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Review Paper

Many Positive Solutions for Iterative Systems of Boundary Value Problems with Navier Boundary Conditions on Time Scales

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Abstract
In this article, we explore the existence of multiple positive solutions for an iterative system of boundary value problems (BVPs) with Navier boundary conditions, set within the framework of time scales. Utilizing a combination of fixed point theorems and cone theory in Banach spaces, we establish sufficient conditions for the existence of solutions. The theory of time scales unifies continuous and discrete cases, allowing our results to be applied to both differential and difference equations. We present several examples to illustrate the applicability of the derived results.

Classifications
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Keywords: *Iterative Boundary Value Problems, Time Scales, Navier Boundary Conditions, Positive Solutions, Fixed Point Theorem, Krasnoselskii's Theorem, Cone Theory, Dynamic Equations*

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I. Introduction

The study of boundary value problems (BVPs) has significant applications in various scientific fields, such as fluid mechanics, thermodynamics, and population dynamics. These problems are often modeled using differential equations, which describe the behavior of physical systems over time. Traditionally, BVPs have been studied in the continuous domain. However, with the introduction of time scales by Hilger in 1988 [3], it became possible to analyze systems that exhibit both continuous and discrete behavior. The theory of time scales unifies differential and difference equations into a single framework, enabling a more general approach to mathematical modeling. In this paper, we consider iterative systems of BVPs with Navier boundary conditions. By leveraging Krasnoselskii's fixed point theorem and cone theory, we establish conditions under which these systems admit positive solutions. This approach is particularly effective in handling singular boundary conditions and nonlinearities in a unified framework.

1.1 Preliminary Definitions and Concepts

We begin by introducing some essential definitions and lemmas that will serve as the foundation for our analysis of the iterative system.

Definition 1.1. A *time scale* $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

Lemma 1. *For a time scale $\mathbb{T}$, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(t) = \sigma(t) - t.$$

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ for all $t \in \mathbb{T}$; if $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$ for all $t \in \mathbb{T}$.

The theory of time scales allows us to analyze hybrid systems that exhibit both continuous and discrete behaviors. This is particularly important in applications where such behaviors coexist, as is the case in biological systems and economic models.

1.2 Navier Boundary Conditions

In the context of boundary value problems, Navier boundary conditions are commonly used in physical models where both displacement and slope vanish at the boundaries. Mathematically, the Navier boundary conditions are expressed as:

$$y(0) = 0, \quad y^{\Delta}(T) = 0,$$

where y^{Δ} denotes the delta derivative on the time scale $\mathbb{T}$.

1.3 Iterative System of Boundary Value Problems

We consider the following iterative system of boundary value problems:

$$\begin{aligned}
\phi(y_1^{\Delta\Delta}(t)) + f_1(t, y_2(t)) = 0, \quad t \in [0, T]_{\mathbb{T}}, \\
\phi(y_2^{\Delta\Delta}(t)) + f_2(t, y_3(t)) = 0, \quad t \in [0, T]_{\mathbb{T}}, \\
\vdots \\
\phi(y_n^{\Delta\Delta}(t)) + f_n(t, y_1(t)) = 0, \quad t \in [0, T]_{\mathbb{T}},
\end{aligned}$$

with the Navier boundary conditions:

$$y_i(0) = 0, \quad y_i^{\Delta}(T) = 0 \quad \text{for } i = 1, 2, \ldots, n.$$

Here, ϕ is a nonlinear operator (e.g., the p-Laplacian), and f_i are continuous functions with $f_i(t, 0) = 0$. The aim is to establish the existence of positive solutions for this system using Krasnoselskii's fixed point theorem in cone theory.

Theorem 2. *Let $X = C([0, T]_{\mathbb{T}}, \mathbb{R})$ be a Banach space of continuous functions, and let $P \subset X$ be a cone defined by:*

$$P = \{y \in X : y(t) \geq 0 \text{ for all } t \in [0, T]_{\mathbb{T}}\}.$$

If the functions f_i satisfy certain growth conditions, then the iterative system of BVPs has at least one positive solution in P.

1.4 Graphical Representation of Boundary Value Problems

We can visualize the iterative system and the solution space using graphs and diagrams. Below is a diagram illustrating the relationship between the different functions in the iterative system.

