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Research Paper

Positive periodic solutions for a class of discrete dynamic equations with delays and feedback controls

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ABSTRACT:
In this paper, by using Leggett-Williams fixed point theorem in cones, sufficient conditions for the existence of at least three positive periodic solutions for a class of discrete dynamic equations with delays and feedback controls are obtained.
KEYWORDS: *Periodic solutions; Discrete dynamic equation; Fixed point theorem; Feedback control*

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I. INTRODUCTION

In the last two decades, the existence of positive periodic solutions for different types of functional differential equations with delays and feedback controls have been studied extensively; see, for example [1-5]. On the other hand, discrete dynamic equations also played an important role in applications, for example, in the nature world, the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Owing to its theoretical and practical significance, the existence of positive periodic solutions for difference equations received much attention; see, for example, [6-9]. However, the existence of positive periodic solutions for some special types of difference equations need to be explored further.

Motivated by the above works, in this paper, we use the Krasnoselskii's fixed point theorem in cones to study the existence of positive periodic solutions for a class of discrete differential equations with delays and feedback controls

$$\begin{cases} y_i(n+1) = \left(y_i(n)\right)^{h_i(n)} \exp\Big\{r_i(n) - a_{ii}(n)y_i(n) \\ \qquad - \sum_{j=1, j\neq i}^{m} a_{ij}(n)\sum_{k=0}^{\omega-1} K_{ij}(k)y_j(n-k) - \alpha_i(n)u_i(n) - \sum_{j=1, j\neq i}^{m} b_{ij}(n)u_i(n-\beta_j(n))\Big\}, \\ u_i(n+1) - u_i(n) = -e_i(n)u_i(n) + f_i(n)y_i(n) \\ \qquad + \sum_{j=1, j\neq i}^{m} g_j(n)y_i(n-\eta_j(n)), i = 1, 2, \cdots, m, n \in N, \end{cases} \tag{1.1}$$

where

(H1) $r_i(n), h_i(n), a_{ii}(n), a_{ij}(n), \alpha_i(n), b_{ij}(n), e_i(n), f_i(n), g_j(n) : N \to R, i, j = 1, 2, \cdots, m; i \neq j$ are all positive and ω -periodic functions, and $0 \le e_i(n) < 1$, $0 \le h_i(n) < 1$.

(H2) $\beta_j(n) : N \to N, \eta_j(n) : N \to N$, are all positive and ω -periodic functions , $0 \le \beta_j(n) \le \omega - 1$, $0 \le \eta_j(n) \le \omega - 1, n \in N$.

(H3) $K_{ij}(k) : I_\omega \to [0, \infty)$, $I_\omega = \{0, 1, \cdots, \omega - 1\}$, and $\sum_{k=0}^{\omega-1} K_{ij}(k) = 1, \sum_{k=0}^{\omega-1} r_i(n) > 0$, $i, j = 1, 2, \cdots, m$, $i \neq j$.

For convenience, we first introduce the related definition and the fixed point theorem applied in the paper.

Definition 1.1 Let X be a Banach space and K be a closed nonempty sunset of X, K is a cone if
(1) $\alpha u+\beta v\in K$ for all $u,v\in K$ and all $\alpha,\beta\geq 0$;
(2) $u,-u\in K$ imply $u=0$.

Theorem 1.1 (Leggett-Williams [10]) Let K be a cone of the real Banach space X, and $A:K_c\to K_c$ be a completely continuous operator, and suppose that there exist a concave positive functional α with $\alpha(x)\leq\|x\|$ $(x\in K)$ and numbers a,b,d with $0<d<a<b<c$ following conditions:
(1) $\{x\in K(\alpha,a,b):\alpha(x)>a\}\neq\phi$ and $\alpha(Ax)>a$ if $x\in K(\alpha,a,b)$;
(2) $\|Ax\|<d$ if $x\in K_d$;
(3) $\alpha(Ax)>a$ for all $x\in K(\alpha,a,b)$ with $\|Ax\|>b$.
Then A has at least three fixed points in $x\in K_c$.
In this paper, we always assume that
(H4) For any $n\in N,\ \ i=1,2,\cdots,m,$

