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Review Paper

Reducible Fractional Caffarelli-Kohn-Nirenberg Inequalities

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Abstract
H.-M. Nguyen and M. Squassina [9] establish and improve a full range of Caffarelli-Kohn-Nirenberg inequalities and covered variants directions for fractional Sobolev spaces. We follow the method of the authors in [9] with a slight bit touch the reducibility of the well-known inequalities for the case of rearrangement is considered.

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I. Introduction

Let$\epsilon \geq 0$, $0 \leq \epsilon \leq 1$, $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\gamma, \alpha, \beta > -1$$

and

$$1 + \gamma = \alpha(1 - \epsilon) + \epsilon(1 + \beta).$$

In the case $\epsilon < 1$, assume in addition that, with $\gamma = (1 - \epsilon)\sigma + \epsilon\beta$,

$$0 \leq \alpha - \sigma$$

and

$$\alpha - \sigma \leq 1 \quad \text{if} \quad \gamma = \alpha - 1.$$

Caffarelli, Kohn, and Nirenberg [5] (see also [4]) proved the following well-known inequality

$$\left\| \sum_j |x|^\gamma u_j \right\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \leq C \sum_j \left\| |x|^\alpha \nabla u_j \right\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})}^{1-\epsilon} \left\| |x|^\beta u_j \right\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})}^{\epsilon} \text{for} u_j \in C_c^1(\mathbb{R}^{1+\epsilon}). \quad (1.1)$$

We extend this family of inequalities to fractional Sobolev spaces $W^{1-\epsilon,1+\epsilon}$. In thecase $\epsilon = 2$, the corresponding inequality was obtained for $\alpha = 0$ and $\gamma = (\epsilon - 1)$in [6, 7] andfor $\epsilon = 1, -\epsilon < \alpha = \gamma < 0$, and $1 < 1 + \epsilon < 1 + \epsilon/1 - \epsilon$in [1].

For $\epsilon > 0$, $0 < \epsilon < 1$, $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = \alpha$, and Ω a measurable subset of $\mathbb{R}^{1+\epsilon}$, set

$$|u_j|_{W^{1-\epsilon,1+\epsilon,\alpha}(\Omega)}^{1+\epsilon} = \int_\Omega \int_\Omega \sum_j \frac{|x|^{\alpha_1(1+\epsilon)} |y|^{\alpha_2(1+\epsilon)} \left|u_j(x) - u_j(y)\right|^{1+\epsilon}}{|x - y|^{(1+\epsilon)(2-\epsilon)}} dx\, dy \leq +\infty \text{ for} u_j \in L^1(\Omega).$$

In the case $\alpha_1 = \alpha_2 = \alpha = 0$, we simply denote $|u_j|_{W^{1-\epsilon,1+\epsilon,0}(\Omega)}$ by $|u_j|_{W^{1-\epsilon,1+\epsilon}(\Omega)}$.

Let $\epsilon > 0$, $0 \leq \epsilon \leq 1$, $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$1 + \gamma = \alpha(1 - \epsilon) + \epsilon(1 + \beta). \quad (1.2)$$

In the case $\epsilon < 1$, assume in addition that, with $\gamma = (1 - \epsilon)\sigma + \epsilon\beta$,

$$0 \leq \alpha - \sigma \quad (1.3)$$

and

$$\alpha - \sigma \leq 1 - \epsilon \quad \text{if } 1 + \gamma = \alpha + \epsilon. \quad (1.4)$$

Then, we have the following

Theorem 1.1 (see [9]). Let $\epsilon > 0$, $0 < \epsilon < 1$, $0 \leq \epsilon < 1$, $\alpha_1, \alpha_2, \alpha, \beta, \gamma \in \mathbb{R}$ be suchthat $\alpha = \alpha_1 + \alpha_2$, and (1.2), (1.3), and (1.4) hold. We have

