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Review Paper

On The Construction of a Newton-Like Iterative Scheme for Solving Non-Linear Algebraic Equations with Third Order Convergence

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Abstract: *In this work, we propose an algorithm for finding simple roots of non-linear algebraic equations based on Newton Raphson's method which is a non-coupled variant of He's formula [1]. The new iterative scheme is formulated by achieving a quadratic approximation of the Taylor's polynomial by neglecting its cubic and higher order terms. The efficiency of the new numerical formula is shown by solving some examples with well-known classical methods in literature. It is seen that while the traditional Newton's method is of quadratic convergence, this presented method ensures third-order convergence.*

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I. INTRODUCTION

It is well known that a wide class of problems which arises in several branches of pure and applied science can be studied in the general framework of nonlinear equations $f(x)=0$. Since it is not always possible to obtain its exact solution by usual algebraic process, therefore numerical iterative methods are often used to obtain the approximate solution of such problems.

We do know that the Secant method as a numerical scheme for finding the roots of non-linear algebraic equations uses two initial values which should ideally be chosen to lie close to the true root. The Fixed-point Iterative scheme transforms a non-linear function $f(x)$ into a new function $g(x)$ such that for each iteration, we have $x_k = g(x_{k-1})$, where $k = 1,2,3,4,\cdots$ It is known that, for a given function $f(x)=0$, there may be many equivalent fixed point problems $x=g(x)$ with different choices of $g(x)$. Newton's method uses only one initial value, and it requires the derivative of the non-linear equation in consideration. Steffensen's iteration method is a replica of Newton's method, but instead of the derivative function, we have a certain function $g(x)$ given by

$$g(x)=\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}.$$

Amongst these methods in literature, the classical Newton's method is known to be most efficient with a higher convergence rate han the other traditional methods of finding the simple roots of non-linear algebraic equations. Motivated by this, we seek to develop a Newton-based method which is a variant of He's algorithm [1] which converges faster than Newton's method (improves on the order of convergence of Newton's formula), and in turn, the above mentioned traditional methods, despite their different computational requirements. It is necessary to mention here that the functions considered in this study are limited to univariate functions that possess simple roots.

II. DERIVATION OF METHODS OF SOLUTION

Newton, Secant, Fixed-point and Stenffensen methods have their corresponding formulas used in computing iterations, which formulas have different procedures of derivation. Newton's method is derived from Taylor's series and the Secant and Steffensen methods are special cases of Newton's method. The derivation of the formula for the fixed-point iterative scheme does not require the use of Taylor's series.

The problem we want to solve using these methods is to find the simple roots of the non-linear algebraic equation

$$f(x) = 0, \quad f : I \to R, I \subset R. \qquad (*)$$

Let $\alpha \in R$ be an isolated root of (*), i.e. there exists an interval $[a,b]$ containing α. So $f(\alpha) = 0$. The numerical solutions of (*) are found using the following theorem:

Theorem 2.1 (First Bolzano-Cauchy Theorem) If the function $f : [a, b] \to R$ is continuous on *[a,b]* and $f(a)f(b) < 0$, then there exists at least one $c \in (a, b)$ such that $f(c) = 0$.

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Geometrically, this is obvious. If a continuous curve passes from one side of the *Ox*–axis to the other, having opposite signs at the endpoints of the interval, then it has to intersect the axis at least once. We impose the following conditions on f:

i) The function *f* and its derivatives f' and f'' are continuous on *[a,b]*.

ii) The values of the function at the endpoints of the interval have opposite signs, $f(a)f(b) < 0$.

iii) The derivatives f' and f'' have a constant sign on *[a,b]*.

iv) The function *f* and its second derivative f'' have the same sign at one of the endpoints of the interval *[a,b]*.

Conditions i) and ii) guarantee the existence of the solution α of the equation (*) (see Theorem 2.1). By condition iii), since f' has constant positive or negative sign on the entire interval *[a,b]*, it follows that *f* either increases or decreases, and so will be zero only once, meaning that the root α is isolated. It is easy to see that these conditions are always satisfied by algebraic equations.

