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Research Paper

Fitting and Comparing Gamma, Lognormal and Weibull Distributions Using Nigeria Rainfall Intensity Data

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ABSTRACT: *This research presents comparison of three probability distribution models to monthly rainfall intensity in Nigeria using 3-parameter Gamma, 3-parameter Lognormal and 3-parameter Weibull distributions. The data for the study spanned from 1901 to 2015, that is, a period of 115 years. The fitted models parameters were estimated using the technique of maximum likelihood estimation (MLE) and two information criteria, Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) were used for testing the models goodness of fit. The descriptive statistics shows that the data has less kurtosis and negative skewness. Also, from the fitted three distributions models results, the estimate of Weibull distribution shows a higher threshold than the other distributions. All the three fitted distributions fit the data but the weibull distribution with the smallest AIC and BIC implies it to be the best fit amongst the three distributions.*

KEYWORDS: *Gamma, Lognormal, Weibull, Information criteria, Rainfall intensity*

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I. INTRODUCTION

One of the most difficult problems in rainfall modeling is often the fitting of theoretical models to data [Richard, 2016] and many researches has been conducted in modeling rainfall distribution in different parts of the world. For instance, [Dikko *et al.*, (2013)], [Azeez and Ayoola (2012)], [Dan'azumi, *et al.*, (2010)], [Mondonedo *et al.*, (2010)], [Deka and Borah (2009)], [Nadarajah and Choi (2007)] and [Cho *et al.*, 2004] used different probability distributions like Gamma, Lognormal, Gumbel, Frecht, Weibull, Exponential, Pareto, etc., in modeling different rainfall data types. The parameters are estimated using either moments, least squares and likelihood methods.

This paper considers the following probability distribution functions to model Nigeria monthly rainfall intensity.

- A 3-parameter Gamma distribution with α, β and λ being the shape, scale and location parameters respectively has a probability density function (PDF) [Hafzullah (1999)] as:

$$f\left(x;\alpha,\beta,\lambda\right)=\frac{\left(x-\lambda\right)^{\alpha-1}\exp\left[-\frac{\left(x-\lambda\right)}{\beta}\right]}{\Gamma\left(\alpha\right)\beta^{\alpha-1}},x>\lambda,\alpha>0,\beta>0,\lambda\in R \qquad \rightarrow(1)$$

- A 3-parameter Lognormal distribution with γ, μ, σ being the location, shape and scale parameters respectively has a PDF [Basak *et al.* (2009)] as:

$$f(x;\gamma,\sigma,\mu)=\frac{1}{(x-\gamma)\sigma\sqrt{2\pi}}exp\left[-\frac{\left(ln(x_i-\gamma)-\mu\right)^2}{2\sigma^2}\right],x>\gamma\geq 0,-\infty<\mu<\infty,\sigma>0 \qquad \rightarrow(2)$$

- A 3-parameter Weibull distribution with non-zero parameters γ, β, α being shape, scale and location parameters respectively has a PDF [(Denis, 2008)] as:

$$f(x;\gamma,\beta,\alpha)=\gamma\beta^{-\gamma}\left(x-\alpha\right)^{\gamma-1}e^{-\left(\frac{x-\alpha}{\beta}\right)^{\gamma}},\gamma,\beta,\alpha>0 \qquad \rightarrow(3)$$

II. MAXIMUM LIKELIHOOD ESTIMATION

The likelihood for the 3-parameter Gamma, Lognormal and Weibull PDFs of equation (1), (2) and (3) respectively, are presented in the following subsections from which each parameter is estimated.

2.1 Gamma PDF Estimation

$$L\left(\alpha,\beta,\lambda \mid x\right)=\prod_{i=1}^{n}\left[f\left(x_i \mid \alpha,\beta,\lambda\right)\right] \qquad \rightarrow(4)$$

$$=\left(\frac{1}{\Gamma\left(\alpha\right)\beta^{\alpha-1}}\right)^{n}\prod_{i=1}^{n}\left(x_i-\lambda\right)^{\alpha-1}\exp\left[-\sum_{i=1}^{n}\left(\frac{x_i-\lambda}{\beta}\right)\right] \qquad \rightarrow(5)$$

Taking the log-likelihood of the above equation (5), we obtained

$$LogL=\left(\alpha-1\right)\sum_{i=1}^{n}\log\left(x_i-\lambda\right)-\sum_{i=1}^{n}\left(\frac{x_i-\lambda}{\beta}\right)-n\left(\alpha-1\right)\log\beta-n\log\Gamma\left(\alpha\right) \qquad \rightarrow(6)$$

By taking the partial derivative of the log-likelihood function with respect to parameters α, β and λ and equate to zero, we have

$$\frac{\partial LogL}{\partial\beta}=0\Rightarrow\sum_{i=1}^{n}\left(\frac{x_i-\lambda}{\beta^2}\right)-\frac{n\left(\alpha-1\right)}{\beta}=0 \qquad \rightarrow(7)$$

Multiply (7) through by β^2

$$=\sum_{i=1}^{n}\left(\frac{x_i-\lambda}{\beta^2}\right)\times\beta^2-\frac{n\left(\alpha-1\right)}{\beta}\times\beta^2=0 \qquad \rightarrow(8)$$

