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Review Paper

Comparing Solutions to the Third-Order Dispersive Partial Differential Equation

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Abstract
In previous decades, many of the practical problems arising in scientific fields such as mathematics, physics, chemistry, biology,and engineering have been related to nonlinear fractional partial differential equations. One of these nonlinear partial differential equations, the third-order dispersive partial differential equation, has been found to have a plethora of useful applications in different fields such as Newtonian fluid mechanics, optimal control, convection diffusion processes, hydrodynamics, and aerodynamics. A special class of solutions has been studied for the third-order dispersive partial differential equation including exact solutions and approximate solutions. The aim of this article is to compare were the Adomian decomposition method, the lines method, an exponential quartic spline and finite difference discretization method, and the non-polynomial spline methodwith the solution of the third-order dispersive partial differential equation. We will conduct a comparison of the stability of the two methods using the Von Neumann stability analysis. In addition, a numerical example will be presented to illustrate the accuracy of these methods.

Keywords: *Dispersive partial differential equation, Cubic B-Spline, Non-Polynomial Spline, Von Neumann Stability.*

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I. Introduction

During the 1800s, water wave problems were extensively investigated by Stokes, Boussinesq, and Korteweg and de Vries (KdV). Since then, significant advances have been made to the domain of non-linear dispersive waves. Reliable asymptotic techniques were proposed in the 1960s for obtaining non-linear wave equations (e.g., KdV equation), which underpin a variety of physical phenomena ([1]. The non-polynomial spline techniques are the most comprehensively explored of the methods designed to solve non-linear differential equations.

A large number of researchers sought to solve non-linear partial differential equations based on the non-polynomial spline method and the cubic B-spline method.

Some approaches for solving the third-order dispersive partial differential equations have been addressed in recent literature; the most prominent of these were the Adomian decomposition method, the lines method, an exponential quartic spline and finite difference discretization method, and the non-polynomial spline method.

Researchers who have used the adomian decomposition method include Wazwaz (2003), who used the adomian decomposition method to solve the third-order dispersive partial differential equations in one- and higher-dimensional spaces [2]. In 2014, Kudu and Amirali studies the convergence of the boundary-value problem for third order partial differential equations by the lines method [3]. The third order dispersive partial differential equations were solved by finite difference discretization to approximate the first order spatial and temporal derivative and using exponential quartic spline to approximate the spatial derivative of third order by Sultana et al. in 2018 [4]. Zaki et al. investigated the approximation solution of the third-order dispersive partial differential equation by the non-polynomial quadratic spline method [5].

As the previous four methods, the Adomian decomposition method, the lines method, an exponential quartic spline and finite difference discretization method, and the non-polynomial spline method, have been used many times in recent years, we wanted to present a comparison between them to assist future researchers. Section one outlines some previous studies on the dispersive partial differential equation. Section two offers basic methods of solutions for the third-order dispersive partial differential equation. The third section showed the local truncation errors. In the fourth section will describe the stability analysis. Using the concept of stability and the von Neumann method. Section five addresses numerical illustration. In this section, we offer an example from each author as well as their results. We havealso presented additional results that we have obtained during our research. In the final section, we offer some conclusions and highlight some areas for further development.

The generalised Third-Order Dispersive Partial Differential Equation of the form [5]:

$\frac{\partial \eta}{\partial t}+\frac{\partial^3 \eta}{\partial x^3}=g(x,t), \quad a\le x\le b, \quad t>0,$(1)
where $g(x,t)$ is a source term. The boundary conditions associated with (1) are assumed to be of the form
$\eta(a,t)=\beta_1(t), \quad \eta(b,t)=\beta_2(t), \ \eta_{xx}(b,t)=\beta_3(t) \quad t>0$,(2)
and the initial condition is
$\eta(x,0)=f(x), \qquad a\le x\le b.$ (3)

II. The Methods of solution

In this section, we will illustrate the Adomian decomposition method, the lines method, an exponential quartic spline and finite difference discretization method, and the non-polynomial spline method.

2.1 The one-dimensional dispersive equation [2]

In an operator form, Eq. (1) becomes

$$\mathrm{L}u=g(x,t)-a\frac{\partial^3 u}{\partial x^3}, \qquad (4)$$

Operating with L^{-1} on both sides of (1) yields

$$u(x,t)=f(x)+\mathrm{L}^{-1}\big(g(x,t)\big)-a\mathrm{L}^{-1}(u_{xxx}). \qquad (5)$$

Decomposes the solution $u(x,t)$ by an infinite series of components

$$u(x,t)=\sum_{n=0}^{\infty} u_n(x,t), \qquad (6)$$

Equation (6) yields

$$\sum_{n=0}^{\infty} u_n(x,t)=f(x)+\mathrm{L}^{-1}\big(g(x,t)\big)-a\mathrm{L}^{-1}\Big(\big(\sum_{n=0}^{\infty} u_n(x,t)\big)_{xxx}\Big). \qquad (7)$$

To determine the components $u_n(x,t)$, the decomposition method identifies the zeroth component $u_0(x,t)$ by all terms that arise from the initial condition and from integrating the source term. Accordingly, the decomposition method introduces the recurrence relation

$$\begin{aligned} &u_0(x,t)=f(x)+\mathrm{L}^{-1}(g(x,t)),\\ &u_{k+1}(x,t)=-a\mathrm{L}^{-1}\big(u_{k_{xxx}}\big), \quad k\geqslant 0, \end{aligned} \qquad (8)$$

The first few components are therefore given by

$$\begin{aligned} &u_0(x,t)=f(x)+\mathrm{L}^{-1}\big(g(x,t)\big)\\ &u_1(x,t)=-a\mathrm{L}^{-1}\big(u_{0_{xxx}}\big)\\ &u_2(x,t)=-a\mathrm{L}^{-1}\big(u_{1_{xxx}}\big)\\ &u_3(x,t)=-a\mathrm{L}^{-1}\big(u_{2_{xxx}}\big). \end{aligned} \qquad (9)$$

Wazwaz found the solution can be enhanced dramatically by increasing the number of computed components.

