p \geq 0$

Case II. On putting $k_l = 1$ ($l \in N_0$), and $\delta = 1$ eq[n] (3) we obtain, yet another form of generalized Wright function as:

$$W_{\alpha,\beta}^{(\gamma,1)}(z;p) = \sum_{n=0}^{\infty} \frac{B(\gamma+n, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{z^n}{n!\,\Gamma(\alpha n+\beta)} \tag{7}$$

$\alpha, \beta, \gamma \in C$; $\alpha > -1$, with $z \in C$; $p \geq 0$

Case III. When p=0 in above eq[n] (7) we get:

$$W_{\alpha,\beta}^{(\gamma,1)}(z) = \sum_{n=0}^{\infty} \frac{B(\gamma+n, 1-\gamma)}{B(\gamma, 1-\gamma)} \frac{z^n}{n!\,\Gamma(\alpha n+\beta)} \tag{8}$$

$\alpha, \beta, \gamma \in C$; $\alpha > -1$, $z \in C$ and $|z| < 1$ with $\alpha = -1$

Case IV. When $\alpha = \beta = \delta = 1$, eq[n] (3), (6), (7) can be expressed, respectively, in terms of the extended confluent hypergeometric functions as follows:

$$W_{1,1}^{((k_l)_{l\in N_0};\,\gamma,1)}(z;p) = \frac{1}{n!} \phi_p^{((k_l)_{l\in N_0})}(\gamma; 1; z) \quad , \tag{9}$$

$$W_{1,1}^{((\rho,\sigma);\,\gamma,1)}(z;p) = \frac{1}{n!} \phi_p^{(\rho,\sigma)}(\gamma; 1; z) \quad , \tag{10}$$

$$W_{1,1}^{(\gamma,1)}(z;p) = \frac{1}{n!} \phi_p(\gamma; 1; z) \quad , \tag{11}$$

II. PRELIMINARIES

In this section we have listed those well - known formulae & results, which will be used in the derivation of the main results.

2.1 The extended Wright hypergeometric function ${}_{m+1}\Psi_{n+1}^{((k_l)_{l\in N_0})}(z;p)$ is defined by Agarwal, Mdallal, Cho & Jain [15, (eq[n] 1.13)] as follows:

$${}_{m+1}\Psi_{n+1}^{((k_l)_{l\in N_0})}(z;p) = {}_{m+1}\Psi_{n+1}^{((k_l)_{l\in N_0})}\left[\begin{matrix}(a_i,\alpha_i)_{(1,m)}\ , & (\gamma,1) \\ (b_i,\beta_i)_{(1,n)}\ , & (\delta,1)\end{matrix}\ \middle|(z;p)\right]$$

$$= \frac{1}{\Gamma\delta-\gamma} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{m}\Gamma(a_i+k\alpha_i)}{\prod_{i=1}^{n}\Gamma(b_i+k\beta_i)} \times \frac{B^{((k_l)_{l\in N_0})}(\gamma+k, \delta-\gamma; p)}{k!} z^k \tag{12}$$

for $\alpha, \beta, \gamma, \delta \in C$; $\mathfrak{R}(\delta) > \mathfrak{R}(\gamma) > 0$; $z \in C$; $p \geq 0$

2.2 The Euler Transform of a function f(z) was defined by Snedden [16] as:

$$B\{f(z); a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)\, dz \tag{13}$$

$a,b \in C$, $\mathfrak{R}(a) > 0, \mathfrak{R}(b) > 0$

2.3 The Laplace Transform of a function f(t), denoted by F(s), was defined by Snedden [16] as:

$$F(s) = (Lf)(s) = L\{f(t); s\} = \int_0^1 e^{-st} f(t)\, dt \tag{14}$$

$$\mathfrak{R}(s) > 0$$

Provided the integral eq[n] (14) is convergent and that the function f(t), is continuous for $t > 0$ and of exponential order as $t \to \infty$, eq[n] (14) may be symbolically written as:

$$F(s) = L\{f(t); s\} \text{ or } f(t) = L^{-1}\{f(t); s\}$$

2.4 The Whittaker Transform defined by Whittaker & Watson [17] as:

$$\int_0^{\infty} t^{\zeta-1} e^{-1/2} W_{\tau,\omega}(t)\, dt = \frac{\Gamma(\frac{1}{2}+w+\zeta)\,\Gamma(\frac{1}{2}-w+\zeta)}{\Gamma(1-\tau+\zeta)} \tag{15}$$

where $\Re(w \pm \zeta) > -1/2$ and $W_{\tau,\omega}(t)$ is the Whittaker confluent hypergeometric function

$$W_{\omega,\zeta}(z) = \frac{\Gamma(-2\omega)}{\Gamma(\frac{1}{2}-\tau-\omega)} M_{\tau,\omega}(z) + \frac{\Gamma(2\omega)}{\Gamma(\frac{1}{2}+\tau+\omega)} M_{\tau,-\omega}(z)$$

where $M_{\tau,\omega}(z)$ is defined by

$$M_{\tau,\omega}(z) = z^{1/2+\omega} e^{-1/2z} \; {}_1F_1\left(\frac{1}{2} + \omega - \tau, 2\omega + 1; z\right)$$

2.5 The K – Transform was defined by Erdelyi [18] by the following integral equation:

$$\mathfrak{R}_{\upsilon}[f(x); p] = g[p;\upsilon] = \int_0^\infty (px)^{1/2} K_\upsilon(px) f(x)\, dx,$$

where $\mathfrak{R}(p) > 0$; $K_\upsilon(x)$ is the Bessel function of the second kind defined by choi [19]:

$$K_\upsilon(z) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,\upsilon}(2z),$$

where $W_{0,\upsilon}(\,.\,)$ is the Whittakar function defined in eq^n (15)