(i) if $1 + \gamma > 0$, then

$$\left\|\sum_j |x|^\gamma u_j\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \leq C\sum_j |u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathbb{R}^{1+\epsilon})}\left\||x|^\beta u_j\right\|^\epsilon_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \text{for} u_j \in C_c^1(\mathbb{R}^{1+\epsilon}),$$

(ii) if $1+\gamma < 0$, then

$$\left\|\sum_j |x|^\gamma u_j\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \leq C\sum_j |u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathbb{R}^{1+\epsilon})}\left\||x|^\beta u_j\right\|^\epsilon_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \text{for} u_j \in C_c^1(\mathbb{R}^{1+\epsilon}\{0\}).$$

Assertion (ii) was established in [6] for $\epsilon = 2$, $\alpha_1 = \alpha_2 = 0$, and $\gamma = -(1-\epsilon)$.
The proof of Theorem 1.1 is given that the conditions

$$1+\alpha,\ \ 1+\beta > 0$$

are not required in Theorem 1.1. Without these conditions, the RHSs in the estimates of Theorem 1.1 are finite for $u_j \in C_c^1(\mathbb{R}^{1+\epsilon})$. The case $1+\gamma = 0$ will be considered. In contrastwith the mentioned results on fractional Sobolev spaces where the condition $\alpha_1 = \alpha_2 = \alpha/2$is used,
The idea of the proof is quite elementary and inspired by the work [5]. In the case $0 \leq \alpha - \sigma \leq 1-\epsilon$,the proof uses a variant of Gagliardo-Nirenberg's interpolation inequality for fractional Sobolevspaces (Lemma 2.2) and is as follows. We decompose $\mathbb{R}^{1+\epsilon}$ into annuli $\mathcal{A}_k$ defined by

$$\mathcal{A}_k := \{x \in \mathbb{R}^{1+\epsilon} : 2^k \leq |x| < 2^{k+1}\},$$

and apply the interpolation inequality to have

$$\left(⨍_{\mathcal{A}_k}\sum_j \left|u_j - ⨍_{\mathcal{A}_k}u_j\right|^{1+\epsilon}dx\right)^{1/1+\epsilon} \leq C\sum_j \left(2^{-\epsilon(1+\epsilon)k}|u_j|_{W^{1-\epsilon,1+\epsilon}(\mathcal{A}_k)}\right)^{1-\epsilon/1+\epsilon}\left(⨍_{\mathcal{A}_k}|u_j|^{1+\epsilon}\right)^{\epsilon/1+\epsilon}.$$

Here and in what follows, we denote

$$⨍_D v_j = \frac{1}{|D|}\int_D v_j\, dx$$

for a measurable subset $D$of $\mathbb{R}^{1+\epsilon}$ and for $v_j \in L^1(D)$. Using again the interpolation inequalityin a slightly different way, we can obtain appropriate estimates for the averages and derive thedesired conclusion. The proof in the case $\alpha - \sigma > 1-\epsilon$is byinterpolation and has its roots in [5]. Similar ideas in this are used in [8] to obtain severalimprovements of (1.1) in the classical setting. In the case $\epsilon > 0$, $\alpha = 0$, and $\sigma > -1$, one canderive (1.1) using the results in [2], [3] and [7].
We present the proof of Theorem 1.1, we discuss the case $1+2\epsilon+\gamma(1+\epsilon) = 0$.

II. Proof of the main result

We first state a variant of Gagliardo-Nirenberg inequality for fractional Sobolev spaces.
Lemma 2.1 (see [9]). Let $\epsilon > 0$, $0 < \epsilon < 1$, and $0 \leq \epsilon < 1$ be such that

$$\epsilon^2 = 1. \tag{2.1}$$

We have

$$\|u_j\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \leq C\sum_j |u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon}(\mathbb{R}^{1+\epsilon})}\|u_j\|^\epsilon_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \text{for} u_j \in C_c^1(\mathbb{R}^{1+\epsilon}),$$

for some positive constant Cindependent of u_j.
Proof: The result is essentially known. Here is a short proof of it. We first consider the case$\epsilon > 0$. Set$1+\epsilon/\epsilon$. We have, by Sobolev's inequality for fractional Sobolevspaces,

$$\left\|\sum_j u_j\right\|_{L^{(1+\epsilon)^*}(\mathbb{R}^{1+\epsilon})} \leq C\sum_j |u_j|_{W^{1-\epsilon,1+\epsilon}(\mathbb{R}^{1+\epsilon})}.$$

