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Review Paper

Solving linear Fractional Schrodinger by Elzaki Homotopy Analysis Method

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Abstract: *In this study, we use the Elzaki homotopy analysis method (EHAM) to identify approximate solutions to fractional Schrodinger PDE. The Caputo fractional operator (CFO) takes into account the method that has been described. There are provided illustrative examples for solving the fractional PDEs. The findings produced are provided to demonstrate the effective features and sample size of the methods for implementing PDEs with CFO that have been described.*
Keywords. *Fractional differential equations; Homotopy analysis method; Caputo fractional operator; Elzaki transform.*

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I. Introduction

Fractional calculus (FC) emerged as a popular academic topic. The latest uses of fractional derivatives in cutting-edge applied science and engineering domains were examined by the mathematicians. The next state of a system depends on its present and past states because the fractional-order differential operator is nonlocal. The primary advantage of non-integer order derivatives is their ability to describe the memory and heredity characteristics of a wide range of occurrences. As a result, fractional-order derivatives and integrals have many uses in both science and technology. For instance, modeling fractional-order fluid dynamic traffic model, chaos theory, signal processing phenomena, electrodynamics, fractional model of cancer chemotherapy, fractional diabetes model, and nonlinear oscillations of earthquakes, among other fields [1-4]. In recent years, many researchers have paid attention to study the behavior of physical problems by using various analytical and numerical techniques which are not described by the common observations, such as the fractional variational iteration method [5-9], fractional differential transform method [10-12], fractional series expansion method [13,14], fractional Sumudu variational iteration method [15,16], fractional Laplace transform method [17], fractional homotopy perturbation method [18], fractional Sumudu decomposition method [19-21], fractional Fourier series method [22], fractional reduced differential transform method [23-25], fractional Adomian decomposition method [26-28], fractional decomposition method [29], fractional homotopy perturbation method (FLHPM) [30], and another method [31–38]. As the main aim of this work the EHAM is implemented to solve fractional PDEs and nonlinear system of fractional PDEs. The paper has been organized as follows. In Section 2, we givei the concepti of FC. In Section 3, we give analysis of the method used. In Section 4, we consider several illustrative examples. Finally, in Section 5, we present our conclusions.

II. Preliminaries

Definition 1. A real function $\Psi(x,\tau), x \in \mathbb{R}, \tau > 0$ is said to be in the space $C_{\varepsilon}, \varepsilon \in \mathbb{R}$ if there exists a real number $q, (q > \varepsilon)$, such that $\Psi(x,\tau) = \tau^{q}\Psi_1(x,\tau)$, where $\Psi_1(x,\tau) \in c[0,\infty]$, and it is said to be in the space C_{ε}^{m} if $\Psi^{(m)}(x,\tau) \in C_{\varepsilon}, m \in \mathbb{N}$.
Definition 2. The Riemann Liouville fractional integral operator of order $\alpha \geq 0$, of a function $\Psi(\tau) \in C_{\varepsilon}, \varepsilon \geq -1$ is defined as

$$I_\tau^\alpha \Psi(\tau) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1} \Psi(s) ds & ,\alpha>0.\tau>0 \\ \Psi(\tau) & ,\alpha = 0 \end{cases} \quad (1)$$

Definition 3. The Caputo fractional derivative (CFD) with order $(\alpha > 0)$ of $\Psi(\tau)$ is defined as follows:

$$^cD_\tau^\alpha \Psi(\tau) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^\tau (\tau-s)^{m-\alpha-1} \Psi^{(m)}(s) ds & ,m-1<\alpha\le m \\ \frac{\partial^n}{\partial \tau^n}\Psi(\tau) & , \quad \alpha = n \in \mathbb{N} \end{cases} \quad (2)$$

The properties of the operator D^α:

1. $D^\alpha I^\alpha \Psi(x,\tau) = \Psi(x,\tau)$

2. $D^\alpha \tau^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \tau^{\beta-\alpha} \quad , \ \alpha > 0$

Definition 4. The Mittag-Leffleri function $E_\alpha(z)$ with $\alpha > 0$ is defined as.

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha+1)} \qquad ,z \in C \quad (3)$$

Definition 5. The Elzaki transform (ET) is defined as:

$$E[\Psi(\tau)] = z\int_0^\infty \Psi(\tau)\, e^{\frac{-\tau}{z}} d\tau, \quad \tau \ge 0, k_1 \le z \le k_2 \quad (4)$$

Some Properties of ET.