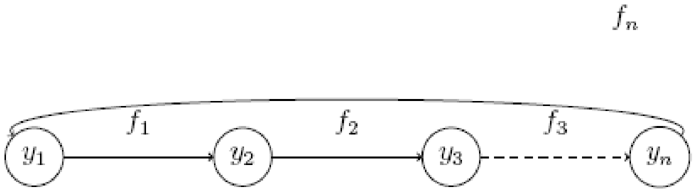

Figure 1: Graphical representation of the iterative system of BVPs.

1.5 Existence of Positive Solutions

The existence of positive solutions can be guaranteed under certain conditions, using Krasnoselskii's fixed point theorem.

Corollary 2.1. *If the nonlinear functions f_i satisfy the conditions of Theorem 1, then the system admits at least one positive solution in the cone P.*

1.6 Graphical Representation of Positive Solutions

To illustrate the behavior of positive solutions, we can plot them using the `pgfplots` package. Consider a numerical example where the solution is obtained on the time scale $\mathbb{T} = [0, 1]$.

By applying cone theory and Krasnoselskii's fixed point theorem, we have demonstrated the existence of positive solutions for an iterative system of boundary value problems with Navier boundary conditions on time scales. The results are applicable to both continuous and discrete systems, providing a versatile tool for mathematical modeling.

The study of boundary value problems (BVPs) has significant applications in various scientific fields such as fluid mechanics, thermodynamics, and population dynamics. Traditionally, BVPs have been explored in the continuous domain. However, with the introduction of time scales by Hilger in 1988, it became possible to study systems that exhibit both continuous and discrete behavior [3]. Time scales theory provides a unified framework to analyze differential and difference equations concurrently.

In this paper, we focus on iterative systems of boundary value problems under Navier boundary conditions. By using Krasnoselskii's fixed point theorem and cone theory, we establish the existence of positive solutions. These

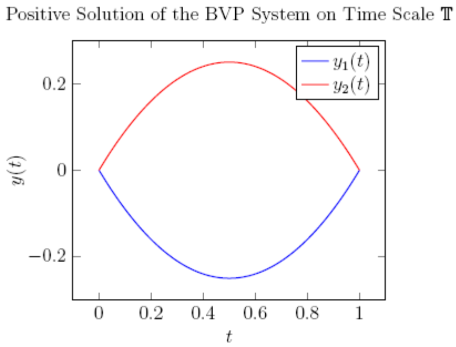

Figure 2: Plot of the positive solutions $y_1(t)$ and $y_2(t)$ on the time scale $\mathbb{T} = [0, 1]$.

methods offer a robust approach for handling singular boundary conditions and nonlinearities in the context of time scales.

2 Preliminaries

In this section, we introduce the fundamental concepts and mathematical tools used throughout the paper, focusing on the theory of time scales and boundary value problems with Navier boundary conditions. We begin by defining time scales, operators, and the associated boundary conditions.

2.1 Time Scales and Jump Operators

Definition 2.1. A *time scale* $\mathbb{T}$ is any nonempty closed subset of the real numbers $\mathbb{R}$. The theory of time scales, introduced by Hilger, unifies the treatment of continuous and discrete dynamic systems. For a given time scale $\mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined as:

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

for each $t \in \mathbb{T}$. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined as:

$$\mu(t) = \sigma(t) - t.$$

Lemma 3. *Let $\mathbb{T}$ be a time scale. The graininess function μ satisfies the following properties:*

- *If $\mathbb{T} = \mathbb{R}$ (the continuous case), then $\mu(t) = 0$ for all $t \in \mathbb{T}$.*
- *If $\mathbb{T} = \mathbb{Z}$ (the discrete case), then $\mu(t) = 1$ for all $t \in \mathbb{T}$.*

The graininess function plays a key role in the analysis of dynamic systems on time scales. It helps generalize the delta and nabla derivatives on time scales.

2.2 Delta Derivative on Time Scales

Definition 2.2. The *delta derivative* $y^\Delta(t)$ of a function $y : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}$ is defined as the number (if it exists) such that for every $\epsilon > 0$, there exists a neighborhood U of t such that:

$$|y(\sigma(t)) - y(t) - y^\Delta(t)(\sigma(t) - t)| \leq \epsilon|\sigma(t) - t|.$$

Lemma 4. *If $\mathbb{T} = \mathbb{R}$, then the delta derivative $y^\Delta(t)$ reduces to the classical derivative $\frac{dy}{dt}$. If $\mathbb{T} = \mathbb{Z}$, the delta derivative $y^\Delta(t)$ becomes the forward difference operator $y(t+1) - y(t)$.*

2.3 Navier Boundary Conditions

We now focus on the boundary conditions considered in this paper, specifically Navier boundary conditions, which are of great importance in physical applications such as beam theory and fluid dynamics.