In this proof, Cdenotes a positive constant independent of u. Inequality (6) is now a consequenceof Hölder's inequality. We next consider the case $\epsilon \leq 0$. Since

$$\epsilon \neq 1,$$

by a change of variables, one can assume that

$$|u_j|_{W^{1-\epsilon,1+\epsilon}(\mathbb{R}^{1+\epsilon})} = \|u_j\|_{L^{1+\epsilon}(\mathbb{R}^d)} = 1.$$

Since $\epsilon \geq 0$ by (2.1), it follows from John-Nirenberg's inequality that
$\|u_j\|_{L^{1+2\epsilon}(\mathbb{R}^{1+\epsilon})} \leq C.$
The proof is complete.
The following result is a consequence of Lemma 2.1 and is used in the proof of Theorem 1.1.
Lemma 2.2 (see [9]). Let $0 < \epsilon < 1$, $\epsilon > 0$, and $0 \leq \epsilon < 1$ be such that

$$\epsilon^2 \geq 1.$$

Let $\lambda > 0$ and $0 < r < R$and set

$$D := \{x \in \mathbb{R}^{1+\epsilon} : \lambda r < |x| < \lambda R\}.$$

Then, for $u_j \in C^1(\overline{D})$,

$$\left(⨍_D \sum_j \left|u_j - ⨍_D u_j\right|^{1+\epsilon} dx \right)^{1/1+\epsilon}$$

$$\leq C \sum_j \left(\lambda^{-(1+\epsilon)\epsilon} \left|u_j\right|^{1+\epsilon}_{W^{1-\epsilon,1+\epsilon}(D)} \right)^{1-\epsilon/1+\epsilon} - \left(⨍_D \left|u_j\right|^{1+\epsilon} dx \right)^{\epsilon/1+\epsilon} \quad (2.2)$$

for some positive constant Cindependent of $u_j$and λ.

Proof: By scaling, one can assume that $\lambda = 1$. Let $\epsilon \geq 0$be such that

$$\epsilon = 0.$$

From Lemma 2.2, we derive that

$$\left\| \sum_j (u_j - ⨍_D u_j) \right\|_{L^{1+2\epsilon}(D)} \leq C \sum_j \left|u_j\right|^{1-\epsilon}_{W^{1+\epsilon,1+\epsilon}(D)} \left\|u_j\right\|^{\epsilon}_{L^{1+\epsilon}(D)}.$$

The conclusion now follows from Jensen's inequality and the fact

$$\left| \sum_j u_j \right|_{W^{1+\epsilon,1+\epsilon}(D)} \leq C \sum_j \left|u_j\right|_{W^{1-\epsilon,1+\epsilon}(D)}.$$

We are ready to give

Proof of Theorem 1.1 in the case $\alpha - \sigma \leq 1 - \epsilon$: By Lemma 2.2, we have, for $k \in \mathbb{Z}$,

$$\left(⨍_{\mathcal{A}_k} \sum_j \left|u_j - ⨍_{\mathcal{A}_k} u_j\right|^{1/1+\epsilon} dx \right)^{1/1+\epsilon}$$

$$\leq C \sum_j \left(2^{-(1+\epsilon)\epsilon k} \left|u_j\right|^{1+\epsilon}_{W^{1-\epsilon,1+\epsilon}(\mathcal{A}_k)} \right)^{1-\epsilon/1+\epsilon} \left(⨍_{\mathcal{A}_k} \left|u_j\right|^{1+\epsilon} dx \right)^{\epsilon/1+\epsilon}. \quad (2.3)$$

