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Review Paper

Locally Attractivity Solution of First Order Nonlinear QFDE

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ABSTRACT: *In this paper, we discuss the Locally Attractivity for Fractional Order Nonlinear Quadratic Functional Differential Equation in* $\mathcal{R}_+$. *For this we consider the first order nonlinear quadratic functional differential equation.*

KEYWORDS: *Banach algebras, hybridfixed point theorem, Quadratic functional differential equation, and Locally Attractivity result.*

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2.1 INTRODUCTION:

In Literature many authors for various aspects of the solutions are studying the nonlinear differential and integral equations. Fractional order differential and integral equations play a very important role in many applications of real word problem. The study of nonlinear fractional differential equations had been made extensively in the literature by several authors all over the world and now it has become the core part of the nonlinear analysis. The development of nonlinear fractional differential and integral equations though vast growing topic in the subject of nonlinear differential and integral functions,Moreover the theory of Differential and Integral equations is rapidly developing using the tools of Topology, Functional Analysis and Fixed point theory[20-25].

In this paper we will study the locally attractivity Solutionof first order nonlinear quadratic functional differential equation.

We consider the following first order nonlinear quadratic functional differential equations:

$$\left.\begin{aligned} \mathcal{D}\left[\frac{x(t)}{f\left(t,x(\alpha(t))\right)}\right] &= g[t,x(\mu(t))], \quad t \in \mathcal{R}_+ \\ x(0) &= 0 \end{aligned}\right\} \tag{2.1.1}$$

Where, $f(t,x)\colon \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R} - \{0\}$, $g(t,x)\colon \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ and $\alpha, \mu\colon \mathcal{R}_+ \to \mathcal{R}$

Here the solution of nonlinear differential equations (2.1.1) we mean a function $x \in BC(\mathcal{R}_+, \mathcal{R})$ such that:

(i) The function $t \to \left[\frac{x(t)}{f\left(t,x(\alpha(t))\right)}\right]$ is bounded and continuous for each $x \in \mathcal{R}$.

(ii) x satisfies (2.1.1)

2.2 PRELIMINARIES:

Let $X = BC(\mathcal{R}_+, \mathcal{R})$ be the space of bounded real valued continuous function on $\mathcal{R}_+$ and S be a subset of X. Let a mapping $\mathcal{A}\colon X \to X$ be an operator and consider the following operator equation in X, namely,

$x(t) = (\mathcal{A}x)(t)$, for all $t \in \mathcal{R}_+$(2.2.1)

We require the following definitions.

Definition 2.2.1[22]: Let X be a Banach space. A mapping $\mathcal{A}\colon X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that, $\|\mathcal{A}x - \mathcal{A}y\| \le \alpha\|x - y\|$ for all $x, y \in X$. If $\alpha < 1$, then $\mathcal{A}$ is called a contraction on X with the contraction constant α.

Definition 2.2.2[18]:.An operator $\mathcal{U}$ on a Banach space X into itself is called compact if for any bounded subset S of X, $\mathcal{U}(S)$ is relatively compact subset of X. If $\mathcal{U}$ is continuous and compact, then it is called completely continuous on X.

Definition 2.2.2.1 [18,22]: We say that solution of the equation (2.2.1) is locally attractive if there exists a closed ball $B_r[0]$ in the space $BC(\mathcal{R}_+,\mathcal{R})$for some $x_0 \in BC(\mathcal{R}_+,\mathcal{R})$and for some real number $r > 0$ such that for arbitrary solution $x = x(t)$ and $y = y(t)$ of equation (2.2.1) belonging to $B_r[0] \cap S$ we have that,

$$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0$$

Definition 2.2.3[18]: Let X be a Banach space with the norm $\|\cdot\|$ and let $\mathcal{U}: X \to X$ be an operator (in general nonlinear). Then $\mathcal{U}$ is called

i. Compact if $\mathcal{U}(X)$ is relatively compact subset of X.
ii. Totally bounded if $\mathcal{U}(S)$ is totally bounded subset of X for any bounded subset S of X.
iii. Completely continuous if it is continuous and totally bounded operator on X

It is clear that every compact operator is totally bounded but the converse need not be true.