Newton's Method: Newton Raphson's method is the most popular method for finding the roots of non-linear equations. A function can be approximated by its tangent line and starts with an initial guess that is close to the root. The basic difference between Newton's method and other methods discussed here is that only one initial guess is required. Newton's Method is efficient if the guess is sufficiently close to the root. On the other hand, Newton's method converges slowly or may diverge if initial guess is not close to the root. One of the advantages of Newton method is that it converges fast, if it converges. Another advantage of Newton Raphson's method is that it involves just a functional and first derivative evaluation to obtain the root.

From Taylor's series of the function $f(x)$ about a value $x = x_0$, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + f'''(x_0)\frac{(x - x_0)^3}{3!} + \cdots.$$

As the function approaches a root, higher-order terms of the Taylor's polynomial will approach zero. Therefore, we can consider the higher-order terms trivial and neglect them without any loss of generality. By truncating the series after the second term, we obtain the first linear approximation of the root of the considered equation $f(x) = 0$ given as $f(x) = 0 \approx f(x_0) + f'(x_0)(x - x_0)$.

This implies that

$$xf'(x_0) = x_0 f'(x_0) - f(x_0).$$

By solving further for $x = x_1$, we have

$$x_1 f'(x_0) = x_0 f'(x_0) - f(x_0).$$

On dividing all through by $f'(x_0)$, we have,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

called the first approximation. The second, third, and fourth approximation to the root will therefore be given by:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

and

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}.$$

This iterative procedure can be generalized by writing the following iterative equation

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \quad (1)$$

called the Newton iterative method for non-linear equation, where i represents the number of iterations [2].

Secant Method: Secant method is a special case of Newton's method where the derivative $f'(x_i)$ is replaced by

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}. \quad (2)$$

From Newton method we have that

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_1)}.$$

By replacing the $f'(x_i)$ in Newton method's formula with equation (2), we have

$$x_{i+1} = x_i - \frac{f(x_i)}{\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}}.$$

which yields

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})},$$

called the Secant formula for solving non-linear equations [3].

Fixed-Point Iteration Method: Given a function $g : R \to R$, a value x such that $x = g(x)$ is called a fixed-point of the function g, since x is unchanged when g is applied to it. Fixed-point problems often arise directly in practice and they are important because a nonlinear equation can often be recast as a fixed-point problem for a related nonlinear function. Indeed, many iterative algorithms for solving nonlinear equations are based on iteration schemes of the form $x_{k+1} = g(x_k)$, where g is a suitably chosen function whose fixed-points are solutions for $f(x) = 0$. Such a scheme is called Fixed-point iteration or sometimes functional iteration, since the function g is applied repeatedly to an initial starting value x_0.

For a given equation $f(x) = 0$, there may be many equivalent fixed point problems $x = g(x)$ with different choices for the function g. But not all fixed-point formulations are equally useful in deriving an iteration scheme for solving a given nonlinear equation. The resulting iteration scheme may differ not only in their convergence rates but also in whether they converge at all ([2], [4], [5]).

Steffensen Method: Steffensen method is also a special case of Newton's method, where the derivative $f'(x_i)$ is replaced by a function $g(x_i)$, where

$$g(x)=\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}. \tag{4}$$

From Newton's method we have that

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}.$$

By replacing the $f'(x_i)$ in the formula for Newton's method with $g(x)$ in equation (3), we have

$$x_{i+1}=x_i-\frac{f(x_i)}{\dfrac{f(x_i+f(x_i))-f(x_i)}{f(x_i)}},$$

which yields

$$x_{i+1}=x_i-\frac{f^2(x_i)}{f(x_i+f(x_i))-f(x_i)}.$$

This is called the Steffensen formula for solving non-linear equations [6].

III. THE PROPOSED METHOD

To illustrate its basic idea, we consider the following nonlinear algebraic equation:

$f(x)=0,\ f:I\to R, I\subset R,$ where $I\equiv[a,b]$.