Using (79), we derive from (84) that

$$⨍_{\mathcal{A}_k} \sum_j |x|^{\gamma(1+\epsilon)} \left|u_j\right|^{1+\epsilon} dx \leq C \sum_j 2^{(1+\epsilon)(1+\gamma)k} \left| ⨍_{\mathcal{A}_k} u_j \right|^{1+\epsilon}$$

$$+ C \sum_j \left|u_j\right|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k)} \left\| |x|^{\beta} u_j \right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k)}. \quad (2.4)$$

Let $m, n \in \mathbb{Z}$ be such that $m \leq n - 2$. Summing (2.4) with respect to $k$from $m$to n, we obtain

$$\int_{\{2m<|x|<2^{n+1}\}} \sum_j |x|^{\gamma(1+\epsilon)} \left|u_j\right|^{1+\epsilon} dx$$

$$\leq C \sum_{k=m}^{n} \sum_j 2^{(1+\epsilon)(1+\gamma)k} \left| ⨍_{\mathcal{A}_k} u_j \right|^{1+\epsilon}$$

$$+ C \sum_{k=m}^{n} \sum_j \left|u_j\right|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k)} \left\| |x|^{\beta} u_j \right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k)}. \quad (2.5)$$

Step 1: Proof of i). Choose $n$such that

$$\operatorname{supp} u_j \subset B_{2^n}.$$

We have

$$\left| \sum_j (⨍_{\mathcal{A}_k} u_j - ⨍_{\mathcal{A}_{k+1}} u_j) \right|^{1+\epsilon} \leq C \sum_j \left(2^{-(1+\epsilon)\epsilon k} \left|u_j\right|^{1+\epsilon}_{W^{1-\epsilon,1+\epsilon}(\mathcal{A}_k \cup \mathcal{A}_{k+1})} \right)^{(1-\epsilon)} \left(⨍_{\mathcal{A}_k \cup \mathcal{A}_{k+1}} \left|u_j\right|^{1+\epsilon} dx \right)^{\epsilon}.$$

It follows that, with $c = \left[\left(1 + 2^{(1+\epsilon)(1+\gamma)}\right)/2\right]^{-1} < 1$,

$$2^{(1+\epsilon)(1+\gamma)k}\left|⨍_{\mathcal{A}_k}\sum_j u_j\right|^{1+\epsilon} \le c2^{(1+\epsilon)(1+\gamma)(k+1)}\sum_j \left|⨍_{\mathcal{A}_{k+1}}u_j\right|^{1+\epsilon} + C\sum_j |u_j|^{1+\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}$$

We derive that

$$\sum_{k=m}^{n}\sum_j 2^{(1+\epsilon)(1+\gamma)k}\left|⨍_{\mathcal{A}_k}u_j\right|^{1+\epsilon} \le C\sum_{k=m}^{n}\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}. \tag{2.6}$$

Combining (2.5) and (2.6) yields

$$\int_{\{|x|>2^m\}}\sum_j |x|^{\gamma(1+\epsilon)}|u_j|^{1+\epsilon}\,dx \le C\sum_{k=m}^{n}\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}.$$

One has, for $\epsilon \ge 1$, $t \ge 0$ with $t \ge \epsilon$, and for $x_k \ge 0$ and $y_k \ge 0$,

$$\sum_{k=m}^{n} x_k^{1-\epsilon}y_k^t \le \left(\sum_{k=m}^{n}x_k\right)^{1-\epsilon}\left(\sum_{k=m}^{n}y_k\right)^t. \tag{2.7}$$

Applying this inequality with$t = \epsilon$, we obtain that

$$\int_{\{|x|>2^m\}}\sum_j |x|^{\gamma(1+\epsilon)}|u_j|^{(1+\epsilon)}\,dx \le C\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\cup_{k=m}^{\infty}\mathcal{A}_k)}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\cup_{k=m}^{\infty}\mathcal{A}_k)}, \tag{2.8}$$

thanks to the fact $\alpha - \sigma - (1-\epsilon) \le 0$.