2.2 The higher-dimensional dispersive equation [2]

The linear, third-order dispersive partial differential equation in a higher dimensional space may be defined by

$$u_t + au_{xxx} + bu_{yyy} + cu_{zzz} = \tilde{g}(x,y,z,t), \quad \mathrm{L}_0 < x,y,z < \mathrm{L}_1,\ t > 0,\ a,b,c > 0, \tag{10}$$

where $\tilde{g}(x,y,z,t)$ is a source term.
The initial condition is$u(x,y,z,0) = \tilde{f}(x,y,z)$,

and the time-dependent boundary conditions associated with (10) are assumed to be prescribed.

The solution $u(x,y,z,t)$ by an infinite series of components

$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t)$,(11)

where the components $u_n(x,y,z,t)$ will be determined by using a recurrence relation. Substituting (11) into both sides of (5) yields

$$\sum_{n=0}^{\infty} u_n = \tilde{f} + \mathrm{L}^{-1}(\tilde{g}) - \mathrm{L}^{-1}\left(a\left(\sum_{n=0}^{\infty} u_n\right)_{xxx} + b\left(\sum_{n=0}^{\infty} u_n\right)_{yyy} + c\left(\sum_{n=0}^{\infty} u_n\right)_{zzz}\right). \tag{12}$$

The first few components are therefore given by

$$\begin{aligned} u_0(x,y,z,t) &= \tilde{f}(x,y,z) + \mathrm{L}^{-1}\big(\tilde{g}(x,y,z,t)\big) = h(x,y,z,t),\\ u_1(x,y,z,t) &= -\mathrm{L}^{-1}\left(au_{0_{xxx}} + bu_{0_{yyy}} + cu_{0_{zzz}}\right),\\ u_2(x,y,z,t) &= -\mathrm{L}^{-1}\left(au_{1_{xxx}} + bu_{1_{yyy}} + cu_{1_{zzz}}\right),\\ u_3(x,y,z,t) &= -\mathrm{L}^{-1}\left(au_{2_{xxx}} + bu_{2_{yyy}} + cu_{2_{zzz}}\right), \end{aligned} \tag{13}$$

so that other components can be obtained in a like manner.

we use the modified recurrence relation

$$\begin{aligned} u_0(x,y,z,t) &= h_1(x,y,z,t),\\ u_1(x,y,z,t) &= h_2(x,y,z,t) - \mathrm{L}^{-1}\left(au_{0_{xxx}} + bu_{0_{yyy}} + cu_{0_{zzz}}\right),\\ u_2(x,y,z,t) &= -\mathrm{L}^{-1}\left(au_{1_{xxx}} + bu_{1_{yyy}} + cu_{1_{zzz}}\right),\\ u_3(x,y,z,t) &= -\mathrm{L}^{-1}\left(au_{2_{xxx}} + bu_{2_{yyy}} + cu_{2_{zzz}}\right). \end{aligned} \tag{14}$$

Although the modification introduces a slight change in the definition of the components u_0 and u_1, but it provides a qualitative effect in accelerating the convergence of the solution.

2.3 Differential-Difference Algorithm and Convergence [3]

Kudu and Amirali constructed first order accurate differential difference scheme for (15) and give an error estimate for its solutions.

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x\, \partial y^2} = f(x, y),$$
$$u(x, 0) = \psi_1(x), u(x, q) = \psi_2(x),$$
$$u(0, y) = g_1(y), u(p, y) = g_2(y), \quad (15)$$
$$\frac{\partial u}{\partial x}(0, y) = g_3(y),$$

where $\psi_1(x), \psi_2(x), g_1(y), g_2(y), g_3(y)$ are sufficiently smooth functions, and the domain are $\Omega\{0 < x < p, 0 < y < q\}$.

After dividing the domain into an *n +1*, approximate the equations by the differential difference problem, obtaining a third-order ordinary differential equation with constant coefficients, find the approximate error solution, respectively.

After that, they used here the inequality

$$\sin x > \frac{2}{\pi} x \left(0 < x < \frac{\pi}{2}\right), (16)$$

and taking into account

$$\sum_{k=1}^{n} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, (17)$$

it follows that

$$|z_k(x)| \leq \frac{C_1 p h^6}{24} (n + 1)^2 \sum_{s=1}^{n} \frac{1}{(n+1-s)^2} \leq \frac{C_1 \pi^2 p q^2}{144} h^4, (18)$$

i.e., fourth order convergence for the approximate solution is established.

2.4 Exponential quartic spline [4]

The exponential quartic spline at the grid point (x_j, t) given by

$$E_j(x, t) = a_{1j}(t) e^{\tau(x - x_j)} + a_{2j}(t) e^{-\tau(x - x_j)} + a_{3j}(t)(x - x_j)^2 + a_{4j}(t)(x - x_j) + a_{5j}(t), (19)$$

for each $j = 0,1, \dots, n$, where $a_{1j}, a_{2j}, a_{3j}, a_{4j}, a_{5j}$ are unknown coefficients and τ is a free parameter.