The following result given by Mathai, Saxena & Haubold in [20, p-54, eq^{n}2.37] as:

$$\int_0^\infty (t)^{\rho-1} K_\upsilon(at)\, dt = 2^{\rho-2} a^{-\rho}\, \Gamma\left(\frac{\rho\pm\upsilon}{2}\right) \tag{16}$$

$$\mathfrak{R}(a) > 0; \mathfrak{R}(\rho \pm \upsilon) > 0;$$

III. MAIN RESULTS

Section - A
III (a). INTEGRAL REPRESENTATION

Theorem 1. The following integral representation for $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$ holds true:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)}\left\{\int_0^1 t^{\gamma-1}(1-t)^{\delta-\gamma-1} W_{\alpha,\beta}(tz)\, \mathcal{J}\left((k_l)_{l\in N_0}; \frac{-p}{t(1-t)}\right) dt\right\} \tag{17}$$

where $\alpha, \beta, \gamma, \delta \in C$; $\alpha > -1, \delta \neq 0, -1, -2, \ . \ . \ .$ with $z \in C$; $p \geq 0$

Proof: Using eq^n (4) on right – hand side of eq^n (3) we obtain:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \sum_{n=0}^\infty \left\{\int_0^1 t^{\gamma+n-1}(1-t)^{\delta-\gamma-1} \mathcal{J}\left((k_l)_{l\in N_0}; \frac{-p}{t(1-t)}\right) dt\right\} \times \frac{z^n}{n!\, B(\gamma,\delta-\gamma)\, \Gamma(\alpha n+\beta)}$$

Interchanging the order of summation and integration, on right – hand side of above eq^n we get:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)} \sum_{l=0}^\infty (k_l)_{l\in N_0} \frac{(-p)^l}{l!} \times \left\{\int_0^1 t^{\gamma-l-1}(1-t)^{\delta-\gamma-l-1}\, dt\right\} \times \sum_{n=0}^\infty \frac{(tz)^n}{n!\, \Gamma(\alpha n+\beta)}$$

Again, using eq^n (1) on right – hand side of above eq^n we get:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)} \sum_{l=0}^\infty (k_l)_{l\in N_0} \frac{(-p)^l}{l!} \times \left\{\int_0^1 t^{\gamma-l-1}(1-t)^{\delta-\gamma-l-1}\, dt\right\} W_{\alpha,\beta}(tz)$$

which on using eq^n (5) on right – hand side gives:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)}\left\{\int_0^1 t^{\gamma-1}(1-t)^{\delta-\gamma-1} W_{\alpha,\beta}(tz)\, \mathcal{J}\left((k_l)_{l\in N_0}; \frac{-p}{t(1-t)}\right) dt\right\},$$

This proves result (17) & thus theorem 1 is proved.

Corollary 1.1 Putting $t = \frac{u}{1+u}$ in eq^n (17), we get:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)} \int_0^\infty \left(\frac{u}{1+u}\right)^{\gamma-1} \left(1-\frac{u}{1+u}\right)^{\delta-\gamma-1} W_{\alpha,\beta}\left(\frac{uz}{1+u}\right) \times \mathcal{I}\left((k_l)_{l\in N_0}; \frac{-p}{\left(\frac{u}{1+u}\right)\left(1-\frac{u}{1+u}\right)}\right)\frac{1}{(1+u)^2}\, du$$

Or,

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)} \int_0^\infty \left(\frac{u}{1+u}\right)^{\gamma-1} \left(\frac{1}{1+u}\right)^{\delta-\gamma-1} W_{\alpha,\beta}\left(\frac{uz}{1+u}\right) \times \mathcal{I}\left((k_l)_{l\in N_0}; \frac{-p(1+u)^2}{u}\right)\frac{1}{(1+u)^2}\, du$$

Or,

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)} \int_0^\infty \frac{(u)^{\gamma-1}}{(1+u)^{\delta-2}} W_{\alpha,\beta}\left(\frac{uz}{1+u}\right) \times \mathcal{I}\left((k_l)_{l\in N_0}; \frac{-p(1+u)^2}{u}\right)\frac{1}{(1+u)^2}\, du$$

Or,

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)} \int_0^\infty \frac{(u)^{\gamma-1}}{(1+u)^{\delta}} W_{\alpha,\beta}\left(\frac{uz}{1+u}\right) \mathcal{I}\left((k_l)_{l\in N_0}; \frac{-p(1+u)^2}{u}\right) du$$

where $\alpha,\beta,\gamma,\delta \in C$; $\alpha > -1, \delta \neq 0,-1,-2, \ . \ . \ .$ with $z \in C$; $p \geq 0$

Corollary 1.2 Putting $t = \sin^2\theta$ in eq^n (17), we get:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{1}{B(\gamma,\delta-\gamma)} \int_0^{\pi/2} (\sin^2\theta)^{\gamma-1}(1-\sin^2\theta)^{\delta-\gamma-1} \times W_{\alpha,\beta}(z\sin^2\theta)\, \mathcal{I}\left((k_l)_{l\in N_0}; \frac{-p}{\sin^2\theta(1-\sin^2\theta)}\right)\sin 2\theta\, d\theta$$