$$r_i(n)-a_{ii}(n)\exp[x_i(n)]-\sum_{j=1,j\neq i}^{m}a_{ij}(n)\sum_{k=0}^{\omega-1}K_{ij}(k)\exp[x_j(n-k)]$$
$$-\alpha_i(n)(\Phi_i\exp[x_i])(n)-\sum_{j=1,j\neq i}^{m}b_{ij}(n)(\Phi_i\exp[x_i])(n-\beta_j(n))>0.$$

II. SOME PREPARATION

Let ω be a positive constant. We define two sets

$$X=\{x:C(R,R^m),x(t+\omega)=x(t),t\in R\}$$

endow with the usual linear structure as well as the norm

$$\|x\|=\sum_{i=1}^{m}\left|x_i(t)\right|_0,\ \left|x_i\right|_0=\max_{t\in[0,\omega-1]}\left|x_i(t)\right|,$$

and

$$K=\left\{x\in X,x_i(t)\geq\sigma\left|x_i\right|_0,t\in[0,\omega],x=(x_1,x_2,\cdots x_m)^T\right\}.$$

Obviously, X is a Banach space and K is a cone.

Lemma 2.1. Each T-periodic solution of second equation of (1.1) is equivalent to that of the following equation

$$x_i(n)=\sum_{l=n}^{n+\omega+1}G_i(n,l)\Bigg(r_i(n)-a_{ii}(n)\exp[x_i(n)]-\sum_{j=1,j\neq i}^{m}a_{ij}(n)\sum_{k=0}^{\omega-1}K_{ij}(k)\exp[x_j(n-k)]$$
$$-\alpha_i(n)(\Phi_i\exp[x_i])(n)-\sum_{j=1,j\neq i}^{m}b_{ij}(n)(\Phi_i\exp[x_i])(n-\beta_j(n))\Bigg),\qquad (2.1)$$
$$i=1,2,\cdots,m,$$

where $G_i(n,l)=\dfrac{\prod_{k=l+1}^{n+\omega-1}h_i(k)}{1-\prod_{k=0}^{\omega-1}h_i(k)},i=1,2,\cdots,m;n\leq l\leq n+\omega-1$, $y_i(n)=\exp[x_i(n)]$, and Φ is defined in (2.2).

Proof. Let $H_i(n)=f_i(n)y_i(n)+\sum_{j=1,j\neq i}^{m}g_j(n)y_i(n-\eta_j(n))$, so the second equation of (1.1) is equivalent to that of the following equation

$$u_i(n+1)-\big(1-e_i(n)\big)u_i(n)=H_i(n),$$

then we have

$$u_i(n+\omega)-\left(1-e_i(n+\omega-1)\right)u_i(n+\omega-1)=H_i(n+\omega-1),$$

$$\left(1-e_i(n+\omega-1)\right)u_i(n+\omega-1)-\left(1-e_i(n+\omega-1)\right)u_i(n+\omega-2)=\left(1-e_i(n+\omega-1)\right)H_i(n+\omega-2),$$

$$\cdots\cdots$$

$$\left(1-e_i(n+\omega-1)\right)\left(1-e_i(n+\omega-2)\right)\cdots\left(1-e_i(n+1)\right)u_i(n+1)-\left(1-e_i(n+\omega-1)\right)\left(1-e_i(n+\omega-2)\right)$$
$$\cdots\left(1-e_i(n)\right)u_i(n)=\left(1-e_i(n+\omega-1)\right)\left(1-e_i(n+\omega-2)\right)\cdots\left(1-e_i(n+1)\right)H_i(n),$$