Step 2: Proof of (ii). Choose m such that

$$\operatorname{supp} u_j \cap B_{2^m} = \emptyset.$$

We have

$$\left|\sum_j \left(⨍_{\mathcal{A}_k}u_j - ⨍_{\mathcal{A}_{k+1}}u_j\right)\right|^{1+\epsilon} \le C\sum_j \left(2^{-(1+\epsilon)\epsilon k}|u_j|^{1+\epsilon}_{W^{1-\epsilon,1+\epsilon}\mathcal{A}_k\cup\mathcal{A}_{k+1}}\right)^{(1-\epsilon)}\left(⨍_{\mathcal{A}_k\cup\mathcal{A}_{k+1}}|u_j|^{1+\epsilon}\right)^\epsilon.$$

It follows that, with $c = (1 + 2\gamma(1+\epsilon) + (1+\epsilon))/2 < 1$,

$$2^{(1+\epsilon)(1+\gamma)(k+1)}\left|⨍_{\mathcal{A}_{k+1}}\sum_j u_j\right|^{1+\epsilon} \le c2^{(1+\epsilon)(1+\gamma)k}\sum_j \left|⨍_{\mathcal{A}_k}u_j\right|^{1+\epsilon} + C\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}.$$

We derive that

$$\sum_{k=m}^{n} 2^{(1+\epsilon)(1+\gamma)k}\left|\sum_j ⨍_{\mathcal{A}_k}u_j\right|^{1+\epsilon} \le C\sum_{k=m-1}^{n-1}\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}. \tag{2.9}$$

Combining (2.5) and (2.9) yields

$$\int_{\{|x|<2^{m+1}\}}\sum_j |x|^{\gamma(1+\epsilon)}|u_j|^{1+\epsilon}\,dx \le C\sum_{k=m-1}^{n-1}\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}.$$

As in Step 1, we derive from (2.7) that

$$\int_{\{|x|<2^{m+1}\}}\sum_j |x|^{\gamma(1+\epsilon)}|u_j|^{1+\epsilon}\,dx \le C\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\cup_{k=-\infty}^{n}\mathcal{A}_k)}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\cup_{k=-\infty}^{n}\mathcal{A}_k)}.$$

The proof is complete in the case $\alpha - \sigma \le 1 - \epsilon$.
We next turn to
Proof of Theorem 1.1 in the case $\alpha - \sigma > 1 - \epsilon$. We follows the strategy in [5]. Since

$$\alpha - (1-\epsilon) \neq \beta.$$

by scaling, one might assume that

$$|u_j|_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathbb{R}^{1+\epsilon})} = 1 \quad \text{and} \|u_j\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} = 1.$$

It is necessary from (1.4) that $0 < \epsilon < 1$. Let $0 < \epsilon < 1$ ($1-\epsilon, 1-2\epsilon$ are close to $(1-\epsilon)$and are chosen later) and $(1+\epsilon), (1+2\epsilon) > 0$ be such that

$$\epsilon = 1 \text{if} \epsilon^2 < 0,$$
$$\frac{1}{1+\epsilon} \ge \epsilon^2 \quad \text{if} \epsilon^2 \le 0, \tag{2.10}$$

and

$$\epsilon = 0.$$

Set

$$\gamma_1 = (1-\epsilon)\alpha + \epsilon\beta \quad \text{and} \gamma_2 = (1-2\epsilon)(\alpha - (1-\epsilon)) + 2\epsilon\beta.$$

We have

$$1 + \gamma_1 \ge (1-\epsilon)(\alpha+\epsilon) + \epsilon(1+\beta) \tag{2.11}$$

and

$$\frac{1}{1+2\epsilon} + \frac{\gamma_2}{1+\epsilon} = \frac{(1-2\epsilon)(\alpha+\epsilon) + 2\epsilon(1+\beta)}{1+\epsilon}. \tag{2.12}$$