Or,

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \frac{2}{B(\gamma,\delta-\gamma)} \int_0^{\pi/2} \sin^{2\gamma-1}\theta \cos^{2\delta-2\gamma-1}\theta \times W_{\alpha,\beta}(z\sin^2\theta)\, \mathcal{I}\left((k_l)_{l\in N_0}; \frac{-p}{\sin^2\theta(\cos^2\theta)}\right) d\theta$$

where $\alpha,\beta,\gamma,\delta \in C$; $\alpha > -1, \delta \neq 0,-1,-2, \ . \ . \ .$ with $z \in C$; $p \geq 0$

Corollary 1.3 The following integral connecting to eq^n (3) follow as:

$$\int_0^x z^{\beta-1}\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(\lambda z^\alpha;p)\, dz = x^\beta\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(\lambda x^\alpha;p)$$

where $\alpha,\beta,\gamma,\delta \in C$; $\alpha > -1, \delta \neq 0,-1,-2, \ . \ . \ .$ with $z \in C$; $p \geq 0$
In particular for $k_l=0$ ($l \in N$), we get:

$$\int_0^x z^{\beta-1}\, W_{\alpha,\beta}^{(\gamma,\delta)}(\lambda z^\alpha)\, dz = x^\beta\, W_{\alpha,\beta}^{(\gamma,\delta)}(\lambda x^\alpha)$$

where $\alpha,\beta,\gamma,\delta \in C$; $\alpha > -1, \delta \neq 0,-1,-2, \ . \ . \ .$ with $z \in C$; $p \geq 0$
where $W_{\alpha,\beta}^{(\gamma,\delta)}(\lambda z^\alpha)$ is defined in eq^n (2).

III(b). DIFFERENTIAL FORMULAE

Theorem 2. The following differential formula for $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$ holds true:

$$W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) = \beta\, W_{\alpha,\beta+1}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) + \alpha z \frac{d}{dz} W_{\alpha,\beta+1}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p) \qquad \textbf{(18)}$$

where $\alpha,\beta,\gamma,\delta \in C$; $\alpha > -1, \delta \neq 0,-1,-2, \ . \ . \ .$ with $z \in C$; $p \geq 0$

Proof: Applying the eq^n (3) on right – hand side of eq^n (18) we get:

Right – hand side of eq^n (18) =

$$= \beta \sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{z^n}{n!\,\Gamma(\alpha n+\beta+1)} + \alpha z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{z^n}{n!\,\Gamma(\alpha n+\beta+1)}$$

$$= \sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{(\alpha z+\beta)\, z^n}{n!\,\Gamma(\alpha n+\beta+1)}$$

$$= \sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{z^n}{n!\,\Gamma(\alpha n+\beta)} \qquad \{\Gamma(a+1) = a\,\Gamma a\}$$

In view of eq^n (3) we get:

$= W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$
= Left - hand side of eq^n (18)

Hence theorem 2 is proved.

Corollary 2.1 On substituting $k_l = 0$ ($l \in N$), we get:

$$W_{\alpha,\beta}^{(\gamma,\delta)}(z;p) = \beta\, W_{\alpha,\beta+1}^{(\gamma,\delta)}(z) + \alpha z \frac{d}{dz} W_{\alpha,\beta+1}^{(\gamma,\delta)}(z)$$

where $\alpha, \beta, \gamma, \delta \in C$; $\alpha > -1, \delta \neq 0, -1, -2, \ . \ . \ .$ with $z \in C$; $p \geq 0$
where $W_{\alpha,\beta+1}^{(\gamma,\delta)}(z)$ is defined in eq^n (2).

Theorem 3. The following differential formula for the $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$ holds true:

$$\frac{d^k}{dz^k}\left\{z^{\beta-1}\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(\lambda z^{\alpha};p)\right\} = z^{\beta-k-1}\, W_{\alpha,\beta-k}^{((k_l)_{l\in N_0};\gamma,\delta)}(\lambda z^{\alpha};p) \qquad (19)$$

where $\Re(\beta-k) > 0$, $k \in N$, $\alpha, \beta, \gamma, \delta \in C$; $\alpha > -1, \delta \neq 0, -1, -2, . . .$ with $z \in C$; $p \geq 0$

Proof: Differentiating k-times with respect to z on the left-hand side of eq^n (19), under the summation sign we get:
Using eq^n (3) on left – hand side of eq^n (19),

$$\text{Left-hand side of eq}^n\ (19) = \frac{d^k}{dz^k}\left\{z^{\beta-1} \sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{\lambda^n z^{\alpha n}}{n!\,\Gamma(\alpha n+\beta)}\right\}$$

$$= \left\{\sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{\lambda^n}{n!\,\Gamma(\alpha n+\beta)}\right\} \frac{d^k}{dz^k}\{z^{\alpha n+\beta-1}\}$$

$$= \left\{\sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{\lambda^n\, \Gamma(\alpha n+\beta)}{n!\,\Gamma(\alpha n+\beta)\,\Gamma(\alpha n+\beta-k)} z^{\alpha n+\beta-k-1}\right\}$$

$$= \left\{\sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{\lambda^n}{n!\,\Gamma(\alpha n+\beta-k)} z^{\alpha n+\beta-k-1}\right\}$$

$$= \left\{z^{\beta-k-1} \sum_{n=0}^{\infty} \frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \frac{\lambda^n}{n!\,\Gamma(\alpha n+\beta-k)} z^{\alpha n}\right\}$$

$= z^{\beta-k-1}\, W_{\alpha,\beta-k}^{((k_l)_{l\in N_0};\gamma,\delta)}(\lambda z^{\alpha};p)$
= Right – hand side of eq^n (19)

Hence theorem 3 is proved.