By applying the continuity conditions of first and second derivatives of (19),

they obtain the following method:

$$h^3(pT_{j+1} + qT_j + qT_{j-1} + pT_{j-2}) = -y_{j+1} + 3y_j - 3y_{j-1} + y_{j-2}, \quad j = 2(1)(n - 1), (20)$$

where the coefficients $p = \beta_1$ and $q = -\alpha_1 + \beta_1 + \beta_2$. As $\tau \to 0$ that is $\theta \to 0$, we have $(p, q) \longrightarrow \left(-\frac{1}{24}, -\frac{11}{24}\right)$. Now, the operator Λ_x for any function W is supposed to have the following form according to Eq. (20):

$$\Lambda_x W_j = pW_{j+1} + qW_j + qW_{j-1} + pW_{j-2}, (21)$$

2.5 Spline solution for linear dispersive equation [4]

Equation (1) is discretized as:

$$\frac{k^{-1}}{2} \delta_t (1 + \sigma \delta_t^2)^{-1} y_j^m + \mu T_j^m = f_j^m, (22)$$

where $T_j^m = E_\Delta^{(3)}(x_j, t_m)$ is the third order spline derivative at (x_j, t_m) w.r.t. the space variable, $f_j^m = f(x_j, t_m), y_j^m$ is the approximate solution of (1) at $(x_j, t_m), \delta_t$ is the central difference operator w.r.t. t and σ is a parameter such that finite difference approximation to the time derivative is of $O(k)$ for arbitrary σ.

Operating Λ_x on both sides of (22) and after some simplifications, we obtain the following method:

$$\delta_t\left(py_{j+1}^m + qy_j^m + qy_{j-1}^m + py_{j-2}^m\right) + \frac{2k\mu}{h^3}(1+\sigma\delta_t^2)\left(-y_{j+1}^m + 3y_j^m - 3y_{j-1}^m + y_{j-2}^m\right)$$
$$= 2k(1+\sigma\delta_t^2)\left(pf_{j+1}^m + qf_j^m + qf_{j-1}^m + pf_{j-2}^m\right), \quad j = 2(1)(n-1). \qquad (23)$$

They discretized the boundary conditions in (2) and develop the following boundary equation of accuracy $O(k+h^2)$:

$-21y_0^m + 24y_1^m - 3y_2^m - 18h(y_0^m)' - 6h^2(y_0^m)'' = 0, \quad j = 1,$(24)

where

$y_0^m = y(a, t_m), \quad (y_0^m)' = \frac{\partial y}{\partial x}(a, t_m), \quad (y_0^m)'' = \frac{\partial^2 y}{\partial x^2}(a, t_m).$(25)

2.6 Spline solution for non-linear dispersive equation [4]

The third order non-linear dispersive equation named as Kortewegde Vries (KdV) equation:

$\frac{\partial y(x,t)}{\partial t} + \varepsilon y(x,t)\frac{\partial y(x,t)}{\partial x} + \mu\frac{\partial^3 y(x,t)}{\partial x^3} = 0, \quad a \le x \le b, t > 0, \mu > 0,$ (26)

with

$y(x,0) = g_2(x), \quad a \le x \le b,$(27)

and

$$\left.\begin{array}{ll} y(x,t) = \gamma_3(t), & x \in \partial\Omega, t > 0, \\ y_x(b,t) = \gamma_4(t), & t > 0, \end{array}\right\} (28)$$

where $\Omega = [a,b] \subset R, \varepsilon$ and μ are positive parameters, and $g_2(x), \gamma_3(t), \gamma_4(t)$ are known functions.

Eq. (29) is discretized as follows:

$\frac{k^{-1}}{2}\delta_t(1+\sigma\delta_t^2)^{-1}y_j^m + \frac{\delta_x}{2h}F_j^m + \mu T_j^m = 0,$(29)

where $F = \frac{\varepsilon}{2}y^2$.
Operating Λ_x on both sides of (29) and after some simplifications, they obtain the following method:

$$\delta_t\left(py_{j+1}^m + qy_j^m + qy_{j-1}^m + py_{j-2}^m\right) + \frac{2k\mu}{h^3}(1+\sigma\delta_t^2)\left(-y_{j+1}^m + 3y_j^m - 3y_{j-1}^m + y_{j-2}^m\right)$$
$$+ \frac{k}{h}(1+\sigma\delta_t^2)\left(pF_{j+2}^m + qF_{j+1}^m - (p-q)F_j^m + (p-q)F_{j-1}^m - qF_{j-2}^m - pF_{j-3}^m\right) \quad ,(30)$$
$$= 0, \quad j = 3(1)(n-2).$$

They discretize the boundary conditions in (28) and develop the following boundary equation of accuracy $O(k+h^2)$:

$$\left.\begin{array}{ll}-y_0^m + 4y_1^m - 6y_2^m + 4y_3^m - y_4^m = 0, & j = 1,\\ -y_1^m + 4y_2^m - 6y_3^m + 4y_4^m - y_5^m = 0, & j = 2,\\ -\frac{4}{3}y_{n-3}^m + 6y_{n-2}^m - 12y_{n-1}^m + \frac{22}{3}y_n^m - 4h(y_n^m)' = 0, & j = (n-1),\end{array}\right\}, (31)$$

where

$$y_0^m = y(a, t_m), \quad y_n^m = y(b, t_m), \quad (y_n^m)' = \frac{\partial y}{\partial x}(b, t_m).(32)$$

2.7 Analysis of the Method [5]

The non-polynomial spline function given by:

$$\begin{aligned} P_i(x, t_j) = \; & a_i(t_j)\cos\omega(x - x_i) + b_i(t_j)\sin\omega(x - x_i) + c_i(t_j)(x - x_i)^2 \\ & + d_i(t_j)(x - x_i) + e_i(t_j), \end{aligned} \quad (33)$$

for each $i = 0,1, \cdots, N$.