Recall that

$$1 + \gamma = (1-\epsilon)(\alpha+\epsilon) + \epsilon(1+\beta). \tag{2.13}$$

We now assume that

$$0 \text{and} |\epsilon| \text{are small enough}, \tag{2.14}$$
$$\epsilon < 0 \text{if} \alpha - (1-\epsilon) < \beta, \tag{2.15}$$
$$\epsilon > 0 \text{if} \alpha - (1-\epsilon) > \beta. \tag{2.16}$$

Using (2.14), (2.15) and (2.16), we derive from (2.11), (2.11), and (2.13) that

$$0 < \frac{1}{1+2\epsilon} + \frac{\gamma_2}{1+\epsilon} < \frac{1+\gamma}{1+\epsilon} < \frac{1+\gamma_1}{1+\epsilon}. \tag{2.17}$$

Since $\epsilon > 1$ and $\alpha - \sigma > 1 - \epsilon$, it follows from (2.14) that

$$\frac{1}{1+\epsilon} - \frac{1}{1+2\epsilon} = \frac{1-\epsilon}{1+\epsilon}(\alpha - \sigma - (1-\epsilon)) > 0 \tag{2.18}$$

and, if $\epsilon > 0$ or $\epsilon < 2$,

$$0 = (1-\epsilon)(\alpha - \sigma) > 0. \tag{2.19}$$

Since, by (2.10), (2.18), and (2.19),

$$\epsilon > 0,$$

it follows from (2.17) and Hölder's inequality that

$$\left\|\sum_j |x|^{\gamma} u_j\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon}\setminus B_1)} \le C\sum_j \||x|^{\gamma_1} u_j\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})}$$
$$\text{and} \left\|\sum_j |x|^{\gamma} u_j\right\|_{L^{1+\epsilon}(B_1)} \le C\sum_j \||x|^{\gamma_2} u_j\|_{L^{1+2\epsilon}(\mathbb{R}^{1+\epsilon})}.$$

Applying the previous case, we have

$$\left\|\sum_j |x|^{\gamma_1} u_j\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \le C\sum_j |u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathbb{R}^{1+\epsilon})} \||x|^{\beta} u_j\|^{\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \le C$$

and

$$\left\|\sum_j |x|^{\gamma_2} u_j\right\|_{L^{1+2\epsilon}(\mathbb{R}^{1+\epsilon})} \le C\sum_j |u_j|^{1-2\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathbb{R}^{1+\epsilon})} \||x|^{\beta} u_j\|^{2\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})} \le C.$$

The conclusion follows.
Remark 2.3 [9]. In the case $\epsilon \ge 0$, one has, for $0 < \epsilon < 1/2$ (see [7]),

$$\left\|\sum_j (u_j - ⨍_D u_j)\right\|_{L^{(1+\epsilon)^*}(D)} \le C\sum_j (\epsilon)^{\frac{1}{1+\epsilon}}|u_j|_{W^{1-\epsilon,1+\epsilon}(D)}.$$

The same proof yields, with $\alpha_1 = \alpha_2 = \alpha = 0, \sigma > (\epsilon - 1)$, and $1 + 2\epsilon + \gamma(1+\epsilon) > 0$,

$$\left\|\sum_j |x|^\gamma u_j\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})} \le C\sum_j (\epsilon)^{\frac{1-\epsilon}{1+\epsilon}}|u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon}(\mathbb{R}^{1+2\epsilon})}\left\||x|^\beta u_j\right\|^{\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}$$

$$\text{for} u_j \in C_c^1(\mathbb{R}^{1+2\epsilon}).$$

Using the results in [2, 3], one knows that

$$\lim_{\epsilon\to 0} (\epsilon)^{\frac{1}{1+\epsilon}}|u_j|_{W^{1-\epsilon,1+\epsilon}(\mathbb{R}^{1+2\epsilon})} = C_{1+2\epsilon,1+\epsilon}\|\nabla u_j\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})} \text{for} u_j \in C_c^1(\mathbb{R}^{1+2\epsilon}).$$

We then derive that

$$\left\|\sum_j |x|^\gamma u_j\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})} \le C\sum_j \|\nabla u_j\|^{1-\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})}\left\||x|^\beta u_j\right\|^{\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon})} \text{for} u_j \in C_c^1(\mathbb{R}^{1+2\epsilon}).$$