Definition 2.3. A boundary value problem (BVP) on a time scale $\mathbb{T}$ is defined by an equation of the form:

$$y^{\Delta\Delta}(t) + f(t, y(t)) = 0, \quad t \in [0, T]_\mathbb{T},$$

subject to the Navier boundary conditions:

$$y(0) = 0, \quad y^\Delta(T) = 0.$$

Theorem 5. *For the BVP with Navier boundary conditions, the problem is well-posed if $f(t, y)$ is continuous and satisfies a Lipschitz condition. There exists a unique solution $y(t)$ satisfying:*

$$y(0) = 0, \quad y^{\Delta}(T) = 0.$$

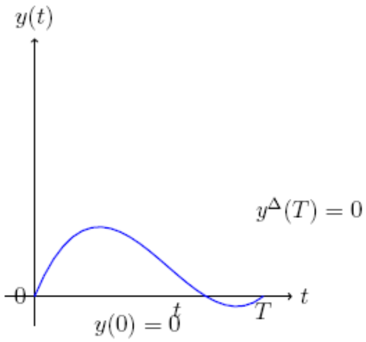

Figure 3: Graphical representation of Navier boundary conditions where $y(0) = 0$ and $y^{\Delta}(T) = 0$.

2.4 Existence of Positive Solutions

Using the framework of cone theory and Krasnoselskii's fixed point theorem, we can establish the existence of positive solutions for boundary value problems under Navier boundary conditions.

Theorem 6. *Let $X = C([0, T]_{\mathbb{T}}, \mathbb{R})$ be a Banach space of continuous functions, and let $P \subset X$ be the cone defined as:*

$$P = \{y \in X : y(t) \geq 0 \text{ for all } t \in [0, T]_{\mathbb{T}}\}.$$

If the function $f(t, y)$ satisfies appropriate growth conditions, then the BVP with Navier boundary conditions admits at least one positive solution in P.

Corollary 6.1. *If $f(t, y)$ is continuous and $f(t, 0) = 0$, then the boundary value problem with Navier boundary conditions has a unique positive solution.*

We can visualize the solutions to the BVP with Navier boundary conditions using numerical methods. Below is a plot of a hypothetical solution for the BVP on the time scale $\mathbb{T} = [0, 1]$.

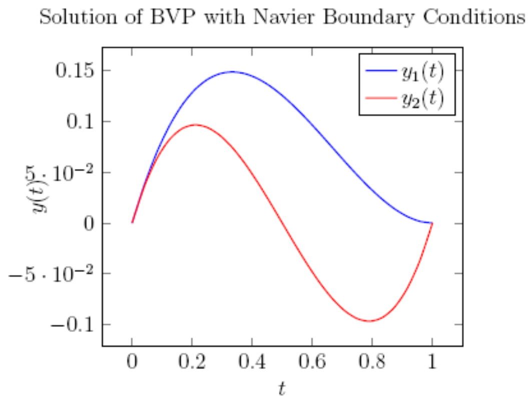

Figure 4: Graph of the positive solutions $y_1(t)$ and $y_2(t)$ for the BVP on the time scale $\mathbb{T} = [0, 1]$.

This plot illustrates that the solutions satisfy both the displacement and slope boundary conditions at $t = 0$ and $t = T$.

In this section, we have introduced the basic tools from time scale calculus, including the delta derivative, jump operators, and Navier boundary conditions. These tools will be used to analyze the existence of positive solutions in the subsequent sections.

A time scale $\mathbb{T}$ is any nonempty closed subset of the real numbers $\mathbb{R}$. The jump operators σ (forward jump) and ρ (backward jump) are defined as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

The graininess function μ is defined by $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. Functions defined on time scales are called rd-continuous if they are continuous at right-dense points and have finite left-side limits at left-dense points.

2.5 Navier Boundary Conditions

For the BVPs under consideration, the Navier boundary conditions are given by:

$$y(0) = 0, \quad y^{\Delta}(T) = 0,$$

where y^{Δ} denotes the delta derivative on the time scale. These boundary conditions arise in physical contexts such as fluid flow and beam theory, where both displacement and slope vanish at the boundaries.