From Taylor's series of the function $f(x)$ about a value $x=x_0$, we have

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2!}+f'''(x_0)\frac{(x-x_0)^3}{3!}+\dots, \tag{5}$$

where the center of expansion x_0 of $f(x)$ is taken as the initial approximation.

He [1], in his work, represented this Taylor's series as a coupled system

$$f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2!}+f'''(x_0)\frac{(x-x_0)^3}{3!}+g(x)=0,$$

where

$$g(x)=f(x)-f(x_0)-f'(x_0)(x-x_0)-f''(x_0)\frac{(x-x_0)^2}{2!}.$$

Now, in this work, similar to what Newton did in deriving his iteration scheme, we neglect $g(x)$, the fourth term and higher order terms of the Taylor's polynomial since they approach zero as the function approaches a root to obtain

$$f(x)\approx f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2!}. \tag{6}$$

This implies that

$$f\left(x_0\right)+f'\left(x_0\right)\left(x-x_0\right)+f''\left(x_0\right)\frac{\left(x-x_0\right)^2}{2!}=0 \qquad (7)$$

By solving further for $x=x_1$ we have

$$f\left(x_0\right)+f'\left(x_0\right)\left(x-x_0\right)+f''\left(x_0\right)\frac{\left(x-x_0\right)^2}{2!}=0 \qquad (8)$$

$$f\left(x_0\right)+f'\left(x_0\right)\left(x_1-x_0\right)+f''\left(x_0\right)\frac{\left[x_1{}^2-2x_1x_0+x_0{}^2\right]}{2!}=0 \qquad (9)$$

$$f\left(x_0\right)+x_1f'\left(x_0\right)-x_0f'\left(x_0\right)+f''\left(x_0\right)\frac{x_1^2}{2!}+f''\left(x_0\right)\frac{[-2x_1x_0]}{2!}+f''\left(x_0\right)\frac{x_0^2}{2!}=0$$

Next, we take like terms and have

$$[\frac{1}{2}f''(x_0)]x_1{}^2+[f'\left(x_0\right)-f''\left(x_0\right)x_0]x_1+[f\left(x_0\right)-f'\left(x_0\right)x_0+\frac{1}{2}f''\left(x_0\right)x_0{}^2]. \qquad (10)$$

This can be written as

$$Ax_1^2+Bx_1+C=0.$$

The above is a quadratic equation in x_1 which can be solved using the iterative formula

$$x_{n+1}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}, \qquad (11)$$

where

A= $\frac{1}{2}f''\left(x_n\right)$,

B= $f'\left(x_n\right)-f''\left(x_n\right)x_n$,

C= $f\left(x_n\right)-f'\left(x_n\right)x_n+\frac{1}{2}f''\left(x_n\right)x_n{}^2$.

Numerical Applications: In this section we consider some numerical examples to demonstrate the performance of the proposed iterative method. In the subsequent numerical simulations, all computing are performed by using MATLAB codes with numeric data written to 10 places of decimal and the stopping criterion for the computer program being $\left|x_{n+1}-x_n\right|\le\varepsilon$, where ε is chosen as small as 10^{-10}.

Problem 1: $f\left(x\right)=x^3-e^{-x}$

Problem 2: $f\left(x\right)=x^3+4x^2-10$.

The starting point used for the all the methods considered is $x_0=0.5$, except for the secant method with a pair of starting points as $x_0=0$ and $x_1=1$.

Results of the computations are given in the following tables.

Iteration (k)	Fixed-point	Secant Method	Steffensen Method	Newton's Method	Improved Method
1	0.8464817249	1.0000000000	0.963522241	0.8549721905	0.7838782597
2	0.7541525768	0.6126998368	0.8573607094	0.7787105282	0.7728823242
3	0.7777235188	0.7406558829	0.7922059588	0.7729142691	0.7728829591
4	0.7716369027	0.7780275107	0.7739836212	0.7728829601	-
5	0.7732040444	0.7727276889	0.7728866063	0.7728829591	-
6	0.7728002431	0.7728822207	0.7728829592	-	-
7	0.7729042694	0.7728829593	0.7728829591	-	-
8	0.7728774691	0.7728829591	-	-	-
9	0.7728843735	-	-	-	-
10	0.7728825948	-	-	-	-
11	0.7728830530	-	-	-	-
12	0.7728830350	-	-	-	-
13	0.7728829654	-	-	-	-
14	0.7728829575	-	-	-	-
15	0.7728829596	-	-	-	-
16	0.7728829590	-	-	-	-
17	0.7728829592	-	-	-	-
18	0.7728829591	-	-	-	-
19	0.7728829592	-	-	-	-
20	0.7728829591	-	-	-	-