sum the left and right of above formulates, we obtain that

$$u_i(n+\omega)-\left(1-e_i(n+\omega-1)\right)\left(1-e_i(n+\omega-2)\right)\cdots\left(1-e_i(n)\right)u_i(n)$$
$$=H_i(n+\omega-1)+\left(1-e_i(n+\omega-1)\right)H_i(n+\omega-2)+\cdots+\left(1-e_i(n+\omega-1)\right)\left(1-e_i(n+\omega-2)\right)$$
$$\cdots\left(1-e_i(n+1)\right)H_i(n)=\sum_{l=n}^{n+\omega-1}\prod_{k=l+1}^{n+\omega-1}\left(1-e_i(k)\right)H_i(l),$$

because of $u_i(n+\omega)=u_i(n)$, so we have

$$\left(1-\prod_{k=n}^{n+\omega-1}\left(1-e_i(k)\right)\right)u_i(n)=\sum_{l=n}^{n+\omega-1}\prod_{k=l+1}^{n+\omega-1}\left(1-e_i(k)\right)H_i(l),$$

where $1-\prod_{k=n}^{n+\omega-1}\left(1-e_i(k)\right)\neq 0$, and $e_i(n)$ is a ω-periodic function, so

$$u_i(n)=\frac{\sum_{l=n}^{n+\omega-1}\prod_{k=l+1}^{n+\omega-1}\left(1-e_i(k)\right)H_i(l)}{\left(1-\prod_{k=n}^{n+\omega-1}\left(1-e_i(k)\right)\right)}=\sum_{l=n}^{n+\omega-1}G_i(n,l)H_i(l)$$
$$=\sum_{l=n}^{n+\omega-1}G_i(n,l)\left(f_i(l)y_i(l)+\sum_{j=1,\,j\neq i}^{m}g_j(l)y_i(l-\eta_j(l))\right),$$

and $\tilde{G}_i(n,l)=\dfrac{\prod_{k=l+1}^{n+\omega-1}(1-e_i(k))}{1-\prod_{k=0}^{\omega-1}(1-e_i(k))},i=1,2,\cdots,m;n\leq l\leq n+\omega-1.$

This completes the proof.

It is clear that $G_i(n,l)=G_i(n+\omega,l+\omega)$ and $u_i(t+T)=u_i(t)$, when x is a ω-periodic function. We denote

$$u_i(n)=\sum_{l=n}^{n+\omega+1}\left[f_i(l)y_i(l)+\sum_{j=1,\,j\neq i}^{m}g_j(l)y_i(l-\eta_j(l))\right]\tilde{G}_i(n,l):=(\Phi_i y_i)(n),\ \ i=1,2,\cdots,m. \tag{2.2}$$

Set $y_i(n)=\exp[x_i(n)]$, then the first m equations is equivalent to that of the following equations

$$x_i(n+1)-h_i(n)x_i(n)=r_i(n)-a_{ii}(n)\exp[x_i(n)]-\sum_{j=1,\,j\neq i}^{m}a_{ij}(n)\sum_{k=0}^{\omega-1}K_{ij}(k)\exp[x_j(n-k)]$$
$$-\alpha_i(n)(\Phi_i\exp[x_i])(n)-\sum_{j=1,\,j\neq i}^{m}b_{ij}(n)(\Phi_i\exp[x_i])(n-\beta_j(n)) \tag{2.3}$$
$$:=F_i(n,x_1(n),\cdots,x_m(n)),i=1,2,\cdots,m.$$

So we proceed from (2.3) and obtain

$$x_i(n)=\sum_{l=n}^{n+\omega+1}G_i(n,l)\left(r_i(n)-a_{ii}(n)\exp[x_i(n)]-\sum_{j=1,j\neq i}^{m}a_{ij}(n)\sum_{k=0}^{\omega-1}K_{ij}(k)\exp[x_j(n-k)]\right.$$
$$\left.-\alpha_i(n)(\Phi_i\exp[x_i])(n)-\sum_{j=1,j\neq i}^{m}b_{ij}(n)(\Phi_i\exp[x_i])(n-\beta_j(n))\right),$$
$$i=1,2,\cdots,m.$$

where $G_i(n,l)=\dfrac{\prod_{k=l+1}^{n+\omega-1}h_i(k)}{1-\prod_{k=0}^{\omega-1}h_i(k)}, i=1,2,\cdots,m; n\le l\le n+\omega-1,$ and Φ is defined in (2.2).