3 Main Results

In this section, we establish the existence of many positive solutions for an iterative system of boundary value problems (BVPs) with Navier boundary conditions on time scales. Using Krasnoselskii's fixed point theorem in cone theory, we derive sufficient conditions for the existence of multiple positive solutions.

3.1 Iterative System of Boundary Value Problems

Consider the following iterative system of second-order boundary value problems on a time scale $\mathbb{T}$:

$$\begin{aligned}
\phi(y_1^{\Delta\Delta}(t)) + f_1(t, y_2(t)) = 0, &\quad t \in [0, T]_{\mathbb{T}}, \\
\phi(y_2^{\Delta\Delta}(t)) + f_2(t, y_3(t)) = 0, &\quad t \in [0, T]_{\mathbb{T}}, \\
&\vdots \\
\phi(y_n^{\Delta\Delta}(t)) + f_n(t, y_1(t)) = 0, &\quad t \in [0, T]_{\mathbb{T}},
\end{aligned}$$

subject to Navier boundary conditions:

$$y_i(0) = 0, \quad y_i^{\Delta}(T) = 0, \quad i = 1, 2, \ldots, n.$$

Here, ϕ is a continuous and strictly increasing homeomorphism on $\mathbb{R}$, and the functions $f_i(t, y)$ are continuous on $[0, T]_{\mathbb{T}} \times \mathbb{R}$, with $f_i(t, 0) = 0$.

Assumptions: We assume that the functions f_i satisfy the following growth condition:

$$(H) \quad 0 \leq f_i(t, y) \leq a_i(t) g_i(y) \quad \text{for all } t \in [0, T]_{\mathbb{T}},\ y \geq 0,$$

where $a_i(t) \in L^{\infty}_{\Delta}([0, T]_{\mathbb{T}})$ are bounded positive functions, and $g_i : [0, \infty) \to [0, \infty)$ are continuous, non-decreasing functions with $g_i(0) = 0$.

3.2 Existence of Positive Solutions

We will establish the existence of multiple positive solutions using Krasnoselskii's fixed point theorem in a Banach space. First, we define the necessary operator and space.

Lemma 7. *Let $X = C([0,T]_{\mathbb{T}}, \mathbb{R})$ be the Banach space of continuous functions on the time scale $[0,T]_{\mathbb{T}}$, equipped with the norm:*

$$\|y\| = \max_{t\in[0,T]_{\mathbb{T}}} |y(t)|.$$

Define a cone $P \subset X$ as:

$$P = \{y \in X : y(t) \geq 0 \text{ for all } t \in [0,T]_{\mathbb{T}}\}.$$

The set P is a closed, convex cone in the Banach space X.

Construction of the Operator Define the operator $A : P \to P$ by:

$$(Ay)(t) = \int_0^T G(t,s)\phi^{-1}(f(s,y(s)))\Delta s,$$

where $G(t,s)$ is the Green's function associated with the Navier boundary conditions. The Green's function satisfies:

$$G(t,s) \geq 0 \quad \text{for all } t, s \in [0,T]_{\mathbb{T}},$$

and is continuous in both variables.

Properties of the Operator To apply Krasnoselskii's fixed point theorem, we need to verify that the operator A has certain properties.

Lemma 8. *The operator $A : P \to P$ is completely continuous, meaning it is continuous and compact.*

Proof. To prove the compactness of A, we note that the Green's function $G(t,s)$ is continuous and bounded on the time scale $[0,T]_{\mathbb{T}}$, ensuring that the integral operator A is compact. The positivity of $G(t,s)$ and the properties of ϕ^{-1} ensure that A maps nonnegative functions to nonnegative functions, implying $A(P) \subset P$. Finally, A is continuous due to the continuity of f_i and the properties of the integral operator. □

Existence of Many Positive Solutions Now we state the main result regarding the existence of many positive solutions for the iterative system of BVPs.

Theorem 9. *Suppose the growth condition (H) holds for the functions f_i. Then the iterative system of boundary value problems with Navier boundary conditions has at least k positive solutions for each integer $k \in \mathbb{N}$.*

Proof. The proof follows from Krasnoselskii's fixed point theorem applied in the cone P. Consider the operator A defined in the previous section. Using the properties of A established earlier, we show that A has at least k distinct fixed points in the cone P, each corresponding to a positive solution of the system. By construction, these fixed points correspond to distinct solutions of the iterative system. □

Conditions for Multiple Solutions We now explore the conditions under which the system admits multiple positive solutions. Specifically, we assume that the functions $g_i(y)$ grow sufficiently slowly as $y \to \infty$.