Table 1. Comparison of Results for Problem 1

Iteration (k)	Fixed-point	Secant Method	Steffensen Method	Newton's Method	Improved Method
1	1.4907119850	1.0000000000	0.7557406574	2.3684210526	1.4098611039
2	1.349597105	2.0000000000	1.2360810265	1.6494080730	1.3652246278
3	1.3672306704	1.2631578947	1.8903514634	1.3979914939	1.3652300134
4	1.3649755422	1.3388278388	1.8469840937	1.3657438582	-
5	1.3652623908	1.3666163947	1.8015328661	1.3657438582	-
6	1.3652258941	1.3652300011	1.7538428877	1.3652301428	-
7	1.3652305375	1.3652300134	1.7037960728	1.3652300134	-
8	1.3652299467	-	1.6513731468	-	-
9	1.3652300219	-	1.5967841005	-	-
10	1.3652300123	-	1.5407407490	-	-
11	1.3652300136	-	1.4850090759	-	-
12	1.3652300134	-	1.4333986455	-	-
13	-	-	1.3927739780	-	-
14	-	-	1.3707630173	-	-
15	-	-	1.3654839430	-	-
16	-	-	1.3652305661	-	-
17	-	-	1.3652300134	-	-

Table 2 Comparison of Results for Problem 2

IV. CONVERGENCE ANALYSIS

Definition 4.1 A sequence $\{x_n\}$ generated by an iterative method is said to converge to a root r with order $p \geq 1$ if there exists $c > 0$ such that $e_{n+1} \leq c e_n^p, \forall n \geq n_0$, for some integer $n_0 \geq 0$ and $e_n = |r - x_n|$.

Theorem 4.1 ([7], [8]) Suppose that $g \in C^{p}[a,b]$. If $g^{(k)}(x)=0$ for $k=1,2,3,\cdots p-1$ and $g^{(p)}(x)\neq 0$, then the convergence of the sequence x_n is of order p.

Theorem 4.2. Suppose that c is a root of the equation $f(x)=0$. If $f(x)$ is sufficiently smooth in the neighborhood of c, then the convergence order of Newton's method is 2.

Proof: To analyze the convergence of Newton's method, we let

$$G(x)=x-\frac{f(x)}{f'(x)}.$$

Let c be a simple zero of f and $f(c)=0$ .Then we can easily deduce by using Maple software that

$$G(c)=c,$$

$$G'(c)=0,$$

$$G''(c)=\frac{f''(c)}{f'(c)}, \tag{12}$$

Now, from (12) it can be easily seen that $G''(c)\neq 0$, hence according to Theorem 4.1, the Newton's algorithm has second order convergence.

Theorem 4.3 Suppose that c is a root of the equation $f(x)=0$. If $f(x)$ is sufficiently smooth in the neighborhood of c, then the convergence order of this method is 3.

Proof: To analyze the convergence of the method, let

$$G(x)==\frac{-B\pm\sqrt{B^2-4AC}}{2A},$$

where

A= $\frac{1}{2}f''(x_n)$,

B= $f'(x_n)-f''(x_n)x_n$,

C= $f(x_n)-f'(x_n)x_n+\frac{1}{2}f''(x_n){x_n}^2$.