1. $E[k] = kz^2 \ , \ k \ constant$
2. $E(\tau^{n\alpha}) = \Gamma(n\alpha+1) z^{n\alpha+2}$.

Lemma 1. The ET of the CFD is defined as

$$E[D_\tau^\alpha \Psi(x,\tau)] = z^{-\alpha} E[\Psi(x,\tau)] - \sum_{k=0}^{m-1} z^{(2-\alpha+k)}\, \Psi^{(k)}(x,0), m-1 < \alpha < m, \quad m \in \mathbb{N} \quad (5)$$

III. FRACTIONAL (EHAM)

Let us consider a general fractional PDE of the form:

$$D_\tau^\alpha\, \Psi(x,\tau) + R\Psi(x,\tau) + N\Psi(x,\tau) = G(x,\tau), m-1 < \alpha \le m, \quad x \in R, \tau > 0 \quad (6)$$

Subject to the initia condition

$$\Psi(x,0) = \Psi^{(k)}(x,0) \quad , \quad k = 1,2,\dots,m-1 \quad (7)$$

where $D_\tau^\alpha\, \Psi(x,\tau)$ is the CFD of the function $\Psi(x,\tau)$ defined as:

$$D_\tau^\alpha\, \Psi(x,\tau) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^\tau (\tau-s)^{m-\alpha-1}\, \frac{\partial^m \Psi(x,s)}{\partial \tau^m} ds & ,m-1<\alpha<m \\ \frac{\partial^m \Psi(x,\tau)}{\partial \tau^m} & ,\alpha = m \in \mathbb{N} \end{cases}$$

and R is the linear differential operator, N represents the general nonlinear differential operator, and $G\,(x,\tau)$ is the source term.

Now taking the ET of both sides of equation (6) we have

$$E[D_\tau^\alpha\, \Psi(x,\tau)] + E[R\, \Psi(x,\tau)] + E[N\, \Psi(x,\tau)] = E[G\,(x,\tau)] \quad (8)$$

Using the differentiation properties of the ET and above initialcondition, we have

$$\frac{E[\Psi(x,\tau)]}{z^\alpha} - \sum_{k=0}^{m-1} z^{2-\alpha+k} \Psi^{(k)}(x,0) + E[R\, \Psi(x,\tau)] + E[N\, \Psi(x,\tau)] = E[G\,(x,\tau)] \quad (9)$$

or

$$E[\Psi(x,\tau)] - \sum_{k=0}^{m-1} z^{2+k} \Psi^{(k)}(x,0) + z^\alpha\{E[R\, \Psi(x,\tau)] + E[N\, \Psi(x,\tau)] - E[G\,(x,\tau)]\}$$

$$= 0 \quad (10)$$

We define the nonlinear operator

$$N[\emptyset(x,\tau;q)] = E[\emptyset(x,\tau;q)] - \sum_{k=0}^{m-1} z^{2+k}\, \Psi^{(k)}(x,0)$$
$$+z^{\alpha}\{E[R\emptyset(x,\tau;q)] + E[N\emptyset(x,\tau;q)] - E[G(x,\tau)]\} \quad (11)$$

where $q \in [0,1]$ and $\emptyset(x,\tau;q)$ is a real function of x, τ *and* q
the so-called zero-order deformation equation of (11) has the form

$$(1-q)E[\emptyset(x,\tau;q) - \Psi_0(x,\tau)]$$
$$= qhH(x,\tau)N[\emptyset(x,\tau;q)] \quad (12)$$

where $q \in [0,1]$ is the embedding parameter, $H(x,\tau)$ denotes a nonzero auxiliary function, $h \neq 0$ is an auxiliary parameter.
$\Psi_0(x,\tau)$ is an initial guess of $\Psi(x,\tau)$ and $\emptyset(x,\tau;q)$ is an unknown function.
Obviously, when the parameter $q = 0$ and $q = 1$, it holds

$$\emptyset(x,\tau;0) = \Psi_0(x,\tau)\,, \emptyset(x,\tau;1)$$
$$= \Psi(x,\tau) \quad (13)$$

respectively. Thus as q increases from 0 to 1
the solution $\emptyset(x,\tau;q)$ variesi from ithe initial guess $\Psi_0(x,\tau)$ to the solution $\Psi(x,\tau)$.
Expanding $\emptyset(x,\tau;q)$ in Taylors serie's with respect to q, we have

$$\emptyset(x,\tau;q) = \Psi_0(x,\tau)$$
$$+\sum_{m=1}^{\infty} \Psi_m(x,\tau)\, q^m \quad (14)$$