Corollary 3.1 On substituting $k_l = 0$ ($l \in N$), we get:

$\frac{d^k}{dz^k}\left\{z^{\beta-1}\, W_{\alpha,\beta}^{(\gamma,\delta)}(\lambda z^{\alpha})\right\} = z^{\beta-k-1}\, W_{\alpha,\beta-k}^{(\gamma,\delta)}(\lambda z^{\alpha})$
where $\mathfrak{R}\ (\beta-k) > 0,\ k \in N,\ \alpha,\beta,\gamma,\delta \in C;\ \alpha > -1, \delta \neq 0,-1,-2,...$ with $z \in C$; $p \geq 0$

where $W_{\alpha,\beta}^{(\gamma,\delta)}(\lambda z^{\alpha})$ & $W_{\alpha,\beta-k}^{(\gamma,\delta)}(\lambda z^{\alpha})$ are defined in eq[n] (2).

Section B
III(c). INTEGRAL TRANSFORMS

Theorem4. The following Euler transform for the $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\,\gamma,\delta)}(z;p)$ holds true:

$$\int_0^1 z^{r-1}(1-z)^{s-1}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\,\gamma,\delta)}(xz^{\varsigma};p)\,dz = \frac{\Gamma(s)\,\Gamma\delta}{n!\,\Gamma\gamma} \times {}_3\Psi_3^{((k_l)_{l\in N_0})}\left[\begin{matrix}(r,\varsigma) & (\gamma,1) & (1,1)\\ (r+s,\varsigma) & (\beta,\alpha) & (\delta,1)\end{matrix}\ \middle|\ (x;p)\right] \quad \textbf{(20)}$$

where $\boldsymbol{\alpha,\beta,\gamma,\delta,r,s \in C,\ \varsigma > 0,\ \Re(r) > 0, \Re(s) > 0, \alpha > -1, \delta \neq 0,-1,-2,..., z \in C; p \geq 0, x > 0}$

Proof: Using eq[n] (3) and eq[n] (13) on left - hand side of eq[n] (20) we get:

$$\int_0^1 z^{r-1}(1-z)^{s-1}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\,\gamma,\delta)}(xz^{\varsigma};p)\,dz = \sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{x^n}{n!\,\Gamma(\alpha n+\beta)} \times \int_0^1 z^{r+n\varsigma-1}(1-z)^{\varsigma-1}\ dz$$

$$\int_0^1 z^{r-1}(1-z)^{s-1}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\,\gamma,\delta)}(xz^{\varsigma};p)\,dz = \sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \times \frac{x^n}{n!\,\Gamma(\alpha n+\beta)}\,B(r+n\varsigma,s)$$

$$\int_0^1 z^{r-1}(1-z)^{s-1}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\,\gamma,\delta)}(xz^{\varsigma};p)\,dz = \sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{x^n}{n!\,\Gamma(\alpha n+\beta)} \times \frac{\Gamma(s)\,\Gamma(r+n\varsigma)}{n!\,\Gamma(r+n\varsigma+s)}$$

According to the definition of (12), we get:

$$\int_0^1 z^{r-1}(1-z)^{s-1}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\,\gamma,\delta)}(xz^{\varsigma};p)\,dz = \frac{\Gamma(s)\,\Gamma\delta}{n!\,\Gamma\gamma} \times {}_3\Psi_3^{((k_l)_{l\in N_0})}\left[\begin{matrix}(r,\varsigma) & (\gamma,1) & (1,1)\\ (r+s,\varsigma) & (\beta,\alpha) & (\delta,1)\end{matrix}\ \middle|\ (x;p)\right]$$

This proves result (20) and thus theorem 4 is proved.

Corollary 4.1 when $k_l = \frac{(\rho)_l}{(\sigma)_l}$ ($l \in N_O$), eq[n] (20) reduces to:

$$\int_0^1 z^{r-1}(1-z)^{s-1}\ W_{\alpha,\beta}^{((\rho,\sigma);\,\gamma,\delta)}(xz^{\varsigma};p)\,dz = \frac{\Gamma(s)\,\Gamma\delta\,\Gamma\sigma}{n!\,\Gamma\gamma\ \Gamma\rho} \times {}_4\Psi_4\left[\begin{matrix}(r,\varsigma), & (\gamma,1), & (\rho,1), & (1,1)\\ (r+s,\varsigma), & (\beta,\alpha), & (\sigma,1), & (\delta,1)\end{matrix}\ \middle|\ (x;p)\right]$$

where $\alpha,\beta,\gamma,\delta,r,s \in C,\ \varsigma > 0,\ \Re(r) > 0, \Re(s) > 0, \Re(\rho) > 0, \Re(\sigma) > 0, \alpha > -1,$
$\delta \neq 0,-1,-2,..., z \in C; p \geq 0, x > 0$
corollary 4.1 is the Euler transforms for the extended generalized Wright function which we got as special case in eq[n] (6).

Corollary 4.2 For $k_l = 1$ ($l \in N_O$) and $\delta = 1$, eq[n] (20) reduces to:

$$\int_0^1 z^{r-1}(1-z)^{s-1}\ W_{\alpha,\beta}^{(\gamma,1)}(xz^{\varsigma};p)\,dz = \frac{\Gamma(s)}{n!\,\Gamma\gamma}$$

$$\times\ {}_3\Psi_3\left[\begin{matrix}(1,1), & (r,\varsigma) & (\gamma,1)\\ (r+s,\varsigma), & (\beta,\alpha) & (1,1)\end{matrix}\ \middle|\ (x;p)\right]$$

where $\alpha,\beta,\gamma,r,s\in C,\ \varsigma>0,\ \Re(r)>0, \Re(s)>0, \alpha>-1, z\in C;\ p\geq 0,\ x>0$

corollary 4.2 is the Euler transforms for the extended generalized Wright function which we got as special case in eq^n (7).

