Quest Journals
Journal of Research in Applied Mathematics
Volume 6 ~ Issue 3 (2020) pp: 06-11
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Analytical solution of the fractional and global stability of multicompartment non-linear epidemic model.

LAID Chahrazed
Department of Mathematics, Faculty of Exacte sciences University Freres Mentouri, Constantine 1, Algeria.

Abstract
In this paper, the Multicompartment epidemiological model assumes that, given a contagious illness, a population can be partitioned into individuals that are susceptible to the illness, infected by the illness, and recovered from the illness. S(t) Number of individuals at time t susceptible to the illness; I(t);i = 1,2,3,4 Number of individuals at time t infected with the illness.$R_S(t)$ Total number of survivors of the illness at time t , $R_D(t)$ Total number of deaths due to the illness at time t.
The stability of a disease-free status equilibrium and the existence of endemic equilibrium can be determine by the ratio called the basic reproductive number. Laplace-Adomian decomposition method is used to compute an analytical solution of the model study. This paper study the equilibrium, local, global stability under certain conditions.
Keywords: *Endemic equilibrium, epidemic model, global stability, lyapunov function, temporary immunity. 2008 MSC: (34D23, 92D30, 37N25)*

Received 07July, 2020; Accepted 22 July, 2020

I. INTRODUCTION

This paper considers the following epidemic model:

$$\begin{cases} \dot{S}(t) = \nu - \rho + \mu\left(S(t) + \sum_{i=1}^{4} I_i(t)\right) - \sum_{i=1}^{4} \beta_i I_i(t) S(t) \\ \dot{I}_1(t) = \sum_{i=1}^{4} \beta_i I_i(t) S(t) - \gamma_1 I_1(t), \\ \dot{I}_2(t) = \gamma_1 I_1(t) - \gamma_2 I_2(t) - \delta_2 I_2(t), \\ \dot{I}_3(t) = \gamma_2 I_2(t) - \gamma_3 I_3(t) - \delta_3 I_3(t), \\ \dot{I}_4(t) = \gamma_3 I_3(t) - \gamma_4 I_4(t) - \delta_4 I_4(t), \\ \dot{R}_S(t) = \gamma_4 I_4(t), \\ \dot{R}_D(t) = \delta_2 I_2(t) + \delta_3 I_3(t) + \delta_4 I_4(t). \end{cases} \quad (1)$$

This epidemiological model assumes that, given a contagious illness, a population partitioned into individuals that are susceptible to the illness, infected by the illness, and recovered from the illness.

- $S(t)$Number of individuals at time t susceptible to the illness;
- $I(t);i$ = 1,2,3,4 Number of individuals at time t infected with the illness.
- $R_S(t)$ Total number of survivors of the illness at time t , $R_D(t)$ Total number of deaths due to the illness at time t.
- The positive constant β_i;i=1,2,3,4, represent the rate at which individuals of the illness cause neighboring susceptible. γ_i,i=1,2,3,4, represent the rate at which individuals in infection .
- The positive constant δ_i,i=1, 2, 3, 4, represent the rate of death due to the illness. The positive constant ν is the parameter of emigration. The positive constant ρ is the parameter of Immigration.
- The positive constant μ represent rate of incidence.

The initial condition of (1) is given as

$$S(\eta) = \Phi_1(\eta),\ I_1(\eta) = \Phi_2(\eta),\ I_2(\eta) = \Phi_3(\eta),\ I_3(\eta) = \Phi_4(\eta) \quad (2)$$
$$I_4(\eta) = \Phi_5(\eta),\ R_S(\eta) = \Phi_6(\eta),\ R_D(\eta) = \Phi_7(\eta);$$
$$-\tau \le \eta \le 0,$$

Where $\Phi = (\Phi 1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \Phi_7)^T \in C$ such that;

$$S(\eta) = \Phi_1(\eta) \ge 0,\ I_1(\eta) = \Phi_2(\eta) \ge 0,\ I_2(\eta) = \Phi_3(\eta) \ge 0,\ I_3(\eta) = \Phi_4(\eta) \ge 0\ I_4(\eta) = \Phi_5(\eta) \ge 0,$$
$$R_S(\eta) = \Phi_6(\eta) \ge 0, R_D(\eta) = \Phi_7(\eta) \ge 0.$$