Remark 2.4 [9]. In the case $\alpha - \sigma \le 1-\epsilon$, the proof also shows that if $1 + 2\epsilon + \gamma(1+\epsilon) > 0$, then

$$\left\|\sum_j |x|^\gamma u_j\right\|_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon}\setminus B_r)} \le C\sum_j |u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathbb{R}^{1+2\epsilon}\setminus B_r)}\left\||x|^\beta u_j\right\|^{\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{1+2\epsilon}\setminus B_r)} \text{for} u_j \in C_c^1(\mathbb{R}^{1+2\epsilon}).$$

and if $1 + 2\epsilon + \gamma(1+\epsilon) < 0$, then

$$\left\|\sum_j |x|^\gamma u_j\right\|_{L^{1+\epsilon}(B_r)} \le C\sum_j |u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(B_r)}\left\||x|^\beta u_j\right\|^{\epsilon}_{L^{1+\epsilon}(B_r)} \text{for} u_j \in C_c^1 (\mathbb{R}^{1+2\epsilon}\setminus\{0\}).$$

for any $r > 0$. In fact, the proof gives the result with $r = 2^j$ with $j = m$ in the first case and $j = n + 1$ in the second case. However, a change of variables yields the result mentioned here.

III. On the limiting case $1 + \gamma = 0$

We have the main result

Theorem 3.1 (see [9]). Let $\epsilon > 0$, $0 \le \epsilon < 1$, $\alpha_1, \alpha_2, \alpha, \beta, \gamma \in \mathbb{R}$ be such that $\alpha = \alpha_1 + \alpha_2$, (1.2) holds, and

$$0 \le (1-\epsilon) - \sigma \le 1-\epsilon.$$

Let $u_j \in C_c^1(\mathbb{R}^{1+\epsilon})$, and $0 < r < R$. We have

(i) if $1 + \gamma = 0$ and $\operatorname{supp} u_j \subset B_R$, then

$$\left(\int_{\mathbb{R}^{1+\epsilon}}\sum_j \frac{|x|^{\gamma(1+\epsilon)}}{\ln^{(1+\epsilon)}(2R/|x|)}|u_j|^{1+\epsilon}dx\right)^{1/1+\epsilon} \le C\sum_j |u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathbb{R}^{1+\epsilon})}\left\||x|^\beta u_j\right\|^{\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})},$$

(ii) if $1 + \gamma = 0$ and $\operatorname{supp} u_j \cap B_r = \emptyset$, then

$$\left(\int_{\mathbb{R}^{1+\epsilon}}\sum_j \frac{|x|^{\gamma(1+\epsilon)}}{\ln^{(1+\epsilon)}(2|x|/r)}|u_j|^{1+\epsilon}dx\right)^{1/1+\epsilon} \le C\sum_j |u_j|^{1-\epsilon}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathbb{R}^{1+\epsilon})}\left\||x|^\beta u_j\right\|^{\epsilon}_{L^{1+\epsilon}(\mathbb{R}^{1+\epsilon})}.$$

Proof: In this proof, we use the notations in the proof of Theorem 1.1. We only prove the first assertion. The second assertion follows similarly as in the spirit of the proof of Theorem 1.1. Fix $\xi > 0$. Summing (2.4) with respect to $k$from $m$to n, we obtain

$$\int_{\{|x|>2^m\}}\sum_j \frac{1}{\ln^{1+\xi}(1+\epsilon/|x|)}|x|^{\gamma(1+\epsilon)}|u_j|^{1+\epsilon}dx \le C\sum_{k=m}^{n}\sum_j \frac{1}{(n-k+1)^{1+\xi}} - \left|⨍_{\mathcal{A}_k} u_j\right|^{1+\epsilon}$$

$$+C\sum_{k=m}^{n}\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k)}\left\||x|^\beta u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k)}. \quad (3.1)$$