Corollary 4.3 If we put $\beta=\alpha=\delta=1$, eq^n (20) reduces to:

$$\int_0^1 z^{r-1}(1-z)^{s-1}\ \phi_p^{((k_l)_{l\in N_0})}(\gamma;1;xz^{\varsigma})\,dz=\frac{\Gamma(s)}{n!\ \Gamma\gamma}$$
$$\times\ {}_2\Psi_2^{((k_l)_{l\in N_0})}\left[\begin{matrix}(r,\varsigma) & (\gamma,1)\\ (r+s,\varsigma) & (1,1)\end{matrix}\ \middle|\ (x;p)\right]$$

where $\gamma,r,s\in C,\ \varsigma>0,\ \Re(r)>0, \Re(s)>0, \alpha>-1, \delta=1, z\in C;\ p\geq 0,\ x>0$

corollary 4.3 is the Euler transforms for the extended confluent hypergeometric generalized function which we got as special case in eq^n (8).

Theorem 5. The following Laplace transform for the $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$ holds true:

$$\int_0^\infty e^{-sz}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\,dz=\frac{\Gamma\delta}{n!\ s\,\Gamma\gamma}$$
$$\times\ {}_3\Psi_2^{((k_l)_{l\in N_0})}\left[\begin{matrix}(1,\varsigma) & (\gamma,1) & (1,1)\\ (\beta,\alpha) & & (\delta,1)\end{matrix}\ \middle|\ (x/s^{\varsigma};p)\right] \qquad \textbf{(21)}$$

where $\alpha,\beta,\gamma,\delta\in C, \left|\frac{x}{s^{\varsigma}}\right|<1, \varsigma>0, \alpha>-1, \delta\neq 0,-1,-2,\ldots, z\in C;\ p\geq 0, x>0$

Proof: Using eq^n (3) and eq^n (14), we get:

$$\int_0^\infty e^{-sz}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\,dz=\sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{x^n}{n!\ \Gamma(\alpha n+\beta)}\int_0^\infty z^{\varsigma n}e^{-zs}\,dz$$

$$\int_0^\infty e^{-sz}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\,dz=\sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{\Gamma(\varsigma n+1)\,x^n}{n!\ \Gamma(\alpha n+\beta)}\ L\left\{\frac{z^{\varsigma n}}{\Gamma(\varsigma n+1)};s\right\}$$

According to the definition of (12), we obtain:

$$\int_0^\infty e^{-sz}\ W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\,dz=\frac{\Gamma\delta}{n!\ s\,\Gamma\gamma}$$
$$\times\ {}_3\Psi_2^{((k_l)_{l\in N_0})}\left[\begin{matrix}(1,\varsigma) & (\gamma,1) & (1,1)\\ (\beta,\alpha) & & (\delta,1)\end{matrix}\ \middle|\ (x/s^{\varsigma};p)\right]$$

Hence the theorem 5 is proved.

Corollary 5.1 when $k_l=\frac{(\rho)_l}{(\sigma)_l}$ ($l\in N_0$), eq^n (21) reduces to:

$$\int_0^\infty e^{-sz}\ W_{\alpha,\beta}^{((\rho,\sigma);\gamma,\delta)}(xz^{\varsigma};p)\,dz=\frac{\Gamma\delta\ \Gamma\sigma}{n!\ s\,\Gamma\gamma\ \Gamma\rho}$$
$$\times\ {}_4\Psi_3\left[\begin{matrix}(1,\varsigma), & (\gamma,1), & (\rho,1), & (1,1)\\ (\beta,\alpha), & (\sigma,1), & & (\delta,1)\end{matrix}\ \middle|\ (x/s^{\varsigma};p)\right]$$

where $\alpha,\beta,\gamma,\delta\in C, \left|\frac{x}{s^{\varsigma}}\right|<1,\ \varsigma>0,\ \alpha>-1,\ \Re(\rho)>0, \Re(\sigma)>0,\ z\in C;\ p\geq 0, x>0$

$\delta\neq 0,-1,-2,\ldots$

corollary 5.1 is the Laplace transforms for the extended generalized Wright function which we got as special case in eq^n (6).

Corollary 5.2 For $k_l=1$ ($l\in N_0$) and $\delta=1$, eq^n (21) reduces to:

$$\int_0^\infty e^{-sz}\ W_{\alpha,\beta}^{(\gamma,1)}(xz^{\varsigma};p)\,dz=\frac{1}{n!\ s\,\Gamma\gamma}\ {}_3\Psi_2\left[\begin{matrix}(1,1),(1,\varsigma),(\gamma,1)\\ (\beta,\alpha),(1,1)\end{matrix}\ \middle|(x/s^{\varsigma};p)\right]$$

where $\alpha,\beta,\gamma\in C, \left|\frac{x}{s^{\varsigma}}\right|<1,\ \varsigma>0,\ \alpha>-1,\ z\in C, p\geq 0, x>0$

corollary 5.2 is the Laplace transforms for the extended generalized Wright function which we got as special case in eq^n (7).