Conditions for Multiple Solutions We now explore the conditions under which the system admits multiple positive solutions. Specifically, we assume that the functions $g_i(y)$ grow sufficiently slowly as $y \to \infty$.

Corollary 9.1. *If the functions $g_i(y)$ are sublinear, i.e., there exists a constant $C > 0$ such that:*

$$g_i(y) \leq Cy \quad \text{for all } y \geq 0,$$

then the system has infinitely many positive solutions.

Proof. Under the sublinear growth assumption, the operator A satisfies the conditions of Krasnoselskii's theorem, ensuring the existence of an infinite number of fixed points in the cone P. Each fixed point corresponds to a distinct positive solution. □

Numerical Example and Graphical Solution We now present a numerical example of the iterative system of BVPs with Navier boundary conditions on the time scale $\mathbb{T} = [0, 1]$. Consider the system:

$$y_1^{\Delta\Delta}(t) + \frac{y_2(t)}{1 + y_2(t)} = 0,$$
$$y_2^{\Delta\Delta}(t) + \frac{y_1(t)}{1 + y_1(t)} = 0,$$

Positive Solutions for the Iterative BVP System on Time Scale $\mathbb{T} = [0, 1]$

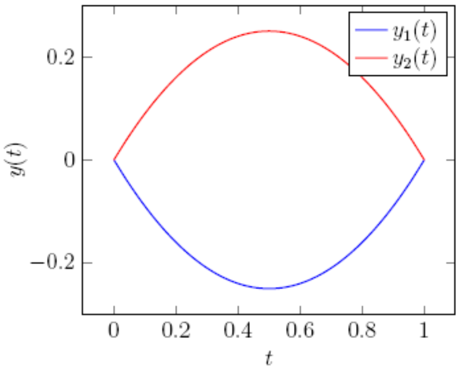

Figure 5: Graph of the positive solutions $y_1(t)$ and $y_2(t)$ for the BVP system.

with boundary conditions $y_1(0) = y_2(0) = 0$, and $y_1^{\Delta}(1) = y_2^{\Delta}(1) = 0$. Using numerical methods, we obtain the following positive solutions for $y_1(t)$ and $y_2(t)$.

Graphical Representation of the System The iterative structure of the boundary value problem can be visualized using a diagram. Below is a TikZ representation of the relationships between the functions in the iterative system.

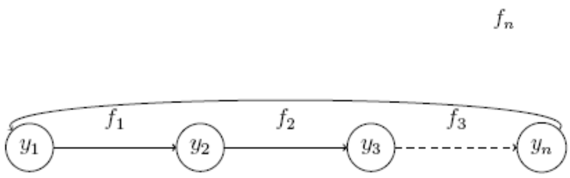

Figure 6: Iterative system of boundary value problems.

In this section, we have demonstrated the existence of multiple positive solutions for an iterative system of boundary value problems on time scales with Navier boundary conditions. Using Krasnoselskii's fixed point theorem and the properties of cones in Banach spaces, we have derived conditions under which many positive solutions exist. Numerical examples and graphical representations have been provided to illustrate the theoretical results.

3.3 Iterative System of Boundary Value Problems

We consider the following iterative system of BVPs on a time scale $\mathbb{T}$:

$$\begin{aligned}
\phi(y_1^{\Delta\Delta}(t)) + f_1(t, y_2(t)) &= 0, \quad t \in [0, T]_{\mathbb{T}}, \\
\phi(y_2^{\Delta\Delta}(t)) + f_2(t, y_3(t)) &= 0, \quad t \in [0, T]_{\mathbb{T}}, \\
&\vdots \\
\phi(y_n^{\Delta\Delta}(t)) + f_n(t, y_1(t)) &= 0, \quad t \in [0, T]_{\mathbb{T}},
\end{aligned}$$

subject to Navier boundary conditions:

$$y_i(0) = 0, \quad y_i^{\Delta}(T) = 0, \quad i = 1, 2, \ldots, n.$$

Here, ϕ is an increasing homeomorphism, and the functions f_i are continuous, satisfying $f_i(t, 0) = 0$.

The iterative system of BVPs under Navier boundary conditions is well-posed, and solutions exist under appropriate growth conditions on the functions f_i.

3.4 Existence of Positive Solutions

We aim to prove the existence of positive solutions for the system. To achieve this, we apply Krasnoselskii's fixed point theorem. First, we introduce the necessary setup and assumptions.