Using the continuity condition of the first and second derivatives, and after some simplifications, then this equation can be rewritten in the following simple form:

$$-Z_{i-2}^j + 3Z_{i-1}^j - 3Z_i^j + Z_{i+1}^j = \alpha S_{i-2}^j + \beta S_{i-1}^j + \beta S_i^j + \alpha S_{i+1}^j, i = 2, \cdots, N.(34)$$

where:

$$\alpha = h^3\left(\frac{\cos\theta - 1}{\theta^3\sin\theta} + \frac{1}{2\theta\sin\theta}\right) \text{ and } \beta = h^3\left(\frac{1-\cos\theta}{\theta^3\sin\theta} - \frac{\cos\theta}{\theta\sin\theta} + \frac{1}{2\theta\sin\theta}\right),$$

As $\omega \to 0$, that is $\theta \to 0$, then $(\alpha, \beta) \to \left(\frac{h^3}{24}, \frac{11h^3}{24}\right)$, and system (34) reduces to ordinary quartic spline:

$$A_i Z_{i-2}^j + B_i Z_{i-1}^j + C_i Z_i^j + D_i Z_{i+1}^j = \alpha Z_{i-2}^{j-1} + \beta Z_{i-1}^{j-1} + \beta Z_i^{j-1} + \alpha Z_{i+1}^{j-1} + \delta_i^j,$$
$i = 2, \ldots, N$. (35)

where
$A_i = -k + \alpha$, $B_i = 3k + \beta, C_i = -3k + \beta, D_i = k + \alpha$

and $\delta_i^j = k(\alpha g_{i-2}^j + \beta g_{i-1}^j + \beta g_i^j + \alpha g_{i+1}^j)$.

Or

$$-Z_{i-2}^j + 3Z_{i-1}^j - 3Z_i^j + Z_{i+1}^j = \frac{h^3}{24}(S_{i-2}^j + 11S_{i-1}^j + 11S_i^j + S_{i+1}^j),(36)$$

III. Truncation error and a class of methods [4-5]

Expanding (30) in a Taylor series in terms of $y(x_j, t_m)$ and its derivatives and using (1) respectively, they obtain the truncation error as follows [4]:

$$
\begin{aligned}
TE_j^m = \Big[& 2(p+q)kD_t - (p+q)khD_tD_x + \frac{1}{2}(5p+q)kh^2D_tD_x^2 \\
& -\frac{1}{6}(7p+q)kh^3D_tD_x^3 + \left(\frac{1}{3}-\sigma\right)2(p+q)k^3D_t^3 - \left(\frac{1}{3}-\sigma\right)(p+q)k^3hD_t^3D_x \\
& +\frac{1}{2}\left(\frac{1}{3}-\sigma\right)(5p+q)k^3h^2D_t^3D_x^2 - \frac{1}{6}\left(\frac{1}{3}-\sigma\right)(7p+q)k^3h^3D_t^3D_x^3 \\
& -2(p+q)kD_x^3 + (p+q)khD_x^4 + \frac{1}{2}(5p+q)kh^2D_x^5 + \frac{1}{6}(7p+q)kh^3D_x^6 \\
& -2(p+q)\sigma k^3D_t^2D_x^3 + (p+q)\sigma k^3hD_t^2D_x^4 - \frac{1}{2}(5p+q)\sigma k^3h^2D_t^2D_x^5 \\
& +\frac{1}{6}(7p+q)\sigma k^3h^3D_t^2D_x^6 - kh^2D_x^3 + \frac{1}{2}kh^3D_x^4 - \sigma k^3h^2D_t^2D_x^3 \\
& +\frac{1}{2}\sigma k^3h^3D_t^2D_x^4 + \cdots\cdots \Big] y_j^m, \qquad (37)
\end{aligned}
$$

where $D_t \equiv \frac{\partial}{\partial t}, D_x \equiv \frac{\partial}{\partial x}, D_t^2 \equiv \frac{\partial^2}{\partial t^2}, D_x^2 \equiv \frac{\partial^2}{\partial x^2}$, and so on.
Here, the following class of methods are obtained:
Case 1. If $p+q \neq 0$, then various methods of $O(k+h)$ for arbitrary values of σ are obtained.

Case 2. If $p+q = 0$, then various methods of $O(k+h^2)$ for arbitrary values of σ are obtained.