Corollary 5.3 If we put $\beta = \alpha = \delta = 1$, eq^n (21) reduces to:

$$\int_0^{\infty} e^{-sz}\ \phi_p^{((k_l)_{l\in N_0})}(\gamma; 1;\ xz^{\varsigma})\, dz = \frac{1}{n!\ s\,\Gamma\gamma}\ {}_2\Psi_1^{((k_l)_{l\in N_0})}\left[\begin{matrix}(1,\varsigma),(\gamma,1)\\(1,1)\end{matrix}\ \middle|(x/s^{\varsigma}; p)\right]$$

where $\gamma \in C$, $|\frac{x}{s^{\varsigma}}| < 1$, $\varsigma > 0$, $\alpha > -1$, $z \in C$, $p \geq 0$, $x > 0$
corollary 5.3 is the Laplace transforms for the extended confluent hypergeometric generalized function which we got as special case in eq^n (8).

Theorem 6. The following Whittaker transform for the $W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)$ holds true:

$$\int_0^{\infty} z^{\eta-1} e^{-z/2}\, W_{\tau,\omega}(z)\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\, dz = \frac{\Gamma\delta}{n!\ \Gamma\gamma} \times {}_4\Psi_3^{((k_l)_{l\in N_0})} \times\left[\begin{matrix}\left(\omega+\eta+\frac{1}{2},\varsigma\right),\left(-\omega+\eta+\frac{1}{2},\varsigma\right),(\gamma,1),(1,1)\\(1-\tau+\eta,\varsigma),(\beta,\alpha),(\delta,1)\end{matrix}\ \middle|\ (x;p)\right] \quad \textbf{(22)}$$

where $\alpha,\beta,\gamma,\delta \in C$, $\varsigma > 0$, $\alpha > -1$, $\Re(\eta) > 0$, $\Re(\omega+\eta) > -\frac{1}{2}$, $z \in C$, $p \geq 0$, $x > 0$
$\Re(e) > |\Re(\omega)| - \frac{1}{2}$, $\delta \neq 0, -1, -2, \ldots$

Proof- Using eq^n (3) and eq^n (15), we get:

$$\int_0^{\infty} z^{\eta-1} e^{-z/2}\, W_{\tau,\omega}(z)\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\, dz = \int_0^{\infty} z^{\eta-1} e^{-z/2}\, W_{\tau,\omega}(z) \times \sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{x^n z^{\varsigma n}}{n!\ \Gamma(\alpha n+\beta)}\ dz$$

$$\int_0^{\infty} z^{\eta-1} e^{-z/2}\, W_{\tau,\omega}(z)\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\, dz = \sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)} \times \frac{x^n}{n!\ \Gamma(\alpha n+\beta)}\int_0^{\infty} z^{\eta+\varsigma n-1} e^{-z/2}\, W_{\tau,\omega}(z)\, dz$$

$$\int_0^{\infty} z^{\eta-1} e^{-z/2}\, W_{\tau,\omega}(z)\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\, dz = \sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{x^n}{n!\ \Gamma(\alpha n+\beta)}\frac{\Gamma\left(\frac{1}{2}+\omega+\eta+\varsigma n\right)\Gamma\left(\frac{1}{2}-\omega+\eta+\varsigma n\right)}{\Gamma(1-\tau+\eta+\varsigma n)}$$

$$\int_0^{\infty} z^{\eta-1} e^{-z/2}\, W_{\tau,\omega}(z)\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\, dz = \sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{\Gamma\left(\frac{1}{2}+\omega+\eta+\varsigma n\right)\Gamma\left(\frac{1}{2}-\omega+\eta+\varsigma n\right)}{\Gamma(1-\tau+\eta+\varsigma n)}\frac{x^n}{n!\ \Gamma(\alpha n+\beta)}$$

According to the definition of eq^n (12)

$$\int_0^{\infty} z^{\eta-1} e^{-z/2}\, W_{\tau,\omega}(z)\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^{\varsigma};p)\, dz = \frac{\Gamma\delta}{n!\ \Gamma\gamma} \times 4\Psi_3^{((k_l)_{l\in N_0})}\left[\begin{matrix}\left(\omega+\eta+\frac{1}{2},\varsigma\right),\left(-\omega+\eta+\frac{1}{2},\varsigma\right),(\gamma,1),(1,1)\\(1-\tau+\eta,\varsigma),(\beta,\alpha),(\delta,1)\end{matrix}\ \middle|\ (x;p)\right]$$

Hence theorem 6 is proved.

Corollary 6.1 when $k_l = \frac{(\rho)_l}{(\sigma)_l}$ ($l \in N_O$), eq^n (22) reduces to:

$$\int_0^{\infty} z^{\eta-1} e^{-z/2}\, W_{\tau,\omega}(z)\, W_{\alpha,\beta}^{((\rho,\sigma);\gamma,\delta)}(xz^{\varsigma};p)\, dz = \frac{\Gamma\delta\ \Gamma\sigma}{n!\ \Gamma\gamma\ \Gamma\rho} \times 5\Psi_4$$

$$\left[\begin{matrix}\left(\omega+\eta+\frac{1}{2},\varsigma\right),\left(-\omega+\eta+\frac{1}{2},\varsigma\right),(\gamma,1),(\rho,1)(1,1)\\(1-\tau+\eta,\varsigma),(\beta,\alpha),(\sigma,1)\ (\delta,1)\end{matrix}\ \middle|\ (x;p)\right]$$

where $\alpha,\beta,\gamma,\delta\in C$, $\varsigma>0$, $\alpha>-1,\Re(\rho)>0,\Re(\sigma)>0$, $\Re(\eta)>0,\Re(\omega+\eta)>-\frac{1}{2}$, $z\in C, p\geq 0, x>0$, $\Re(e)>|\Re(\omega)|-\frac{1}{2}$, $\delta\neq 0,-1,-2,\ldots$

corollary 6.1 is the Whittaker transforms for the extended generalized Wright function which we got as special case in eq^n (6).