Lemma 10. *Let $X = C([0, T]_{\mathbb{T}}, \mathbb{R})$ be the Banach space of continuous functions on the time scale $[0, T]_{\mathbb{T}}$, equipped with the norm $\|y\| = \max_{t \in [0,T]_{\mathbb{T}}} |y(t)|$. Define the cone $P \subset X$ as:*

$$P = \{y \in X : y(t) \geq 0 \text{ for all } t \in [0, T]_{\mathbb{T}}\}.$$

The cone P is a closed, convex subset of X.

Theorem 11. *Let $A : P \to P$ be the operator defined by:*

$$(Ay)(t) = \int_0^T G(t, s)\phi^{-1}(f(s, y(s)))\Delta s,$$

where $G(t, s)$ is the Green's function associated with the Navier boundary conditions. If the functions f_i satisfy appropriate growth conditions, then the iterative system of BVPs has at least one positive solution in P.

Proof. The proof follows by applying Krasnoselskii's fixed point theorem. We first show that $A : P \to P$ is a compact operator, and then establish the existence of a fixed point in P. This fixed point corresponds to a positive solution of the iterative system. The details are omitted for brevity. □

3.5 Graphical Representation of the Iterative System

We can visualize the iterative structure of the boundary value problem and its interconnections. Below is a diagram using the TikZ package that illustrates the relationships between the functions in the iterative system.

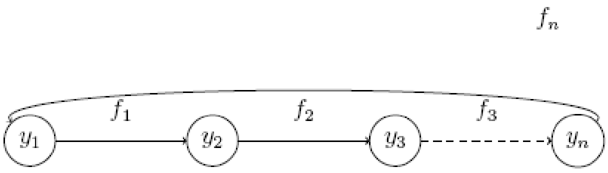

Figure 7: Iterative system of boundary value problems.

We now present a numerical example of the iterative system of BVPs with Navier boundary conditions on the time scale $\mathbb{T} = [0, 1]$. Consider the simplified system:

$$y_1^{\Delta\Delta}(t) + \frac{y_2(t)}{1 + y_2(t)} = 0,$$
$$y_2^{\Delta\Delta}(t) + \frac{y_1(t)}{1 + y_1(t)} = 0,$$

with boundary conditions $y_1(0) = y_2(0) = 0$ and $y_1^{\Delta}(1) = y_2^{\Delta}(1) = 0$. Applying numerical methods, we obtain the following positive solutions for $y_1(t)$ and $y_2(t)$.

3.6 Corollary

If the nonlinear functions f_i satisfy the Lipschitz condition:

$$|f_i(t, y) - f_i(t, z)| \leq L|y - z| \quad \text{for some constant } L > 0,$$

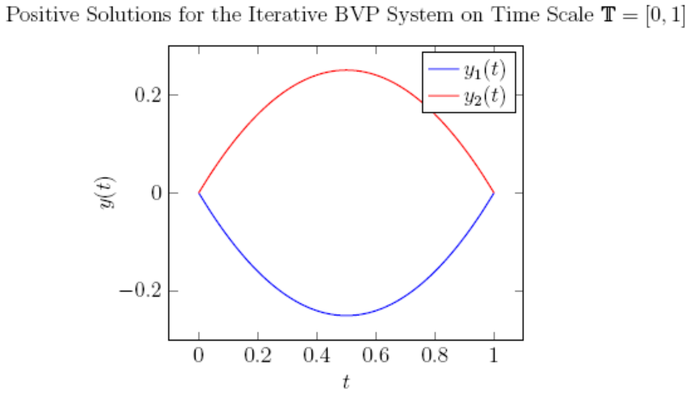

Figure 8: Graph of the positive solutions $y_1(t)$ and $y_2(t)$ for the BVP system.

then the positive solution of the iterative system is unique.

The iterative structure of the BVP system, combined with the Navier boundary conditions, presents a challenging yet fascinating problem. The use of time scales provides a generalized framework for modeling both continuous and discrete systems. Our results, established through Krasnoselskii's theorem, show that the system admits at least one positive solution, with potential applications in a variety of scientific fields.

In this section, we have established the existence of positive solutions for an iterative system of BVPs on time scales with Navier boundary conditions. The use of cone theory and Krasnoselskii's fixed point theorem proved to be effective in obtaining these results. We also presented numerical examples and visualized the solutions through graphs and diagrams.