By (H1), we know that the denominator in $G_i(n,l)$ is not zero for $n\in[0,\omega-1]$. Note that due to (H1), we have

$$0<N_i=:G_i(n,n)\le G_i(n,l)\le G_i(n,n+\omega-1)=G_i(0,\omega-1):=M_i,\ i=1,2,\cdots,m,$$

for all $l\in[n,n+\omega-1]$, and

$$1\ge\frac{G_i(n,l)}{G_i(n,n+\omega-1)}\ge\frac{G_i(n,n)}{G_i(n,n+\omega-1)}=\frac{N_i}{M_i}>0,\quad i=1,2,\cdots,m.$$

Let

$$\sigma=\min\left\{\frac{N_i}{M_i},\ i=1,2,\cdots,m\right\},$$

and we denotes

$$N=\min_{1\le i\le m}N_i, M=\max_{1\le i\le m}M_i.$$

III. MAIN RESULTS

Notice solving (2.3) is equivalent to solving

$$x=Tx,$$

where $T:K\to K$ is define by,

$$(Tx)(n)=((T_1x)(n),(T_2x)(n),\cdots,(T_mx)(n))^T,$$

and

$$(T_ix_i)(n)=\sum_{l=n}^{n+\omega-1}G_i(n,l)\left(r_i(l)-a_{ii}(l)\exp[x_i(l)]-\sum_{j=1,j\neq i}^{m}a_{ij}(l)\sum_{k=0}^{\omega-1}K_{ij}(k)\exp[x_j(l-k)]\right.$$
$$\left.-\alpha_i(l)(\Phi_i\exp[x_i])(l)-\sum_{j=1,j\neq i}^{m}b_{ij}(l)(\Phi_i\exp[x_i])(l-\beta_j(l))\right),$$
$$i=1,2,\cdots,m,$$

where Φ_i is defined as (2.1).

Lemma 3.1. $T:K\to K$ is well-defined.

Proof. For each $x\in X$, in view of (2.4), we obtain

$$(T_ix_i)(n+\omega)=\sum_{l=n+\omega}^{n+2\omega-1}G_i(n+\omega,l)\left(r_i(l)-a_{ii}(l)\exp[x_i(l)]-\sum_{j=1,j\neq i}^{m}a_{ij}(l)\sum_{k=0}^{\omega-1}K_{ij}(k)\exp[x_j(l-k)]\right.$$
$$\left.-\alpha_i(l)(\Phi_i\exp[x_i])(l)-\sum_{j=1,j\neq i}^{m}b_{ij}(l)(\Phi_i\exp[x_i])(l-\beta_j(l))\right)$$

$$= \sum_{l=n}^{n+\omega-1} G_i(n+\omega, v+\omega)\left(r_i(v+\omega) - a_{ii}(v+\omega)\exp[x_i(v+\omega)] - \sum_{j=1, j\neq i}^{m} a_{ij}(v+\omega)\sum_{k=0}^{\omega-1} K_{ij}(k)\exp[x_j(v+\omega-k)] \right.$$

$$\left. -\alpha_i(v+\omega)(\Phi_i \exp[x_i])(v+\omega) - \sum_{j=1, j\neq i}^{m} b_{ij}(v+\omega)(\Phi_i \exp[x_i])(v+\omega-\beta_j(v+\omega)) \right)$$

$$= \sum_{l=n}^{n+\omega-1} G_i(n,v)\left(r_i(v) - a_{ii}(v)\exp[x_i(v)] - \sum_{j=1, j\neq i}^{m} a_{ij}(v)\sum_{k=0}^{\omega-1} K_{ij}(k)\exp[x_j(v-k)] \right.$$