Let C denote the Banach space $C([-\tau, 0], R^7)$ of continuous functions mapping the interval $[-\tau, 0]$ into R^7. With a biological meaning, we further assume that $\Phi_i(\eta) = \Phi_i(0) \ge 0$ for $i = 1,2,3,4,5,6,7$.
Hence, system (1) is rewritten as

$$\begin{cases} \dot{S}(t) = \nu - \rho + \mu\left(S + \sum_{i=1}^{4} I_i\right) - \sum_{i=1}^{4} \beta_i S I_i \\ \dot{I}_1(t) = \sum_{i=1}^{4} \beta_i I_i S - \gamma_1 I_1, \\ \dot{I}_i(t) = \gamma_{i-1} I_{i-1} - (\gamma_i + \delta_i) I_i, i = 2,3,4, \\ \dot{R}_S(t) = \gamma_4 I_4, \\ \dot{R}_D(t) = \sum_{i=2}^{4} \delta_i I_i. \end{cases} \quad (3)$$

With the initial conditions in (2).
We study the following reduced system:

$$\begin{cases} \dot{S}(t) = \nu - \rho + \mu\left(S + \sum_{i=1}^{4} I_i\right) - \sum_{i=1}^{4} \beta_i S I_i \\ \dot{I}_1(t) = \sum_{i=1}^{4} \beta_i I_i S - \gamma_1 I_1, \\ \dot{I}_i(t) = \gamma_{i-1} I_{i-1} - (\gamma_i + \delta_i) I_i, i = 2,3,4, \end{cases} \quad (4)$$

Where ;

$$\Phi_i(0) \ge 0, -\tau \le \eta < 0; \textit{for}\text{"}i = [1, 7]\text{"}. \quad (5)$$

II. EQUILIBRIUM AND STABILITY

An equilibrium point of system (4), with condition (5) satisfies,

$$\begin{cases} \nu - \rho + \mu\left(S + \sum_{i=1}^{4} I_i(t)\right) - \sum_{i=1}^{4} \beta_i S I_i = 0 \\ \sum_{i=1}^{4} \beta_i I_i S - \gamma_1 I_1 = 0, \\ \gamma_{i-1} I_{i-1} - (\gamma_i + \delta_i) I_i = 0, i = 2,3,4, \end{cases} \quad (6)$$

We calculate the points of equilibrium in the absence and presence of infection.
In the absence of infection $I_i = 0$, i=1, 2, 3, 4; the system (4) has a disease-free equilibrium E_0.

$$E_0 = \left(\hat{S}, \hat{I}_1, \hat{I}_i\right)^T = \left(\frac{\nu - \rho}{\mu}, 0, 0\right)^T, i = 2,3,4 \quad (7)$$

Theorem 2.1. *The disease-free equilibrium E_0 of the system* (4) *is locally asymptotically stable if $R_0 < 1$.*
Proof. The eigenvalues can be determined by solving the characteristic equation of the linearization of (4) near E_0.

$$A^2 + A\left[(\gamma_2 + \delta_2) + \left(\gamma_1 + \beta_1 \frac{\nu - \rho}{\mu}\right)\right] + \frac{\nu - \rho}{\mu}\left(\beta_1(\gamma_2 + \delta_2) - \gamma_1 \beta_2\right) = 0 \quad (8)$$

□

E_0 of the system (4) is locally asymptotically stable if and only if the trace of the jacobian matrix near E_0.is strictly negative and its determinant is strictly positive.

$$\begin{cases} (\gamma_2+\delta_2)+\left(\gamma_1+\beta_1\frac{\nu-\rho}{\mu}\right)\prec 0 \\ \frac{\nu-\rho}{\mu}\left(\beta_1\left(\gamma_2+\delta_2\right)-\gamma_1\beta_2\right)\succ 0 \end{cases} \tag{9}$$

Then we define the basic reproduction number of the infection R_0 as follows:

$$R_0=\frac{\beta_2}{\beta_1}\times\frac{\gamma_1}{\gamma_2+\delta_2}. \tag{10}$$