In the other hand the local truncation error by Zaki et al. can be simplified as:

$$
\begin{aligned}
T_i^j = & -k\left(-\frac{2\square}{1!}D_x + \frac{(2\square)^2}{2!}D_x^2 - \cdots\right)\eta_i^j + 3k\left(-\square D_x + \frac{\square^2}{2!}D_x^2 - \cdots\right)\eta_i^j + \\
& k\left(\square D_x + \frac{\square^2}{2!}D_x^2 + \frac{\square^3}{3!}D_x^3 + \cdots\right)\eta_i^j - 2(\beta+\alpha)\left(-kD_t + \frac{k^2}{2!}D_t^2 - \frac{k^3}{3!}D_t^3 + \cdots\right)\eta_i^j + \\
& (\beta+\alpha)\square\left(\frac{-1}{2!}\binom{2}{1}kD_t + \frac{1}{3!}\binom{3}{1}k^2D_t^2 - \cdots\right)D_x\eta_i^j + \\
& (\beta+5\alpha)\square^2\left(\frac{1}{3!}\binom{3}{2}kD_t - \frac{1}{4!}\binom{4}{2}k^2D_t^2 + \cdots\right)D_x^2\eta_i^j + \\
& (\beta+7\alpha)\square^3\left(\frac{-1}{4!}\binom{4}{3}kD_t + \frac{1}{5!}\binom{5}{3}k^2D_t^2 - \cdots\right)D_x^3\eta_i^j + \cdots - \\
& k\alpha\left(-2\square D_x + \frac{(2\square)^2}{2!}D_x^2 - \cdots\right)(D_t + D_x^3)\eta_i^j - k\beta\left(-\square D_x + \frac{\square^2}{2!}D_x^2 - \cdots\right)(D_t + D_x^3)\eta_i^j - \\
& k\alpha\left(\square D_x + \frac{\square^2}{2!}D_x^2 + \cdots\right)(D_t + D_x^3)\eta_i^j - 2k(\beta+\alpha)(D_t + D_x^3)\eta_i^j.
\end{aligned}
$$

$$
\begin{aligned}
T_i^j = & k\left(\square^3 - 2(\beta+\alpha)\right)D_x^3\eta_i^j + k\square\left(\frac{-\square^3}{2} + (\beta+\alpha)\right)D_x^4\eta_i^j \\
& +k\square^2\left(\frac{\square^3}{4} - \frac{1}{2}(\beta+5\alpha)\right)D_x^5\eta_i^j + k\square^3\left(\frac{-\square^3}{12} + \frac{1}{6}(\beta+7\alpha)\right)D_x^6\eta_i^j \\
& +k\square^4\left(\frac{\square^3}{40} - \frac{1}{24}(\beta+17\alpha)\right)D_x^7\eta_i^j + \cdots + \\
& 2(\beta+\alpha)\left(-\frac{k^2}{2!}D_t^2 + \frac{k^3}{3!}D_t^3 - \cdots\right)\eta_i^j \\
& + (\beta+\alpha)\square\left(\frac{1}{3!}\binom{3}{1}k^2D_t^2 - \frac{1}{4!}\binom{4}{1}k^3D_t^3 + \cdots\right)D_x\eta_i^j \\
& +(\beta+5\alpha)\square^2\left(-\frac{1}{4!}\binom{4}{2}k^2D_t^2 + \frac{1}{5!}\binom{5}{2}k^3D3_t - \cdots\right)D_x^2\eta_i^j \\
& +(\beta+7\alpha)\square^3\left(\frac{1}{5!}\binom{5}{3}k^2D_t^2 - \frac{1}{6!}\binom{6}{3}k^3D_t^3 + \cdots\right)D_x^3\eta_i^j + \ldots \qquad (38)
\end{aligned}
$$

For $\beta + \alpha = \frac{h^3}{2}$, the local truncation error is of order $o(kh^2 + k^2h^3)$ but for $\beta + \alpha = \frac{h^3}{2}$, *and* $\alpha = 0$ it is of $o(kh^4 + k^2h^3)$.

IV. Stability analysis and convergence [4-5]

Using the Von Neumann method, the stability of the method can be investigated. According to this method, the solution of the difference equation (39)can be written in the form

$Z_i^j = \zeta^j \, exp(q\phi i □),$ (39)

where φ is the wave number, $q = \sqrt{-1}$, h is the element size, and ζ^j is the amplification factor at time level j. We obtain the characteristic equation in the form:

$$\zeta^j\{A_i \, exp((i-2)q\phi h) + B_i \, exp\big((i-1)q\phi h\big) + C_i \, exp(iq\phi h) + D_i \, exp\big((i+1)q\phi h\big)\} = \zeta^{j-1}\{\alpha \, exp((i-2)q\phi h) + \beta \, exp\big((i-1)q\phi h\big) + \beta \, exp(iq\phi h) + \alpha \, exp\big((i+\ 1)q\phi h\big)\}.$$
$$A_i = -k + \alpha, \; B_i = 3k + \beta, C_i = -3k + \beta, A_i = k + \alpha.$$

Equation (39) becomes

$\zeta = \frac{X^* + qY^*}{X + qY},$ (40)

where

$X^* = \alpha \cos 2\,\varphi + (\beta + \alpha)\cos\varphi + \beta,$
$Y^* = -\alpha \sin 2\,\varphi + (\alpha - \beta)\sin\varphi,$
$X = (\alpha - k)\cos 2\,\varphi + (\beta + \alpha + 4k)\cos\varphi + (\beta - 3k),$
$Y = (k - \alpha)\sin 2\,\varphi + (-2k + \alpha - \beta)\sin\varphi.$

For stability, we must have $|\zeta| \le 1$ (otherwise ζ^j in (40) would grow in an unbounded manner). Using equation (40), we can say that the stability condition, that is $|\zeta| \le 1$, is satisfied.