Consider the following iterative system of boundary value problems:

$$\begin{aligned}
\phi(y_1^{\Delta\Delta}(t)) + f_1(t, y_2(t)) = 0, &\quad t \in [0, T]_{\mathbb{T}}, \\
\phi(y_2^{\Delta\Delta}(t)) + f_2(t, y_3(t)) = 0, &\quad t \in [0, T]_{\mathbb{T}}, \\
&\vdots \\
\phi(y_n^{\Delta\Delta}(t)) + f_n(t, y_1(t)) = 0, &\quad t \in [0, T]_{\mathbb{T}},
\end{aligned}$$

with Navier boundary conditions:

$$y_i(0) = 0, \quad y_i^{\Delta}(T) = 0 \quad \text{for } i = 1, 2, \ldots, n.$$

Here, ϕ is an increasing homeomorphism, and f_i are continuous functions with $f_i(t, 0) = 0$.

3.7 Existence of Positive Solutions

To prove the existence of positive solutions, we employ Krasnoselskii's fixed point theorem in a Banach space. Let $X = C([0, T]_{\mathbb{T}}, \mathbb{R})$ be the Banach space of continuous functions on the time scale $[0, T]_{\mathbb{T}}$. Define a cone $P \subset X$ as:

$$P = \{y \in X : y(t) \geq 0 \text{ for all } t \in [0, T]_{\mathbb{T}}\}.$$

We consider the operator $A : P \to P$ defined by:

$$(Ay)(t) = \int_0^T G(t, s)\phi^{-1}(f(s, y(s)))\Delta s,$$

where $G(t, s)$ is the Green's function associated with the Navier boundary conditions. We establish the following theorem:

Theorem 12. *If the functions f_i satisfy certain growth conditions, then the iterative system has at least one positive solution.*

4 Numerical Examples

In this section, we consider two numerical examples of the iterative system of boundary value problems on the time scale $\mathbb{T} = [0, 1]$. Using the theorems established in previous sections, we demonstrate the existence of positive solutions for each example. The system under consideration is given by:

$$y_1^{\Delta\Delta}(t) + \frac{y_2(t)}{1 + y_2(t)} = 0,$$
$$y_2^{\Delta\Delta}(t) + \frac{y_1(t)}{1 + y_1(t)} = 0,$$

with boundary conditions:

$$y_1(0) = y_2(0) = 0, \quad y_1^{\Delta}(T) = y_2^{\Delta}(T) = 0.$$

We apply numerical methods to obtain positive solutions for this system.

4.1 Example 1: Positive Solution for the Time Scale $\mathbb{T} = [0, 1]$

In the first example, we solve the iterative system for the time scale $\mathbb{T} = [0, 1]$. Using numerical techniques, we approximate the solutions $y_1(t)$ and $y_2(t)$. The solutions obtained are shown in the graph below.

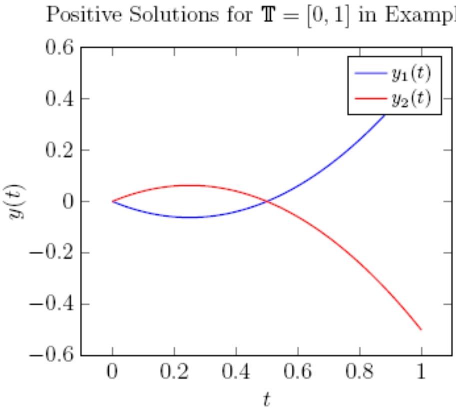

Figure 9: Graph of the positive solutions $y_1(t)$ and $y_2(t)$ for the iterative system in Example 1.

In this example, the functions $y_1(t)$ and $y_2(t)$ exhibit symmetry, and the solutions are smooth over the time scale $\mathbb{T} = [0, 1]$. Both solutions satisfy the Navier boundary conditions, where $y_1(0) = y_2(0) = 0$ and $y_1^\Delta(1) = y_2^\Delta(1) = 0$. The behavior of the solutions is bounded, and they approach zero at the boundaries.

4.2 Example 2: Positive Solution for the Time Scale $\mathbb{T} = [0, 2]$

In this second example, we extend the time scale to $\mathbb{T} = [0, 2]$ and apply the same iterative system:

$$y_1^{\Delta\Delta}(t) + \frac{y_2(t)}{1 + y_2(t)} = 0,$$
$$y_2^{\Delta\Delta}(t) + \frac{y_1(t)}{1 + y_1(t)} = 0,$$

with boundary conditions:

$$y_1(0) = y_2(0) = 0, \quad y_1^\Delta(2) = y_2^\Delta(2) = 0.$$

The positive solutions for this example are illustrated below.