Where

$$\Psi_m(x,\tau)$$
$$= \frac{1}{m!}\frac{\partial^m \emptyset(x,\tau;q)}{\partial q^m}\,\Big|_{q=0} \quad (15)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are properly chosen.
The series (14) converges at $q = 1$, then we has

$$\Psi(x,\tau) \;= \Psi_0(x,\tau)$$
$$+\sum_{m=1}^{\infty} \Psi_m(x,\tau) \quad (16)$$

which must be one of the solution of the original nonlinear equations.
According to the definition (16), the governing equation can be deduced from the zero-order deformation (12)
Define the vectors

$$\vec{\Psi}_m(x,\tau)$$
$$= \{\Psi_0(x,\tau), \Psi_1(x,\tau), \dots, \Psi_m(x,\tau)\} \quad (17)$$

Differentiating the zero order deformation equation (12) m-times with respect to q and then dividing by m! and finally setting q=0 we get the following $m^{th} -$ order deformation equation :

$$E[\Psi_m(x,\tau) - x_m\Psi_{m-1}(x,\tau)]$$
$$= hH(x,\tau)R_m(\vec{\Psi}_{m-1}(x,\tau)) \quad (18)$$

Applying the inverse Elzaki transform, we have

$$\Psi_m(x,\tau) = x_m\Psi_{m-1}(x,\tau)$$
$$+E^{-1}\left[hH(x,\tau)R_m\left(\vec{\Psi}_{m-1}(x,\tau)\right)\right], \quad (19)$$

where

$$R_m(\vec{\Psi}_{m-1})$$
$$= \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\emptyset(x,\tau;q)]}{\partial q^{m-1}}\,\Big|_{q=0} \quad (20)$$

and $X_m = \begin{cases} 0 & , x \leq 1 \\ 1 & , x > 1 \end{cases}$ (21)

In this way, it is easily to obtain $\Psi_m(x,\tau)$ for mi ≥ 1, iat m^{th} - order, $h = -1$, we have

$$\Psi(x,\tau) = \sum_{m=0}^{\infty} \Psi_m(x,\tau) \quad (22)$$

IV. Applications of (EHAM)

Example : consider the following fractional Schrodinger equation in (EHAM).

$$iD_\tau^\alpha \Psi + \Psi_{xx} = 0 \quad , \quad 0 < \alpha \leq 1 \quad (23)$$

with the initial condition

$$\Psi(x,0) = \sin(x) \quad (24)$$

Multiplying Eq.(1) by (-i) so we have Eq.(1) as follows

$$D_\tau^\alpha \Psi - i\Psi_{xx} = 0 \quad (25)$$

Applying Elzaki transform on both sides in Eq.(3) and after using the differentiation property of Elzaki transform for fractional derivative we get.

$$\frac{E(\Psi)}{z^\alpha} - \frac{\Psi(x,0)}{z^{\alpha-2}} - iE[\Psi_{xx}] = 0 \quad (26)$$

On simplifying and using the Eq.(2) we have

$$E(\Psi) - z^2 \sin(x) - iz^\alpha E[\Psi_{xx}] = 0 \quad (27)$$

we now define a nonlinear operator as :

$$N[\phi(x,\tau)] = E[\phi(x,\tau)] - z^2 \sin(x) - iz^\alpha E\left[\left(\phi(x,\tau)\right)_{xx}\right] \quad (28)$$

And thus

$$R_m(\vec{\Psi}_{m-1}) = E(\Psi_{m-1}) - (1 - x_m)z^2 \sin(x) - iz^\alpha E[(\Psi_{m-1})_{xx}] \quad (29)$$

The m^{th}-order deformation Eq. is

$$E[\Psi_m - x_m\Psi_{m-1}] = hH(x,\tau)R_m(\vec{\Psi}_{m-1}) \quad (30)$$

Applying the invers Elzaki we have.

$$\Psi_m = x_m\Psi_{m-1} + hE^{-1}\left[H(x,\tau)R_m(\vec{\Psi}_{m-1})\right] \quad (31)$$

Solving above Eq.(9) for m=1,2,… and choosing $H(x,\tau) = 1$

Let us take the initial condition.