In the other hand, Sultana et al. by (39)get the following equation:

$U\xi^2 + V\xi + W = 0,$ (41)

where

$$U = Pe^{i\theta} + Q + Re^{-i\theta} + Se^{-2i\theta}$$
$$V = N\big(e^{i\theta} + 3 - 3e^{-i\theta} + e^{-2i\theta}\big)$$
$$W = -\big(Se^{i\theta} + R + Qe^{-i\theta} + Pe^{-2i\theta}\big)$$

The necessary and sufficient condition to be stable is $|\xi| \le 1$. For this, we obtain the following condition:

$$(4N\sin^3\phi)/((p+q)^2 - 4(10pq + 9p^2 + q^2 + 12\sigma^2 r^2)\sin^2\phi + 16(4p^2 + 2(3p - q)\sigma r + 8\sigma^2 r^2)\sin^4\phi - 32(pq - p^2 - (3p - q)\sigma r - 2\sigma^2 r^2)\sin^6\phi)^{1/2} \le 1.$$

Simplifying and putting $p + q = 0$, we deduce the method is conditionally stable for $\sigma \ge \left(\frac{1}{2} - \frac{1}{2r}\right)$, where $r > 0$ and $\phi = \frac{\theta}{2}$.

The present method is convergent by Lax theorem as the stability criterion is satisfied.

V. Numerical Examples [2, 4-5]:

In this section, we obtain numerical solutions of equation (1) for a numerical example. Consider the non-homogeneous third-order dispersive partial differential equation

$\frac{\partial\eta}{\partial t} + \frac{\partial^3\eta}{\partial x^3} = -sin(\pi x)sin\, t - \pi^3 cos(\pi x)cos\, t, \quad 0 \le x \le 1, t \ge 0,$ (42)

with boundary conditions

$$\eta(0,t) = \eta(1,t) = \eta_{xx}(1,t) = 0, \quad t > 0,$$

and the initial condition

$$\eta(x,0) = \sin \pi x, \quad 0 \le x \le 1$$

The exact solution of this problem is

$$\eta(x,t) = \sin \pi x \cos t$$

5.1 The Results of [2]:

Following the analysis gives the recurrence relation

$$u_0(x,t) = \sin(\pi x)\cos t - \pi^3\cos(\pi x)\sin t,$$

$$u_{k+1}(x,t) = -\mathrm{L}^{-1}\left(u_{k_{xx}}\right), \quad k \geqslant 0.$$

This relation enables us to determine first few components as follows:

$$u_0(x,t) = \sin(\pi x)\cos t - \pi^3\cos(\pi x)\sin t$$

$$u_1(x,t) = -\mathrm{L}^{-1}\left(u_{0_{xxx}}\right) = \pi^3\cos(\pi x)\sin t - \pi^6\sin(\pi x)\cos t + \pi^6\sin(\pi x)$$

It is easily observed that the noise terms $-\pi^3\cos(\pi x)\sin t$ and $\pi^3\cos(\pi x)\sin t$ appear in the components u_0 and u_1 respectively. As stated before, canceling this term from u_0, and justifying that the remaining non-canceled term justifies the dispersive equation (42), we obtain the exact solution

$$u(x,t) = \sin(\pi x)\cos t.$$

This confirms our belief that noting the appearance of the noise terms, if exist, will accelerate the convergence of the solution.

In what follows, the modified decomposition method presented before in the analysis will be employed to confirm the power of this method in accelerating the convergence of the solution.

5.2 The Results of [4]:

The computational results of this example for$\mu = 1$are tabulated in Table 1. and Table 2.

Table 1 shows$L_\infty, L_2$and RMS errors for$h = \frac{1}{20}, \frac{1}{40}; r = \frac{1}{100}, \sqrt{\frac{7}{60}}; \sigma = \frac{1}{12}$and time steps $= 50{,}100$ for different values of parameters p and q.

(p,q,σ)	r	*Time steps*	$h = \frac{1}{20}$			$h = \frac{1}{40}$		
			L_∞	L_2	*RMS*	L_∞	L_2	*RMS*
$(25,-25,\frac{1}{12})$	$\frac{1}{100}$	50	***6.4058(−6)***	***1.4539(−5)***	***3.3356(−6)***	***5.2848(−6)***	***2.0991(−5)***	***3.3613(−6)***
	$\sqrt{\frac{7}{6}}$		***1.4202(−4)***	***3.5828(−4)***	***8.2196(−5)***	***1.8686(−4)***	***7.2452(−4)***	***1.1601(−4)***
	$\frac{1}{100}$	100	***6.4058(−6)***	***1.4539(−5)***	***3.3356(−6)***	***5.2848(−6)***	***2.0991(−5)***	***3.3613(−6)***
	$\sqrt{\frac{7}{60}}$		***1.4202(−4)***	***3.5828(−4)***	***8.2196(−5)***	***1.8686(−4)***	***7.2452(−4)***	***1.1601(−4)***
$(30,-30,\frac{1}{12})$	$\frac{1}{100}$	50	***9.3013(−6)***	***2.1326(−5)***	***4.8926(−7)***	***3.9930(−6)***	***1.5921(−5)***	***2.5494(−6)***
	$\sqrt{\frac{7}{60}}$		***2.4187(−4)***	***6.1364(−4)***	***1.4078(−4)***	***1.4128(−4)***	***5.4912(−4)***	***8.7930(−5)***
	$\frac{1}{100}$	100	***9.3013(−6)***	***2.1326(−5)***	***4.8926(−7)***	***3.9930(−6)***	***1.5921(−5)***	***2.5494(−6)***
	$\sqrt{\frac{6}{60}}$		***2.4187(−4)***	***6.1364(−4)***	***1.4078(−4)***	***1.4128(−4)***	***5.4912(−4)***	***8.7930(−5)***