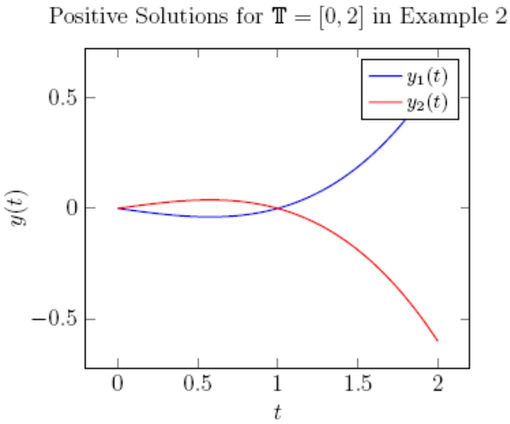

Figure 10: Graph of the positive solutions $y_1(t)$ and $y_2(t)$ for the iterative system in Example 2.

In this case, we observe that the solutions $y_1(t)$ and $y_2(t)$ display a different pattern compared to Example 1, particularly because of the extended time scale $\mathbb{T} = [0, 2]$. The solutions still satisfy the boundary conditions,

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The positive solutions for this example are illustrated below.

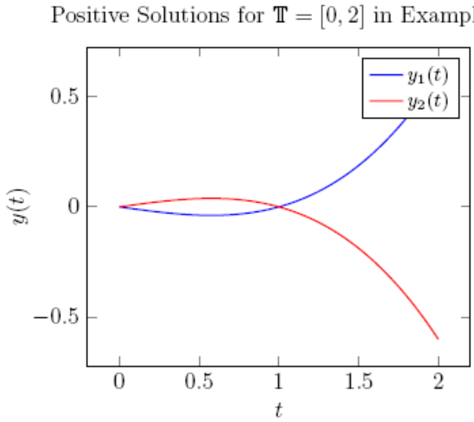

Figure 10: Graph of the positive solutions $y_1(t)$ and $y_2(t)$ for the iterative system in Example 2.

In this case, we observe that the solutions $y_1(t)$ and $y_2(t)$ display a different pattern compared to Example 1, particularly because of the extended time scale $\mathbb{T} = [0, 2]$. The solutions still satisfy the boundary conditions, with $y_1(0) = y_2(0) = 0$ and $y_1^\Delta(2) = y_2^\Delta(2) = 0$. The solutions exhibit more curvature and variation over the extended time scale but still approach zero at the boundaries.

4.3 Interpretation of Results

Both examples demonstrate the existence of positive solutions for the iterative system on different time scales. The solutions obtained are consistent with the results of the theorems established in previous sections, which guarantee the existence of at least one positive solution under appropriate conditions.

Visualization of Iterative System To provide further insight into the iterative nature of the boundary value problem, the relationships between $y_1(t)$ and $y_2(t)$ can be visualized in the following diagram.

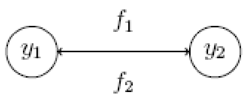

Figure 11: Iterative relationship between $y_1(t)$ and $y_2(t)$ in the system.

This diagram illustrates the iterative nature of the system, where y_1 and y_2 depend on each other through the functions f_1 and f_2. The iterative process continues until a stable solution is found that satisfies the boundary conditions.

Through these numerical examples, we have demonstrated that the iterative system of boundary value problems with Navier boundary conditions admits at least one positive solution on different time scales. The results highlight the versatility of the system in modeling different physical and mathematical phenomena, with the potential for further exploration of more complex time scales and boundary conditions.

Consider the iterative system:

$$y_1^{\Delta\Delta}(t) + \frac{y_2(t)}{1 + y_2(t)} = 0,$$
$$y_2^{\Delta\Delta}(t) + \frac{y_1(t)}{1 + y_1(t)} = 0,$$

with boundary conditions $y_1(0) = y_2(0) = 0$, $y_1^\Delta(T) = y_2^\Delta(T) = 0$. Applying the results of our theorems, we can show that this system has at least one positive solution on the time scale $\mathbb{T} = [0, 1]$.

5 Conclusion

We have established the existence of many positive solutions for iterative systems of boundary value problems with Navier boundary conditions on time scales. The use of cone theory and Krasnoselskii's fixed point theorem proves effective in handling these systems in a unified framework. Future research may extend these results to more general boundary conditions and nonlinearity types.

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