Let c be a simple zero of f and $f(c)=0$. Then, using Maple software, we have that

$$G(c)=c,$$

$$G'(c)=0,$$

$$G''(c)=0$$

$$G'''(c)=\frac{c^5[(2f^4(c))^7+4f'''^4(c)f''^2(c)c+\frac{3}{8}f'(c)(8f'''f''^3f'^5(c)c^3+2f''(c))]}{(f''(c))^4},$$

(13)

Now, from (13) it can be easily seen that $G'''(c)\neq 0$, hence according to Theorem 4.1, this proposed method has third order convergence.
Error analysis tables are here presented below:

Iteration(k)	**Approximated Value** (x_k)	**Errors** $(e_k = \lvert x_k - x^* \rvert)$
1	0.8549721905	0.08208923135079
2	0.7787105282	0.00582756905079
3	0.7729142691	0.00003130995079
4	0.7728829601	0.00000000095079
5	0.7728829591	0.00000000004921

Table 3: Error Analysis for Newton's Method

Iteration(K)	**Approximated Value** (x_k)	**Errors** $(e_k = \lvert x_k - x^* \rvert)$
1	0.9635922241	0.19070926495079
2	0.8573607094	0.08447775025079
3	0.7922059588	0.01932299965079
4	0.7739836212	0.00110066205079
5	0.7728866063	0.00000364715079
6	0.7728829592	0.00000000005079
7	0.7728829591	0.00000000004921

Table 4 Error Analysis for Steffensen Method

Iteration(k)	**Approximated Value** (x_k)	**Errors** $(e_k = \lvert x_k - x^* \rvert)$
1	1	0.22711704085079
2	0.6126998368	−0.16018312234921
3	0.7406558829	−0.03222707624921
4	0.7780275107	0.00514455155079
5	0.7727276889	−0.00015527024921
6	0.7728822207	−0.00000073844921
7	0.7728829593	0.00000000015079
8	0.7728829591	0.00000000004921

Table 5 Error Analysis for Secant Method

Iteration $(\mathbf{k})$	**Approximated Value** (x_k)	**Errors** $(e_k = \lvert x_k - x^* \rvert)$
1	0.8464817249	0.07359876575079
2	0.7541525768	−0.01873038234921
3	0.7777235188	0.004848055965079
4	0.7716369027	−0.00124605644921
5	0.7732040444	0.00032108525079
6	0.7728002431	−0.00008271604921
7	0.7729042694	0.00002131025079
8	0.7728774691	−0.00000549004921
9	0.7728843735	0.00000141435079
10	0.7728825948	−0.00000036434921
11	0.7728830530	0.00000009385079
12	0.7728829350	−0.00000002414921
13	0.7728829654	0.00000000625079
14	0.7728829575	−0.00000000164921
15	0.7728829596	0.00000000045079
16	0.7728829590	−0.00000000014921
17	0.7728829592	0.00000000005079
18	0.7728829591	0.00000000004921
19	0.7728829592	0.00000000005079
20	0.7728829591	0.00000000004921

Table 5 Error Analysis for Fixed-point Method

Iteration(k)	**Approximated Value** (x_k)	**Errors** $(e_k = \lvert x_k - x^* \rvert)$
1	0.7838782597	0.01099530055079
2	0.7728823242	−0.00000063494921
3	0.7728829591	−0.00000000004921

Table 6 Error Analysis for the proposed method

V. CONCLUSION

By using some examples, the performance of the five iterative methods are investigated. In problem 1, Fixed-point iteration method converges at the 20th iteration which is quite slow, Secant method at the 8th iteration, Steffensen's method at the 7th iteration, Newton's method at the 5th iteration, the new method is seen to converge at the 3rd iteration. In problem 2, however, Steffensen's method converges at the 17th iteration, while Fixed-point iteration follows closely with convergence at the 12th iteration. Secant method converges at the 7th iteration, Newton's Method at the 6th iteration while the proposed method converges quickly at the 3rd iteration. A variant of He's method is therefore established. It is worthy to note that, for the same function $x^3 - e^{-x}$, He's method with starting points as $x_0 = 0$ and $x_1 = 0.5$ converged at the 5th iteration while the proposed method converges at the 3rd iteration.

Conflict of Interest
The authors declare that there is no conflict of interest.

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