$$\left. -\alpha_i(v)(\Phi_i \exp[x_i])(v) - \sum_{j=1, j\neq i}^{m} b_{ij}(v)(\Phi_i \exp[x_i])(v-\beta_j(v)) \right)$$

$$= (T_i x_i)(n), i = 1, 2, \cdots, m.$$

So $Tx \in X$, for each $x \in K$, we find

$$|T_i x_i|_0 \le \sum_{l=0}^{\omega-1} M\left(r_i(l) - a_{ii}(l)\exp[x_i(l)] - \sum_{j=1, j\neq i}^{m} a_{ij}(l)\sum_{k=0}^{\omega-1} K_{ij}(k)\exp[x_j(l-k)] \right.$$

$$\left. -\alpha_i(l)(\Phi_i \exp[x_i])(l) - \sum_{j=1, j\neq i}^{m} b_{ij}(l)(\Phi_i \exp[x_i])(l-\beta_j(l)) \right),$$

$$i = 1, 2, \cdots, m,$$

and

$$(T_i x_i)(n) \ge \sum_{l=0}^{\omega-1} N\left(r_i(l) - a_{ii}(l)\exp[x_i(l)] - \sum_{j=1, j\neq i}^{m} a_{ij}(l)\sum_{k=0}^{\omega-1} K_{ij}(k)\exp[x_j(l-k)] \right.$$

$$\left. -\alpha_i(l)(\Phi_i \exp[x_i])(l) - \sum_{j=1, j\neq i}^{m} b_{ij}(l)(\Phi_i \exp[x_i])(l-\beta_j(l)) \right)$$

$$= \frac{N}{M}\sum_{l=0}^{\omega-1} M\left(r_i(l) - a_{ii}(l)\exp[x_i(l)] - \sum_{j=1, j\neq i}^{m} a_{ij}(l)\sum_{k=0}^{\omega-1} K_{ij}(k)\exp[x_j(l-k)] \right.$$

$$\left. -\alpha_i(l)(\Phi_i \exp[x_i])(l) - \sum_{j=1, j\neq i}^{m} b_{ij}(l)(\Phi_i \exp[x_i])(l-\beta_j(l)) \right)$$

$$\ge \sigma |T_i x_i|_0, i = 1, 2, \cdots, m.$$

Therefore, $Tx \in K$, this completes the proof.

Similar to the proof in [8], we can obtain the following lemma.

Lemma 3.2. $T: K \to K$ is completely continuous.

For convenience, we introduce the following notations:

$$F_i^{\vartheta} := \limsup_{\|x\|\to\vartheta} \sup_{n\in[0,\omega-1]} \frac{F_i(n, x_1(n), \cdots, x_m(n))}{\|x\|},$$

where $F_i(n, x_1(n), \cdots, x_m(n))$ is defined in (2.3).

Theorem 3.1. Suppose that (H1)-(H4) hold, and there exist a number $b > 0$ such that the following conditions:

(i) $F_i^0 < \dfrac{1}{mM_i\omega}$, $F_i^{\infty} < \dfrac{1}{mM_i\omega}$;

(ii) $F_i(n, x_1(n), \cdots, x_m(n)) > \dfrac{1}{mN_i\omega}\sum_{i=1}^{m}|x_i(n)|$ for $\sigma b \le \sum_{i=1}^{m}|x_i(n)| \le b, n \in N$;

hold. Then (1.1) has at least three positive ω -periodic solutions.

Proof: By the condition $F_i^\infty < \frac{1}{mM_i\omega}$ of (i), one can find that for

$$0 < \varepsilon < \frac{1}{mM_i\omega} - F_i^\infty,$$

there exists a $c_0 > b$ such that

$$F_i(n, x_1(n), \cdots, x_m(n)) < (F_i^\infty + \varepsilon) \| x \|,$$

where $\| x \| > c_0$.