Table1.L_∞, L_2 and RMS errors for(42)

Table2. L_∞ errors for(42)

Time steps	*r*	*h*	*x*	(p,q,σ) $(25,-25,\frac{1}{12})$	(p,q,σ) $(50,-50,\frac{1}{12})$
100	*1*	*0.05*	*0.1*	*2.19(−4)*	*6.94(−4)*
			0.3	*2.72(−4)*	*8.63(−4)*
			0.5	*1.28(−6)*	*1.58(−6)*
			0.7	*2.74(−4)*	*8.66(−4)*
			0.9	*2.20(−4)*	*6.96(−4)*
		0.1	*0.1*	*1.20(−3)*	*1.37(−3)*
			0.3	*2.81(−3)*	*3.29(−3)*
			0.5	*6.59(−3)*	*7.21(−3)*
			0.7	*1.60(−2)*	*1.77(−2)*
			0.9	*1.43(−2)*	*1.57(−2)*

Where $L_\infty = \max_{1\le i\le n} \left|y_{\text{ana}}(i) - y_{\text{app}}(i)\right|$,

$L_2 = \sqrt{\sum_{i=1}^{n}\left(y_{\text{ana}}(i) - y_{\text{app}}(i)\right)^2}$, $RMS = \sqrt{\left(\sum_{i=1}^{n}\left(y_{\text{ana}}(i) - y_{\text{app}}(i)\right)^2\right)/n}$,

y_{ana} is analytical and y_{app} is approximate solution of third order dispersive equation for our method.

Figure 1. shows the graphical comparison between analytical and approximate solution for $h = \frac{1}{64}, r = \frac{1}{100}$ and time steps $= 100$.

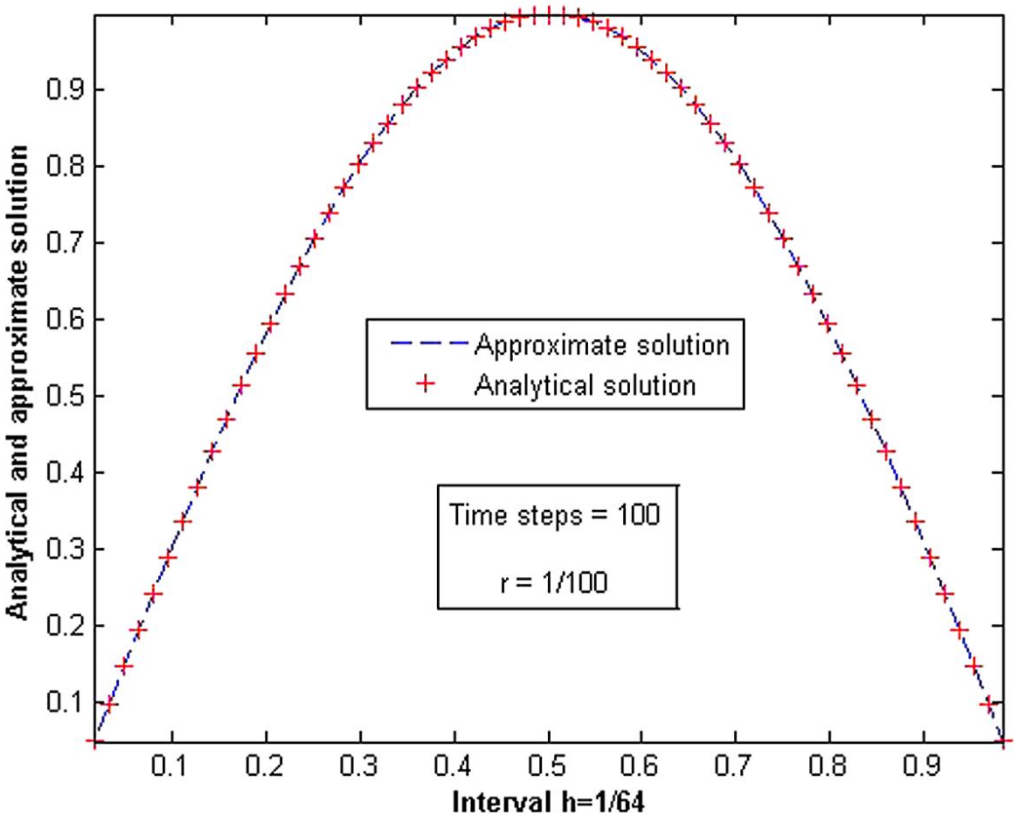

Figure 1. Comparison between analytical and approximate solution

5.3 The Results of [5]:

The obtained numerical results are listed in the tables below, where all calculations are carried out using Mathematica. The accuracy of method is measured by computing L∞ - error norm, Max. Absolute error, as shown in Table 3. Table 4. illustrate numerical and exact solutions for $h = 0.025, k = 0.0005, and\ \beta = -\alpha + \frac{h^3}{2}$.

Table 3.$h = 0.025, k = 0.0005, \alpha = 0, and\ \beta = -\alpha + \frac{h^3}{2}$.

Time	**0.500**	**1.500**	**2.00**	**2.500**
$L_\infty error$	4.59312×10^{-6}	5.05911×10^{-7}	2.01782×10^{-6}	4.047×10^{-6}

The reason that the accuracy in Table 3. is the best is because

For$\alpha = 0, \beta = -\alpha + \frac{h^3}{2}$, the local truncation error is of order $o(kh^4 + k^2h^3)$ but for $\beta + \alpha = \frac{h^3}{2}$, it is of order$o(kh^2 + k^2h^3)$.