Theorem 2.2.1 [6] :(Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $C(\mathcal{R}_+,\mathcal{R})$, then it has a convergent subsequence.

Theorem 2.2.2[6]: A metric space X is compact iff every sequence in X has a convergent subsequence.

Theorem 2.2.3[5, 6, and 17]: Let S be a non-empty, bounded and closed-convex subset of the Banach space X and let $\mathcal{A}: X \to X$ and $\mathcal{B}: S \to X$ are two operators satisfying

a) $\mathcal{A}$ is Lipschitz with a lipschitz constant α,
b) $\mathcal{B}$ is completely continuous, and
c) $\mathcal{A}x\mathcal{B}x \in S$ for all $x \in S$, and
d) $\alpha M < 1$,where $M = \|\mathcal{B}(S)\|: \sup\{\|\mathcal{B}x\|: x \in S\}$.

Then the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution in S.

2.3 EXISTENCE THEORY:

Now we want the solution of (2.2.1) in the space $BC(\mathcal{R}_+,\mathcal{R})$of bounded and continuous realvalued functions defined on $\mathcal{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication *"$\cdot$ "* in $BC(\mathcal{R}_+,\mathcal{R})$by,$\|x\| = sup\{|x(t)|: t \in \mathcal{R}_+\}$, $(xy)(t) = x(t)y(t)$, $t \in \mathcal{R}_+$

Clearly, $BC(\mathcal{R}_+,\mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ we denote the space of Lebesgue-integrable function in$\mathcal{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by

$$\|x\|_{\mathcal{L}} = \int_0^\infty |x(t)|dt$$

Hypothesis:

We require the following hypothesis.

($\mathcal{H}_1$) The function $\alpha, \mu: \mathcal{R}_+ \to \mathcal{R}$ are continuous.

($\mathcal{H}_2$) The function $f(t,x): \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and bounded with bound F= $sup_{(t,x)\epsilon\mathcal{R}_+\times\mathcal{R}}|f(t,x)|$there exist a bounded function $l: \mathcal{R}_+ \to \mathcal{R}$ with bound L satisfying
$|f(t,x) - f(t,y)| \le l(t)\{|x(t) - y(t)|\}a.e.\ t \in \mathcal{R}_+$, for all $x, y \in \mathcal{R}$.

($\mathcal{H}_3$) The function $g(t,x) = g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is satisfying caratheodory condition with continuous function $h(t): \mathcal{R}_+ \to \mathcal{R}$ such that $g(t,x) \le h(t)\ \ \forall t \in \mathcal{R}_+$and $x, y \in \mathcal{R}$.

($\mathcal{H}_4$) The function $v: \mathcal{R}_+ \to \mathcal{R}$ defined by the formulas $v(t) = \int_0^t \frac{h(s)}{(t-s)^{1-\beta}} ds$ is bounded on $\mathcal{R}_+$ and vanish at infinity, that is $\lim_{t\to\infty} v(t) = 0$.

Remark 2.4.1: Note that the $(\mathcal{H}_3)$and$(\mathcal{H}_4)$hold, then there exists a constant $K_1 > 0$ such that $K_1 = \underbrace{\sup}_{t\ge 0}\left\{\frac{v(t)}{\Gamma(\beta)}: t \in \mathcal{R}_+\right\}$

Lemma 2.4.1: Suppose that $\zeta \in (0,1)$ and the function f, g satisfying FNFDE (2.1.1) then x is the solution of the FNFDE (2.1.1) if and only if it is the solution of integral equation

$x(t) = [f(t,x(\alpha(t)))]\left[\int_0^t g\left(s, x(\mu(s))\right)ds\right],\ t \in \mathcal{R}_+$(2.4.1)

Theorem 2.4.1: Assume that condition ($\mathcal{H}_1$-$\mathcal{H}_4$) hold. Further if $LK_1 < 1$, where K_1 is defined in remark (2.4.1). Then FNFDE (2.1.1) has a solution in the space $BC(\mathcal{R}_+,\mathcal{R})$.