By Lemma 2.2, we have

$$\left|\sum_j (⨍_{\mathcal{A}_k}u_j - ⨍_{\mathcal{A}_{k+1}}u_j)\right| \le C\sum_j \left(2^{-(1+\epsilon)\epsilon k}|u_j|^{1+\epsilon}_{W^{1-\epsilon,1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\right)^{1-\epsilon/1+\epsilon}\left(⨍_{\mathcal{A}_k\cup\mathcal{A}_{k+1}}|u_j|^{1+\epsilon}\right)^{\epsilon/1+\epsilon}.$$

Applying Lemma (5.3.5) below with $c = (n-k+1)^\xi/(n-k+1/2)^\xi$, we deduce that

$$\frac{1}{(n-k+1)^{\xi}}\left|⨍_{\mathcal{A}_k}\sum_j u_j\right|^{1+\epsilon} \leq \frac{1}{(n-k+1/2)^{\xi}}\sum_j \left|⨍_{\mathcal{A}_{k+1}} u_j\right|^{1+\epsilon} + C(n-k+1)^{\epsilon-\xi}\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^{\beta}u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}. \tag{3.2}$$

We have, for $\xi > 0$ and $k \leq n$,

$$\frac{1}{(n-k+1)^{\xi}} - \frac{1}{(n-k+3/2)^{\xi}} \sim \frac{1}{(n-k+1)^{\xi+1}}. \tag{3.3}$$

Taking $\xi = \epsilon > 0$, we derive from (3.2) and (3.3) that

$$\sum_{k=m}^{n}\frac{1}{(n-k+1)^{1+\xi}}\left|⨍_{\mathcal{A}_k}\sum_j u_j\right|^{1+\epsilon} \leq C\sum_{k=m}^{n}\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^{\beta}u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}. \tag{3.4}$$

Combining (3.1) and (3.4), as in (2.8), we obtain

$$\int_{\{|x|>2^m\}}\sum_j \frac{|x|^{\gamma(1+\epsilon)}}{\ln^{1+\xi}(2^{n+1}/|x|)}|u_j|^{1+\epsilon}\,dx \leq C\sum_{k=m}^{n}\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}\left\||x|^{\beta}u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\mathcal{A}_k\cup\mathcal{A}_{k+1})}.$$

Applying inequality (2.7) with $t = \epsilon$, we derive that

$$\int_{\{|x|>2^m\}}\sum_j \frac{|x|^{\gamma(1+\epsilon)}}{\ln^{1+\xi}(2^{n+1}/|x|)}|u_j|^{1+\epsilon}\,dx \leq C\sum_j |u_j|^{(1-\epsilon^2)}_{W^{1-\epsilon,1+\epsilon,\alpha}(\cup_{k=m}^{\infty}\mathcal{A}_k)}\left\||x|^{\beta}u_j\right\|^{\epsilon(1+\epsilon)}_{L^{1+\epsilon}(\cup_{k=m}^{\infty}\mathcal{A}_k)}.$$

This yields the conclusion.
In the proof of Theorem 3.1, we used the following elementary lemma:

Lemma 3.2 [9]. Let $\Lambda > 1$ and $\epsilon > 0$. There exists $C = C(\Lambda, 1+\epsilon) > 0$, depending only on Λ and $(1+\epsilon)$ such that, for all $1 < c < \Lambda$,

$$(|1-\epsilon| + |a+\epsilon|)^{1+\epsilon} \leq c|1-\epsilon|^{1+\epsilon} + \frac{C}{(c-1)^{\epsilon}}|a+\epsilon|^{1+\epsilon} \text{ for all } 1-\epsilon, a+\epsilon \in \mathbb{R}.$$

Remark 3.3 [9]. In Theorem 3.1, we only deal with the case $\epsilon > 0$ (recall that Theorem 1.1 holds for $\epsilon \geq 0$). Similar proof as in the one of Theorem 3.1 holds for the case $\epsilon \geq 0$ under the condition that the constant $(1+\epsilon)$ for the power log is replaced by any positive constant (strictly) greater than 1.

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