$$\Psi_0 = \sin(x) \quad (32)$$

$$\begin{aligned} \Psi_1 &= x_1\Psi_0 + hE^{-1}\left[R_1(\vec{\Psi_0})\right] \\ &= (0)(\sin(x)) + hE^{-1}[E(\Psi_0) - (1-0)z^2 \sin(x) - iz^\alpha E((\Psi_0)_{xx})] \\ &= hE^{-1}[z^2 \sin(x) - z^2 \sin(x) - iz^\alpha E(-\sin(x))] \\ &= hE^{-1}[iz^{\alpha+2} \sin(x)] \\ &= \frac{ih\tau^\alpha \sin(x)}{\Gamma(\alpha+1)} \end{aligned} \quad (33)$$

$$\begin{aligned} \Psi_2 &= x_2\Psi_1 + hE^{-1}\left[R_2(\vec{\Psi_1})\right] \\ &= (1)\left(\frac{ih\tau^\alpha \sin(x)}{\Gamma(\alpha+1)}\right) + hE^{-1}\left[E(\Psi_1) - (1-1)z^2 \sin(x) - iz^\alpha E[(\Psi_1)_{xx}]\right] \\ &= \frac{ih\tau^\alpha \sin(x)}{\Gamma(\alpha+1)} + hE^{-1}\left[ihz^{\alpha+2} \sin(x) - iz^\alpha E\left[\frac{-ih\tau^\alpha \sin(x)}{\Gamma(\alpha+1)}\right]\right] \\ &= \frac{ih\tau^\alpha \sin(x)}{\Gamma(\alpha+1)} + hE^{-1}[ihz^{\alpha+2} \sin(x) - hz^{2\alpha+2} \sin(x)] \\ &= \frac{ih\tau^\alpha \sin(x)}{\Gamma(\alpha+1)} + \frac{ih^2\tau^\alpha \sin(x)}{\Gamma(\alpha+1)} - \frac{h^2\tau^{2\alpha} \sin(x)}{\Gamma(2\alpha+1)} \end{aligned} \quad (34)$$

$\vdots$

And so on

Then we have

$$\Psi(x,\tau) = \Psi_0 + \Psi_1 + \Psi_2 + \cdots$$

substitute h=-1 to obtain $\Psi_1, \Psi_2, \cdots$

$$\Psi_1 = \frac{-i\tau^{\alpha}\sin(x)}{\Gamma(\alpha+1)}$$

$$\Psi_2 = \frac{-\tau^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)}$$

$$\vdots$$

Then

$$\Psi(x,\tau) = \sin(x) - \frac{i\tau^{\alpha}\sin(x)}{\Gamma(\alpha+1)} - \frac{\tau^{2\alpha}\sin(x)}{\Gamma(2\alpha+1)} + \cdots \qquad (35)$$

put $\alpha = 1$ to obtain the exact solution

$$\Psi(x,\tau) = \sin(x)\, e^{-i\tau} \qquad (36)$$

In Figure 1, we plot the graph of the exact and approximate solutions for Eq.(23) when $\alpha = 0.9, 0.95, 1$. In Figure 2, 3D surface solution for (23) when $\alpha = 0.9, 0.95, 1$.

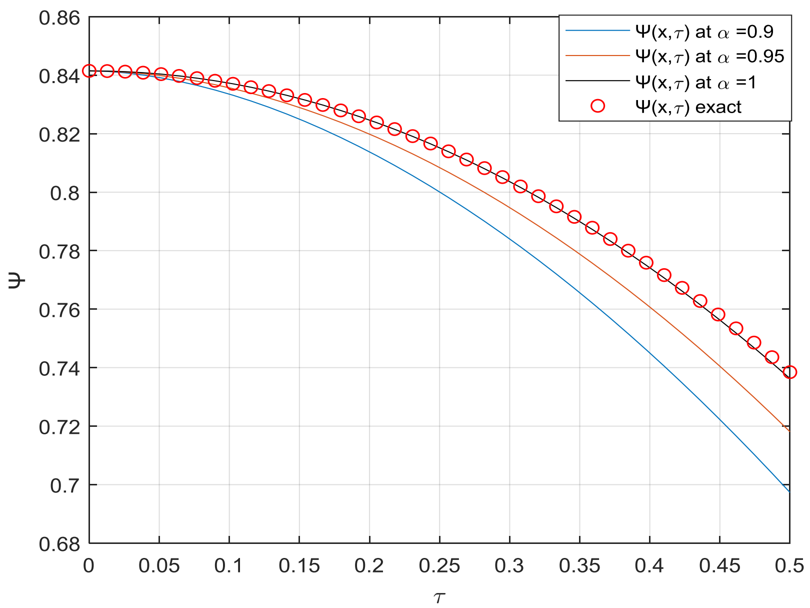

Figure 1: Plots of the exact and approximate solution $\Psi(x,\tau)$ for different values of α with fixed value x.