Table 4.$h = 0.025, t = 2, \alpha = \frac{h^3}{160}, and\ \beta = -\alpha + \frac{h^3}{2}$.

x	**Exact Solution**	**Numerical Solution**
0.1 π	-0.128596	-0.129126
0.2 π	-0.244605	-0.2445870
0.3 π	-0.336669	-0.3366450
0.4 π	-0.395779	-0.3957500
0.5 π	-0.416147	-0.4161160
0.6 π	-0.395779	-0.3957500
0.7 π	-0.336669	-0.3366450
0.8 π	-0.244605	-0.2445870
0.9 π	-0.128596	-0.129126

The following figure show the relation between the numerical and exact solutions of the dispersive equation for virus time and the same discretization's (***h***).

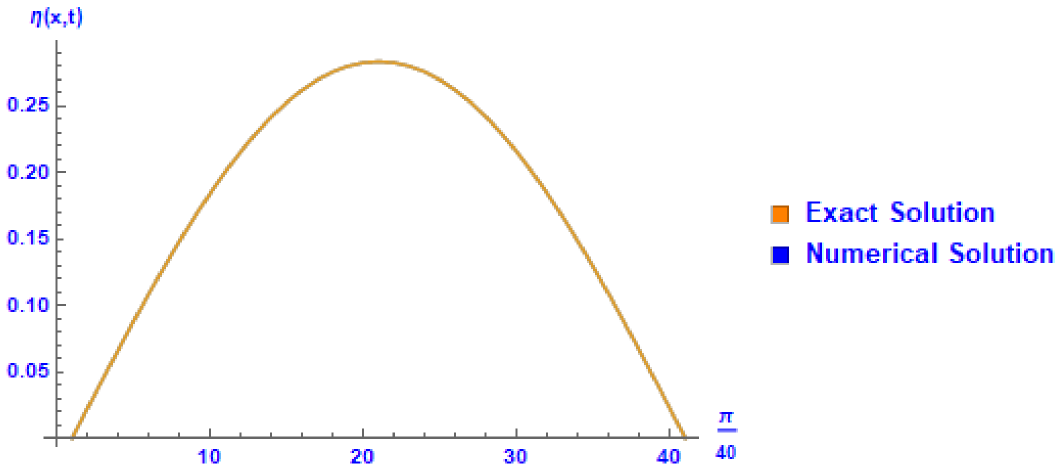

Figure2. the relation between the numerical and exact solutions of the dispersive equation at

$$h = 0.025, k = 0.0005, \alpha = \frac{h^3}{160}, \beta = -\alpha + \frac{h^3}{2}, and\ t = 5.0.$$

The following figureshow the 3D of the numerical solutions of the dispersive equation for virus time and the same discretization's (*h*).

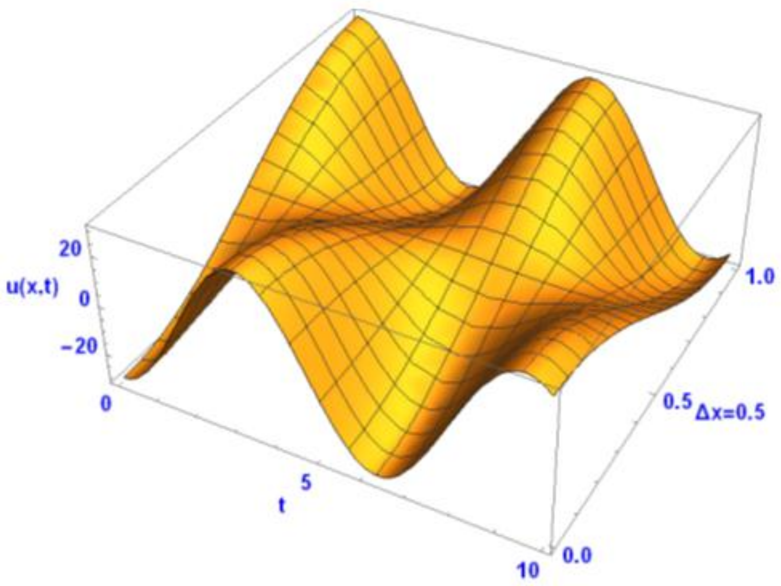

Figure3. 3D for the numerical solutions of the dispersive, the time from *t=0.00 to t=10.0*

VI. Conclusion

In this article, we discussed the Adomian decomposition method, the lines method, anexponential quartic spline and finite difference discretization method, and the non-polynomial spline method for solving the third order of dispersive partial differential equation and its truncation errors. The stability analysis of these methods was shown to be conditionally stable. Furthermore, the obtained approximate numerical solutions maintaingood accuracy compared with the exact solutions, especially for small values.The results obtained by [4] and [5] demonstrate that solving the dispersive partial differential equation using the non-polynomial spline method is more accurate than using the Adomian decomposition method and an exponential quartic spline and finite difference methods.A large set of values was used to treat the third order of dispersive partial differential equation using the non-polynomial spline method, and both 2D and 3D graph representations were provided.Our conclusion was that the non-polynomial spline method is more useful than the Adomian decomposition method and an exponential quartic spline and finite difference methods when the researcher wants to obtain stability and convergent, for solving the third order of the dispersive partial differential equation

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