If $R_0 < 1$, Then E_0 of the system (4) is locally asymptotically stable.2

In the presence of infection I_i 6= 0, substituting in the system contains a unique positive, endemic equilibrium $E^*=(S^*,I_1^*,I_2^*,I_3^*,I_4^*,)^T$ where

(11)

$$\begin{cases} S^*=\frac{\gamma_1}{\beta_1+\sum_{i=2}^4\beta_ic_i}, \\ I_1^*=\frac{\nu-\rho+\mu\left(\frac{\gamma_1}{\beta_1+\sum_{i=2}^4\beta_ic_i}\right)}{\left(\beta_1+\sum_{i=1}^4\beta_ic_i\right)\left(\frac{\gamma_1}{\beta_1+\sum_{i=2}^4\beta_ic_i}\right)-\mu-\sum_{i=2}^4c_i} \\ I_2^*=c_2\times I_1^*, \\ I_3^*=c_3\times I_1^*, \\ I_4^*=c_4\times I_1^*, \\ c_2=\left[\frac{\beta_1}{\beta_2}\times R_0\right], c_3=\left[\frac{\beta_1}{\beta_2}\times\frac{\gamma_2}{\gamma_3+\delta_3}\times R_0\right], \\ c_4=\left[\frac{\beta_1}{\beta_2}\times\frac{\gamma_2}{\gamma_3+\delta_3}\times\frac{\gamma_3}{\gamma_4+\delta_4}\times R_0\right]. \end{cases}$$

So $E^*=(S^*,I_1^*,I_2^*,I_3^*,I_4^*,)^T$ is the unique positive endemic equilibrium point which exists if $R_0 > 1$.

Theorem 2.2. *With $R_0 > 1$, system (4) has, a unique non-trivial equilibrium E^* is locally asymptotically stable.*

III. THE FRACTIONAL EPIDEMIC

The new system is describe by the system of fractional differential equations as follows:

$$\begin{cases} D^{a_1}S(t)=\nu-\rho+\mu\left(S(t)+\sum_{i=1}^4I_i(t)\right)-\sum_{i=1}^4\beta_iS(t)I_i(t) \\ D^{a_2}I_1(t)=\sum_{i=1}^4\beta_iS(t)I_i(t)-\gamma_1I_1(t), \\ D^{a_j}I_i(t)=\gamma_{i-1}I_{i-1}(t)-(\gamma_i+\delta_i)I_i(t), i=2,3,4.j=3,4,5. \end{cases} \tag{12}$$

Where $a_1,a_2,a_j\succ 0$. j=3 , 4, 5.

With the initial conditions

$S(0)=N_1, I_1(0)=N_2, I_i(0)=N_k, i=2,3,4, k=3,4,5.$ (13)

For this model, the initial conditions are not independents, since they must satisfy the condition

$$N=\sum_{k=1}^{5}N_k \tag{14}$$

Where N is the total number of the individuals in the population.

*3.1. **The Laplace-Adomian Decomposition Method***

We have the fractional-order epidemic model (13) with (14).

Applying the Laplace transform on (13), we obtain

$$\begin{cases} L\{D^{a_1}S\}=\nu-\rho+\mu\left(L\{S\}+\sum_{i=1}^4L\{I_i\}\right)-\sum_{i=1}^4\beta_iL\{SI_i\}, \\ L\{D^{a_1}I_1\}=\sum_{i=1}^4\beta_iL\{SI_i\}-\gamma_1L\{I_1\}, \\ L\{D^{a_1}I_i\}=\gamma_{i-1}L\{I_{i-1}\}-(\gamma_i+\delta_i)L\{I_i\}, i=2,3,4.j=3,4,5. \end{cases} \tag{15}$$

Applying the properties of the Laplace transform to (15) ; we obtain

$$\begin{cases} P^{a_1} L\{S\} - P^{a_1-1}\{S(0)\} = \nu - \rho + \mu\left(L\{S\} + \sum_{i=1}^{4} L\{I_i\}\right) - \sum_{i=1}^{4} \beta_i L\{SI_i\} \\ P^{a_2} L\{I_1\} - P^{a_2-1}\{I_1(0)\} = \sum_{i=1}^{4} \beta_i L\{SI_i\} - \gamma_1 L\{I_1\}, \\ P^{a_j} L\{I_i\} - P^{a_j-1}\{I_i(0)\} = \gamma_{i-1} L\{I_{i-1}\} - (\gamma_i + \delta_i) L\{I_i\}, i = 2,3,4. j = 3,4,5. \end{cases} \quad (16)$$