Proof: By a solution of FNFDE (2.1.1) we mean a continuous function $x: \mathcal{R}_+ \to \mathcal{R}$ that satisfies FNFDE (2.1.1) on $\mathcal{R}_+$.Let$X = BC(\mathcal{R}_+,\mathcal{R})$

$B_r[0]$isthe closed ballin X centred at origin0 and radius r as

$B_r[0] = \{x \in X: \|x\| \le r\}$where r satisfies the inequality $FK_1 \le r$.

Let $X = BC(\mathcal{R}_+,\mathcal{R})$ be Banach algebras of all absolutely continuous real-valued function on $\mathcal{R}_+$ with the norm,$\|x\| = sup|x(t)|, t \in \mathcal{R}_+$(2.4.2)

Now the FNFDE (2.1.1) is equivalent to the FNFIE

$x(t) = \left[f\left(t, x(\alpha(t))\right)\right]\left[\int_0^t g\left(s, x(\mu(s))\right) ds\right]$(2.4.3)

Let us define the two mapping $\mathcal{A}: X \to X$ and $\mathcal{B}: B_r[0] \to X$ by

$\mathcal{A}x(t) = f\left(t, x(\alpha(t))\right), t \in \mathcal{R}_+$(2.4.4)

$\mathcal{B}x(t) = \int_0^t g\left(s, x(\mu(s))\right) ds \ , t \in \mathcal{R}_+$(2.4.5)

Thus from the FNDE (2.1.1), we obtain the operator equation as follows:

$x(t) = \mathcal{A}x(t)\mathcal{B}x(t), \ t \in \mathcal{R}_+$(2.4.6)

By use of all above preliminaries and Hypothesiswe have already proved the operator $\mathcal{A}$ and $\mathcal{B}$ satisfy all the hypothesis of theorem (2.2.3), so the operator equation (2.4.6) has a solution on $B_r[0]$.

Here we only show Locally Attractivity solution.

Step I: Here we show the locally attractivity Solution for FNFDE (2.1.1).

For this, Let x and y be two solutions of FNFDE (2.1.1) in $B_r[0]$defined on $\mathcal{R}_+$.

Then we have

$$|x(t) - y(t)| = \left| \left[f\left(t, x(\alpha(t))\right)\right]\left[\int_0^t g\left(s, x(\mu(s))\right) ds\right] - \left[f\left(t, y(\alpha(t))\right)\right]\left[\int_0^t g\left(s, y(\mu(s))\right) ds\right] \right|$$

$$|x(t) - y(t)| \le \left| \left[f\left(t, x(\alpha(t))\right)\right]\left[\int_0^t g\left(s, x(\mu(s))\right) ds\right] \right| + \left| \left[f\left(t, y(\alpha(t))\right)\right]\left[\int_0^t g\left(s, y(\mu(s))\right) ds\right] \right|$$

$$|x(t) - y(t)| \le \left|f\left(t, x(\alpha(t))\right)\right| \int_0^t \left|g\left(s, x(\mu(s))\right)\right| ds + \left|f\left(t, y(\alpha(t))\right)\right| \int_0^t \left|g\left(s, y(\mu(s))\right)\right| ds$$

$$|x(t) - y(t)| \le F\left\{\int_0^t h(s)\, ds\right\} + F\left\{\int_0^t h(s)\, ds\right\}$$

$$|x(t) - y(t)| \le 2F\int_0^t h(s)\, ds \le 2F[v(t)]$$

Since$\lim_{t\to\infty} v(t) = 0$

this gives that

$\lim_{t\to\infty} |x(t) - y(t)| = 0$

This completes the proof.

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