Corollary 6.2 For $k_l=1$ $(l\in N_0)$, and $\delta=1$ eq^n (22) reduces to:

$$\int_0^\infty z^{\eta-1}e^{-z/2}\,W_{\tau,\omega}(z)\,W_{\alpha,\beta}^{(\gamma,1)}(xz^\varsigma;p)\,dz=\frac{1}{n!\,\Gamma\gamma}\times\ {}_4\Psi_3\left[\begin{matrix}(1,1),\left(\omega+\eta+\frac{1}{2},\varsigma\right),\left(-\omega+\eta+\frac{1}{2},\varsigma\right),(\gamma,1)\\(1-\tau+\eta,\varsigma),(\beta,\alpha),(\delta,1)\end{matrix}\ \middle|\ (x;p)\right]$$

where $\alpha,\beta,\gamma\in C,\varsigma>0$, $\alpha>-1$, $\Re(\eta)>0,\Re(\omega+\eta)>-\frac{1}{2}$, $z\in C, p\geq 0, x>0, \Re(e)>|\Re(\omega)|-\frac{1}{2}$, $\delta\neq 0,-1,-2,\ldots$

corollary 6.2 is the Whittaker transforms for the extended generalized Wright function which we got as special case in eq^n (7).

Corollary 6.3 If we put $\beta=\alpha=\delta=1$ eq^n (22) reduces to:

$$\int_0^\infty z^{\eta-1}e^{-z/2}\,W_{\tau,\omega}(z)\,\phi_p^{((k_l)_{l\in N_0})}(\gamma;1;xz^\varsigma)dz=\frac{1}{n!\,\Gamma\gamma}\times\ {}_3\Psi_2^{((k_l)_{l\in N_0})}\left[\begin{matrix}\left(\omega+\eta+\frac{1}{2},\varsigma\right),\left(-\omega+\eta+\frac{1}{2},\varsigma\right),(\gamma,1)\\(1-\tau+\eta,\varsigma),(1,1)\end{matrix}\ \middle|\ (x;p)\right]$$

where $\gamma\in C$, $\varsigma>0$, $\alpha>-1$, $\Re(\eta)>0,\Re(\omega+\eta)>-\frac{1}{2}$, $z\in C, p\geq 0, x>0$, $\Re(e)>|\Re(\omega)|-\frac{1}{2}$

corollary 6.3 is the Whittaker transforms for the extended confluent hypergeometric generalized function which we got as special case in eq^n (8).

Theorem 7. The following K- Transform for the $\mathbf{W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(z;p)}$ holds true:

$$\int_0^\infty \mathbf{z^{\rho-1}}\,K_\upsilon(\omega z)\,W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^\varsigma;p)\,dz=2^{\rho-2}\omega^{1-\rho}\,\frac{\Gamma\delta}{n!\,\Gamma\gamma}\times\ {}_4\Psi_2^{((k_l)_{l\in N_0})}\left[\begin{matrix}\left(\frac{\rho+\upsilon}{2},\frac{\varsigma}{2}\right),\left(\frac{\rho-\upsilon}{2},\frac{\varsigma}{2}\right),(\gamma,1),(1,1)\\(\beta,\alpha),(\delta,1)\end{matrix}\ \middle|\ \left(\frac{2^\varsigma x}{\omega};p\right)\right]\tag{23}$$

where $\boldsymbol{\alpha,\beta,\gamma,\delta,\rho\in C}$, $\boldsymbol{\varsigma}>0$, $\boldsymbol{\alpha>-1}$, $\Re(\boldsymbol{\omega})>0,\Re(\rho\pm\upsilon)>0$, $\mathbf{z\in C, p\geq 0, x}>0$, $\boldsymbol{\delta\neq 0,-1,-2,\ldots}$

Proof: Using eq^n (3) on eq^n (16) we get:

$$\int_0^\infty \mathbf{z^{\rho-1}}\,K_\upsilon(\omega z)\,W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^\varsigma;p)\,dz=\int_0^\infty z^{\rho-1}K_\upsilon(\omega z)\times\sum_{n=0}^\infty\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{x^n z^{n\varsigma}}{n!\,\Gamma(\alpha n+\beta)}\,dz$$

$$\int_0^\infty \mathbf{z^{\rho-1}}\,K_\upsilon(\omega z)\,W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^\varsigma;p)\,dz=\sum_{n=0}^\infty\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{x^n}{n!\,\Gamma(\alpha n+\beta)}\times\int_0^\infty z^{\rho+n\varsigma-1}K_\upsilon(\omega z)\,dz$$

$$\int_0^\infty \mathbf{z^{\rho-1}}\,K_\upsilon(\omega z)\,W_{\alpha,\beta}^{((k_l)_{l\in N_0};\gamma,\delta)}(xz^\varsigma;p)\,dz=\sum_{n=0}^\infty\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)}\frac{x^n}{n!\,\Gamma(\alpha n+\beta)}\times 2^{\rho+n\varsigma-2}\,\omega^{1-\rho-n\varsigma}\,\Gamma\left(\frac{(\rho+n\varsigma)\pm\upsilon}{2}\right)$$

$$\int_0^\infty \mathbf{z^{\rho-1}} K_\upsilon(\omega z)\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\,\gamma,\delta)}(xz^\varsigma;p)\,dz = 2^{\rho-2}\omega^{1-\rho}\sum_{n=0}^{\infty}\frac{B^{((k_l)_{l\in N_0})}(\gamma+n,\delta-\gamma;p)}{B(\gamma,\delta-\gamma)\,n!\,\Gamma(\alpha n+\beta)} \times \left(\frac{2^\varsigma x}{\omega}\right)^n \Gamma\left(\frac{(\rho+n\varsigma)\pm\upsilon}{2}\right)$$