Then

$$\begin{cases} P^{a_1} L\{S\} = P^{a_1-1}\{S(0)\} + \nu - \rho + \mu\left(L\{S\} + \sum_{i=1}^{4} L\{I_i\}\right) - \sum_{i=1}^{4} \beta_i L\{SI_i\} \\ P^{a_2} L\{I_1\} = P^{a_2-1}\{I_1(0)\} + \sum_{i=1}^{4} \beta_i L\{SI_i\} - \gamma_1 L\{I_1\}, \\ P^{a_j} L\{I_i\} = P^{a_j-1}\{I_i(0)\} + \gamma_{i-1} L\{I_{i-1}\} - (\gamma_i + \delta_i) L\{I_i\}, i = 2,3,4. j = 3,4,5. \end{cases} \quad (17)$$

Using (14) and (15) we obtain

$$\begin{cases} L\{S\} = \frac{N_1}{S} + \frac{1}{P^{a_1}}\left[\nu - \rho + \mu\left(L\{S\} + \sum_{i=1}^{4} L\{I_i\}\right) - \sum_{i=1}^{4} \beta_i L\{SI_i\}\right], \\ L\{I_1\} = \frac{N_2}{I_1} + \frac{1}{P^{a_2}}\left[\sum_{i=1}^{4} \beta_i L\{SI_i\} - \gamma_1 L\{I_1\}\right], \\ L\{I_i\} = \frac{N_k}{I_i} + \frac{1}{P^{a_j}}\left[\gamma_{i-1} L\{I_{i-1}\} - (\gamma_i + \delta_i) L\{I_i\}\right], i = 2,3,4. j = 3,4,5, k = [3,5] \end{cases} \quad (18)$$

The method has a solution as follows:

$$S = \sum_{m=0}^{\infty} S_m, I_1 = \sum_{m=0}^{\infty} (I_1)_m, I_i = \sum_{m=0}^{\infty} (I_i)_m, i = 2,3,4. \quad (19)$$

The non-linearity $\sum_{i=1}^{4} S(t) I_i(t)$ is defined as follows

$$\sum_{i=1}^{4} S(t) I_i(t) = \sum_{m=0}^{\infty} C_m \quad (20)$$

With C_m; is called Adomian polynomials witch is defined as

$$C_m^i = \frac{1}{(m)!} \times \frac{d^m}{d\lambda^m}\left[\sum_{n=0}^{m} \lambda^n S_n \sum_{n=0}^{m}\left(\sum_{i=1}^{4} \lambda^n (I_i)_n\right)\right] |_{\lambda=0}; i = 1,2,3,4. \quad (21)$$

Substituting from (20), (22) into (19); then we obtain

$$\begin{cases} L\{(S)_0\} = \frac{N_1}{P} \\ L\{(I_1)_0\} = \frac{N_2}{P} \quad i = 2,3,4; k = [3,5] \\ L\{(I_i)_0\} = \frac{N_k}{P} \end{cases} \quad (22)$$

We have

$$\begin{cases} L\{S_1\} = \frac{1}{P^{a_1}}\left[\nu - \rho + \mu\left(L\{S_0\} + \sum_{i=1}^{4} L\{(I_i)_0\}\right) - \sum_{i=1}^{4} \beta_i L\{C_0^i\}\right], i = 1,2,3,4, \\ \vdots \\ L\{S_{m+1}\} = \frac{1}{P^{a_1}}\left[\nu - \rho + \mu\left(L\{S_m\} + \sum_{i=1}^{4} L\{(I_i)_m\}\right) - \sum_{i=1}^{4} \beta_i L\{C_m^i\}\right], i = 1,2,3,4 \end{cases} \quad (23)$$

(24)