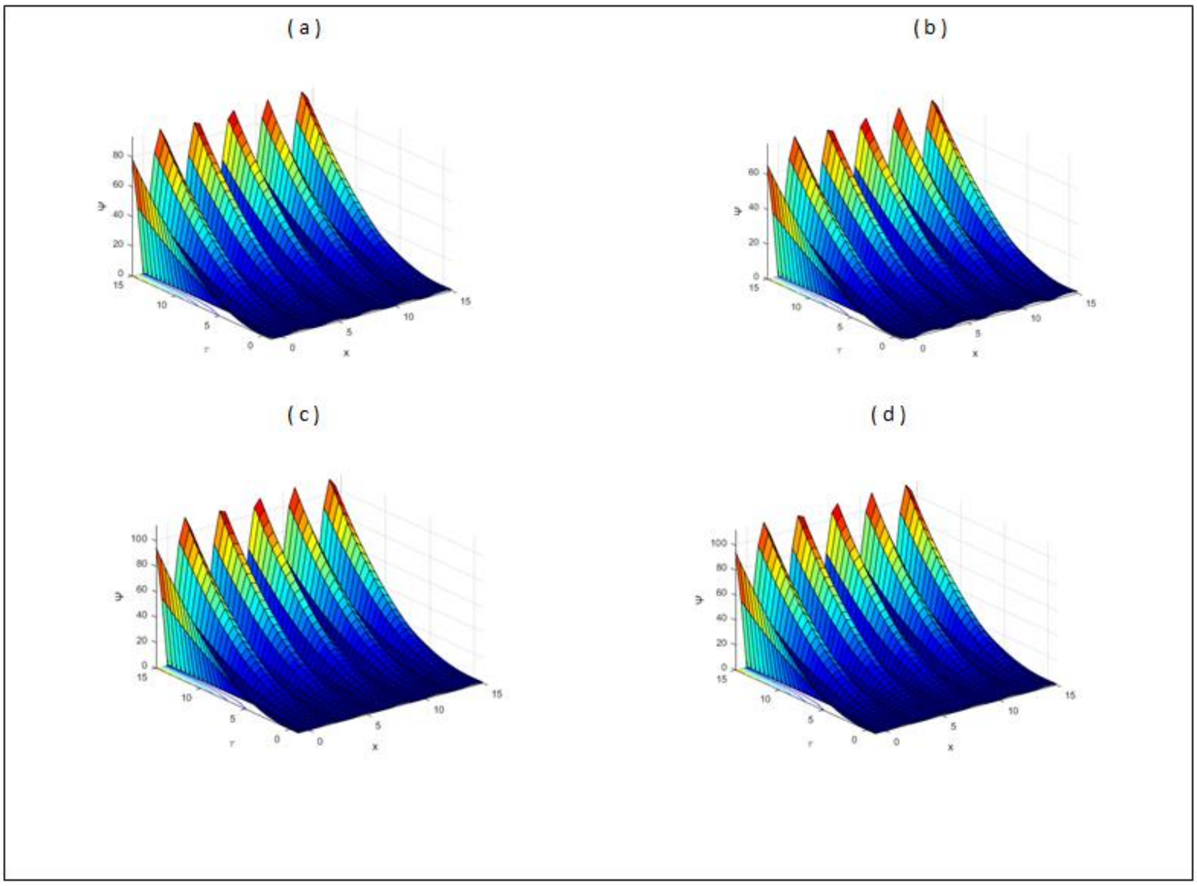

Figure 2: The surface graph of the approximate solution $\Psi(\mathrm{x},\tau)$ of (23): (a) $\Psi(\mathrm{x},\tau)$ when $\alpha = 0.9$, (b) $\Psi(\mathrm{x},\tau)$ when $\alpha = 0.95$, (c) $\Psi(\mathrm{x},\tau)$ when $\alpha = 1$, (d) $\Psi(\mathrm{x},\tau)$ exact solution.

V. Conclusions

This work has produced the approximate analytical solutions of the linear Fractional Schrodinger by using CF D and Elzaki Homotopy Analysis Method PDE.The solutions that were found had the shape of infinite power series, which have a closed form.We can conclud from the results that this method is an effective mathematical tool for solving fractional PDEs. It can also be used to get an approximative solution to other problems.

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