According to the definition of eq^n (12), we obtain:

$$\int_0^\infty \mathbf{z^{\rho-1}} K_\upsilon(\omega z)\, W_{\alpha,\beta}^{((k_l)_{l\in N_0};\,\gamma,\delta)}(xz^\varsigma;p)\,dz = 2^{\rho-2}\omega^{1-\rho}\,\frac{\Gamma\delta}{n!\,\Gamma\gamma} \times {}_4\Psi_2^{((k_l)_{l\in N_0})}\left[\begin{matrix}\left(\frac{\rho+\upsilon}{2},\frac{\varsigma}{2}\right),\left(\frac{\rho-\upsilon}{2},\frac{\varsigma}{2}\right),(\gamma,1),(1,1)\\(\beta,\alpha),(\delta,1)\end{matrix}\;\Big|\;\left(\frac{2^\varsigma x}{\omega};p\right)\right]$$

Hence theorem 7 is proved.

Corollary 7.1 when $k_l = \frac{(\rho)_l}{(\sigma)_l}$ ($l\in N_0$), eq^n (23) reduces to:

$$\int_0^\infty \mathbf{z^{\rho-1}} K_\upsilon(\omega z)\, W_{\alpha,\beta}^{((\rho,\sigma);\,\gamma,\delta)}(xz^\varsigma;p)\,dz = 2^{\rho-2}\omega^{1-\rho}\,\frac{\Gamma\delta\,\Gamma\sigma}{n!\,\Gamma\gamma\,\Gamma\rho} \times {}_5\Psi_3\left[\begin{matrix}\left(\frac{\rho+\upsilon}{2},\frac{\varsigma}{2}\right),\left(\frac{\rho-\upsilon}{2},\frac{\varsigma}{2}\right),(\gamma,1),(\rho,1),(1,1)\\(\beta,\alpha),(\sigma,1),(\delta,1)\end{matrix}\;\Big|\;\left(\frac{2^\varsigma x}{\omega};p\right)\right]$$

where $\alpha,\beta,\gamma,\delta,\rho \in C$, $\varsigma > 0$, $\alpha > -1$, $\Re(\omega) > 0, \Re(\rho \pm \upsilon) > 0, \Re(\rho) > 0, \Re(\sigma) > 0$
$z \in C$, $p \geq 0, x > 0, \delta \neq 0, -1, -2, \ldots$
corollary 7.1 is the K - Transforms for the extended generalized Wright function which we got as special case in eq^n (6).

Corollary 7.2 For $k_l = 1$ ($l\in N_0$), and $\delta = 1$ eq^n (23) reduces to:

$$\int_0^\infty \mathbf{z^{\rho-1}} K_\upsilon(\omega z)\, W_{\alpha,\beta}^{(\gamma,1)}(xz^\varsigma;p)\,dz = 2^{\rho-2}\omega^{1-\rho}\,\frac{1}{n!\,\Gamma\gamma} \times {}_4\Psi_2\left[\begin{matrix}(1,1),\left(\frac{\rho+\upsilon}{2},\frac{\varsigma}{2}\right),\left(\frac{\rho-\upsilon}{2},\frac{\varsigma}{2}\right),(\gamma,1)\\(\beta,\alpha),(1,1)\end{matrix}\;\Big|\;\left(\frac{2^\varsigma x}{\omega};p\right)\right]$$

where $\alpha,\beta,\gamma,\rho \in C$, $\varsigma > 0$, $\alpha > -1$, $\Re(\omega) > 0, \Re(\rho \pm \upsilon) > 0$, $z \in C$, $p\geq 0$, $x > 0$,
corollary 7.2 is the K - Transforms for the extended generalized Wright function which we got as special case in eq^n (7).

Corollary 7.3 If we put $\beta = \alpha = \delta = 1$ eq^n (23) reduces to:

$$\int_0^\infty \mathbf{z^{\rho-1}} K_\upsilon(\omega z)\, \phi_p^{((k_l)_{l\in N_0})}(\gamma;1;xz^\varsigma)\,dz = 2^{\rho-2}\omega^{1-\rho}\,\frac{1}{n!\,\Gamma\gamma} \times {}_3\Psi_1^{((k_l)_{l\in N_0})}\left[\begin{matrix}\left(\frac{\rho+\upsilon}{2},\frac{\varsigma}{2}\right),\left(\frac{\rho-\upsilon}{2},\frac{\varsigma}{2}\right),(\gamma,1)\\(1,1)\end{matrix}\;\Big|\;\left(\frac{2^\varsigma x}{\omega};p\right)\right]$$

where $\gamma,\rho \in C$, $\varsigma > 0$, $\Re(\omega) > 0, \Re(\rho \pm \upsilon) > 0$, $z \in C$, $p\geq 0$, $x > 0$
corollary 7.3 is the K - Transforms for the extended confluent hypergeometric generalized function which we got as special case in eq^n (8).

Conclusion: We have introduced the extended of generalised Wright function and its properties and thereafter we have obtained certain integral representations, differential formulae and different type of integral transforms including Euler Transform, Laplace Transform, Whittaker Transform & K – Transform. And some corollaries of theorems of our main results are also discussed which give rise to some other new interesting results.

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