And

$$\begin{cases} L\{(I_1)_1\} = \frac{1}{P^{a_2}}\left[\sum_{i=1}^{4} \beta_i L\{C_0^i\} - \gamma_1 L\{(I_1)_0\}\right], \\ \vdots \\ L\{(I_1)_{m+1}\} = \frac{1}{P^{a_2}}\left[\sum_{i=1}^{4} \beta_i L\{C_m^i\} - \gamma_1 L\{(I_1)_m\}\right] \end{cases}$$

$$\begin{cases} L\left\{(I_i)_1\right\} = \frac{1}{P^{a_j}}\left[\gamma_{i-1}L\left\{(I_{i-1})_0\right\} - (\gamma_i+\delta_i)L\left\{(I_i)_0\right\}\right], i=2,3,4.j=3,4,5, \\ \vdots \\ L\left\{(I_i)_{m+1}\right\} = \frac{1}{P^{a_j}}\left[\gamma_{i-1}L\left\{(I_{i-1})_m\right\} - (\gamma_i+\delta_i)L\left\{(I_i)_m\right\}\right], i=2,3,4.j=3,4,5 \end{cases} \quad (25)$$

IV. GLOBAL ASYMPTOTIC STABILITY

Theorem 4.1. *The disease-free equilibrium E_0 of the system* (4) *is globally asymptotically stable if $R_0 < 1$.*
Proof. Choose the Lyapunov functional

$$V(x_1,x_2,x_3,x_4,x_5) = x_1\left(S-\hat{S}\right)+x_2\left(I_1-\hat{I}_1\right)+x_3\left(I_2-\hat{I}_2\right)+x_4\left(I_3-\hat{I}_3\right)+x_5\left(I_4-\hat{I}\right) \quad (26)$$

The derivative $V\left(x_1,x_2,x_3,x_4,x_5\right)$ is

$$\dot{V}\left(x_1,x_2,x_3,x_4,x_5\right) = x_1\dot{S}+x_2\dot{I}_1+x_3\dot{I}_2+x_4\dot{I}_3+x_5\dot{I}_4. \quad (27)$$

Where x_1,x_2,x_3,x_4,x_5 are positive constants

$$\dot{V}(x_1,x_2,x_3,x_4,x_5) = \begin{array}{c} x_1\left[\nu-\rho+\mu\left(S+\sum_{i=1}^4 I_i\right)\right]-\left[x_1-x_2\right]\left[\sum_{i=1}^4\beta_i S I_i\right] \\ -\left[x_2-x_3\right]\gamma_1 I_1-\left[x_3-x_4\right]\gamma_2 I_2-\left[x_4-x_5\right]\gamma_3 I_3-x_3\delta_2 I_2-x_4\delta_3 I_3-x_5\delta_4 I_4 \end{array} \quad (28)$$

Let choose $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.
We obtain

$$\dot{V}(x_1,x_2,x_3,x_4,x_5) = -\left[\left[\rho-\nu-\mu\left(S+\sum_{i=1}^4 I_i\right)\right]+\sum_{j=2}^4\delta_i I_j\right]. \quad (29)$$

$\dot{V}\left(x_1,x_2,x_3,x_4,x_5\right)<0$, E_0 is globally asymptotically stable if $R_0 < 1$.□

V. CONCLUSION

This paper addresses a the equilibrium and stability of the **multicompartment** epidemic model , in the absence of infection, the system has a disease-free equilibrium, in the presence of infection the system, has a unique positive, endemic equilibrium. Both trivial and endemic equilibrium are founded. The disease-free equilibrium is locally asymptotically stable if $R_0 < 1$. In the paper, we have the epidemic nonlinear model, describing the spread of an epidemic in a population. We to use the Laplace-Adomian Decomposition method for obtaining the solution analytic of the multicompartment epidemic model. Finally, we study global stability under some conditions.