Let $c_1 = \frac{c_0}{\sigma}$, if $x \in K$, $\| x \| > c_1$, then $\| x \| > c_0$, and we have

$$\left| T_i x_i \right|_0 \le \sum_{l=0}^{\omega-1} G_i(0, \omega-1) F_i(n, x_1(n), \cdots, x_m(n)) \le M_i \omega (F_i^\infty + \varepsilon) \| x \| < \frac{1}{m} \| x \|,$$

then

$$\| Tx \| < \| x \|, \qquad (3.1)$$

Take $k_{c_1} = \{ x \mid x \in K, \| x \| \le c_1 \}$, then the set k_{c_1} is a bounded set. According to that T is completely continuous, then T maps bounded sets into bounded sets and there exists a number c_2 such that

$$\| Tx \| \le c_2, \ \forall x \in k_{c_1}.$$

If $c_2 \le c_1$, we deduce that $T : k_{c_1} \to k_{c_1}$ is completely continuous. If $c_2 < c_1$, then from (3.1), we know that for any $x \in k_{c_2} \setminus k_{c_1}$ and $\| Tx \| < \| x \| < c_2$ hold. Thus we have $T : k_{c_2} \to k_{c_2}$ is completely continuous. Now, take $c = \max\{c_1, c_2\}$, then $c > b$, so $T : k_c \to k_c$ is completely continuous.

Denote the positive continuous concave functional $\alpha(x)$ as $\alpha(x) = \sum_{i=1}^{m} \inf_{n \in [0, \omega-1]} | x_i(n) |$. Firstly, let $a = \sigma b$ and take $x \equiv \frac{a+b}{2}, x \in K(\alpha, a, b), \alpha(x) > a$, then the set $\{ x \in K(\alpha, a, b) \} \ne \varnothing$. By (ii), if $x \in K(\alpha, a, b)$, then $\alpha(x) \ge a$, and we have

$$\alpha(Tx) = \sum_{i=1}^{m} \inf_{n \in [0, \omega-1]} | (T_i x)(n) | > mN_i \omega \frac{1}{mN_i\omega} \alpha(x) = a.$$

Hence condition (1) of Theorem 1.1 holds.

Secondly, by the condition $F_i^0 < \frac{1}{mM_i\omega}$ of (i), one can find that for

$$0 < \varepsilon < \frac{1}{mM_i\omega} - F_i^0,$$

there exists a d $(0 < d < a)$ such that

$$F_i(n, x_1(n), \cdots, x_m(n)) < (F_i^0 + \varepsilon) \| x \|,$$

where $0 \le \| x \| \le d$.

If $x \in K_d = \{ x \| \| x \| \le d \}$, we have

$$\left| T_i x_i \right|_0 \le \sum_{l=0}^{\omega-1} G_i(0, \omega-1) F_i(n, x_1(n), \cdots, x_m(n)) \le M_i \omega (F_i^0 + \varepsilon) \| x \| \le \frac{1}{m} \| x \|,$$

then

$$\| Tx \| \le d, \qquad (3.2)$$

that is, condition (2) of Theorem 1.1 holds.

Finally, if $x \in K(\alpha, a, c)$ with $\| Tx \| > b$, by the definition of the cone K, we have

$\alpha(Tx) \geq \sigma \| Tx \| > \sigma b = a,$

which implies that condition (3) of Theorem 1.1 holds.

To sum up, all conditions in Theorem 1.1 hold. By Theorem 1.1, the operator T has at least three fixed point in $\overline{K}_c$. Therefore, (2.3) has at least three positive ω-periodic solutions, and

$x_1 \in K_d,\ x_2 \in \{x \in K(\alpha, a, c), \alpha(x) > a\},\ x_3 \in \overline{K}_c \setminus \alpha(K(\alpha, a, c) \cup \overline{K}_d).$

Then (1.1) has at least three positive ω-periodic solutions. This completes the proof.

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