REFERENCES

[1] Anderson R. M and Medley R M and Jhonson A K. A Preliminary Study of the Transmission Dynamics of the Human Immunodeficiency Virus (HIV), the Causative Agent of AIDS. IMA. J. Math. Appl. Med. Biol 3, 229-263. (1986).
[2] Abta A and Kaddar A and Talibi H. A. Global Stability for Delay SIR and SEIR Epidemic Models with Saturated Incidence Rates. Electronic Journal of Differential Equations, 23, 1-13. (2012).
[3] Bailley. N.T.J. Some Stochastic Models for Small Epidemics in Large Population. Appl. Statist.13, 9-19. (1964).
[4] Bailley. N.T.J. The Mathematical Theory of Infection Diseases and its Application. Applied Statistics, 26, N1, 85-87. (1977).
[5] Batiha, M. S. M. Noorani and I. Hashim. Numerical solutions of the nonlinear integro-differential equations, Int. J. Open Probl. Compt. Math, 34-42. (2008).
[6] Becker.N.G. The Uses of Epidemic Models**.** *Biometrics 35, 295-305.* (1979).
[7] Ben O'Donnell, Brandon McPhail, Duy Tran, Patrick Carlisle. Computing the SIR Model: Fractal dimension leads to vaccination strategies.pp 1-16. (2014).https://www.reed.edu/epi/papers/sirpaper.pdf.
[8] Billard.L. A Stochastic General Epidemic in m Sub-Population. *J. Appl. Prob. 13, 567-572.* (1976).
[9] Jinliang W, Xinxin Tian. Global Stabilty of a Delay Differential Equation of Hepatitis B Virus Infection with Immune Response. Electronic Journal of Differential Equations, 94, 1-11. (2013).
[10] Jin. Z, Zhien. M and Maoan. H. Globale stability of an SIRS epidemic model with delay, Acta Matimatica Scientia. 26 B. 291-306. (2006).
[11] Kuang Y. Delay-Differential Equations with application in population biology. Academic Press, New York (1993).

[12] Lahrouz A and El Maroufy H. Qualitative Behaviour of a Model of an SIRS Epidemic Stability and Permanence. Applied Mathematics & Information Sciences. An International Journal 5 (2), 220-238. (2011).
[13] Luo Q and Mao X. Stochastic population dynamics under regime switching. J. Math. Anal. Appl.334, 69-84. (2007)
[14] Michael Steel J. Stochastic calculus and financial applications. Springer Verlag. (2003).
[15] Ray Waston. A useful Random Time-Seal Transformation for the Standard Epidemic Model. J. Appl. Prob.17, 324-332. (1980).
[16] Ray Waston, 1980. On The Size Distribution for Some Epidemic Models. J. Appl. Prob.17, 912-921. (1980).
[17] Robert N and May. Population Biology of infectious diseases I. International centre of theoretical physics.1-9. (1982).
[18] Ruoyan Sun. Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence. Computers and Mathematics with Applications 60. 2286-2291. (2010).
[19] S. Z. Rida, A. A. M. Arafa, Y. A. Gaber. Solution of the fractional epidemic model by L-ADM. Journal of Fractional Calculus and Applications.Vol. 7(1), pp. 189-195. (2016).
[20] Takeuchi and W. Ma. Stability analysis on a delayed SIR epidemic model with density dependent birth process, Dy-nam. Contin. Discrete Impuls. Systems, 5. 171-184. (1999).
[21] Volodymyr Makarov, Denis Dragunov. A numeric-analytical method for solving the Cauchy problem for ordinary differential equations. Applied Mathematics and Computation, 1-26. (2010).
[22] W. Ma, Y. Takeuchi, T. Hara and E. Beretta, Permanence of are SIR epidemic model with distributed time delays, Tohoku Math. J. 54, 581591.(2002)
[23] W. Wang, Global behavior of an SEIR epidemic model with time delay, Appl. Math. Letters.15, 423-428. (2002).
[24] Wen L and Yang X. Global stability of a delayed SIRS model with temporary immunity. Chaos, Solitons and Fractals 38, 221-226. (2008)
[25] Xiao, L Chen and F. ven den Bosch, Dynamical behavior for a stage structured SIR infectious disease model, Nonlinear Anal. Real World Appl 3,175-190. (2002).
[26] Z. Ma, J. Liu and J. Li, Stability analysis for differential infectivity epidemic models, Nonlinear Anal. Real World Appl 4, 841-856. (2003).
[27] Zhang F and Zhen Li and Zhang F. Global stability of an SIR epidemic model with constant infectious period. Applied Mathematics and Computation 199, 285-291. (2008).