$\frac{\partial LogL}{\partial\beta}$ can be rewritten in a simplest form as:

$$\frac{\partial LogL}{\partial\beta}=\sum_{i=1}^{n}\left(x_i-\lambda\right)-n\left(\alpha-1\right)\beta=0$$

$$\sum_{i=1}^{n}\left(x_i-\lambda\right)=n\left(\alpha-1\right)\beta \qquad \rightarrow(9)$$

$$\hat{\beta}=\frac{\sum_{i=1}^{n}\left(x_i-\hat{\lambda}\right)}{n\left(\hat{\alpha}-1\right)} \qquad \rightarrow(10)$$

$$\frac{\partial LogL}{\partial\alpha}=0\Rightarrow\sum_{i=1}^{n}\log\left(x_i-\lambda\right)-0-n\log\beta-n\Psi\left(\alpha\right)=0 \qquad \rightarrow(11)$$

where, $\Psi\left(\alpha\right)$ is the digamma function defined as $\Psi\left(\alpha\right)=\frac{\partial Log\left(\Gamma\alpha\right)}{\partial\alpha}$

$$\frac{\partial LogL}{\partial\lambda}=0\Rightarrow-\frac{-\alpha n}{\lambda}+\frac{n}{\lambda}+\frac{n}{\beta}=0 \qquad \rightarrow(12)$$

$$=\frac{n\left(1-\alpha\right)}{\lambda}+\frac{n}{\beta}=0 \qquad \rightarrow(13)$$

$$=\frac{n\beta\left(1-\alpha\right)+\lambda n}{\lambda\beta}=0\Rightarrow n\beta\left(1-\alpha\right)+\lambda n=0 \qquad \rightarrow(14)$$

$$= \beta(1-\alpha)+\lambda = 0 \qquad \rightarrow (15)$$

$$\hat{\lambda} = \hat{\beta}\left(1-\hat{\alpha}\right) \qquad \rightarrow (16)$$

Equation (11) does not exist in a closed form hence the estimation of α can only be obtained through numerical solution.

2.2 Lognormal PDF Estimation

$$L(y,\theta) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \prod_{i=1}^{n}(x_i-\gamma)^{-1}\exp\left[-\frac{\sum_{i=1}^{n}\left(\ln(x_i-\gamma)-\mu\right)^2}{2\sigma^2}\right] \qquad \rightarrow (17)$$

taking the log of the likelihood function of 16 we have

$$\ln(L(y,\theta)) = -n\ln\sigma - n\ln\sqrt{2\pi} - \sum_{i=1}^{n}\ln(x_i-\gamma) - \frac{\sum_{i=1}^{n}\left(\ln(x_i-\gamma)-\mu\right)^2}{2\sigma^2} \qquad \rightarrow (18)$$

Differentiating equation (18) with respect to μ and setting the derivative to zero we have

$$\frac{d\,ln(Lf(y;\gamma,\mu,\sigma))}{d\mu} = 0-0-0-\left[-2\frac{\sum_{i=1}^{n}\left(ln(x_i-\gamma)-\mu\right)}{2\sigma^2}\right] = 0$$

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}[ln(x_i-\gamma)-\mu] = 0 \qquad \rightarrow (19)$$

Multiplying both sides of equation (19) by σ^2 and forcing the summation into the bracket we have

$$\sum_{i=1}^{n}\ln(x_i-\gamma) - n\mu = 0$$

$$\sum_{i=1}^{n}\ln(x_i-\gamma) = n\mu$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\ln(x_i-\gamma) = \left(\frac{1}{n}\sum_{i=1}^{n}\ln x_i - \gamma\right) \qquad \rightarrow (20)$$

Differentiating equation (18) with respect to σ and setting the derivative to zero we have

$$\frac{d\ln(L(y,\theta))}{d\sigma} = -\frac{n}{\hat{\sigma}} - 0 - 0 - \left(-\frac{1}{\hat{\sigma}^3}\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu]^2\right) = 0$$

$$-\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu]^2 = 0$$

$$\frac{\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu]^2}{\sigma^3} - \frac{n}{\sigma} = 0$$

$$\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu]^2 - n\sigma^2 = 0$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu]^2}{n}$$

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu]^2}{n}} \rightarrow (21)$$

Differentiating equation (18) with respect to γ and equating the derivative to zero

$$\frac{d\ln(L(y,\theta)}{d\gamma} = 0-0-\sum_{i=1}^{n}\frac{1}{(x_i-\gamma)} + 4\sigma^2\sum_{i=1}^{n}\frac{[\ln(x_i-\gamma)-\mu]}{4\sigma^4(x_i-\gamma)} = 0 \rightarrow (22)$$

$$\sum_{i=1}^{n}\frac{1}{(x_i-\gamma)} + \frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{[ln(x_i-\gamma)-\mu]}{(x_i-\gamma)} = 0 \rightarrow (23)$$

Equation (23) can be rewritten as follows

$$\frac{n}{\sum_{i=1}^{n}(x_i-\gamma)} + \frac{1}{\sigma^2}\left[\frac{\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu]}{\sum_{i=1}^{n}(x_i-\gamma)}\right] = 0 \rightarrow (24)$$

Multiply equation (24) by $\sigma^2\sum_{i=1}^{n}(x_i-\gamma)$ and expand the equation we have

$$\frac{n\sigma^2\sum_{i=1}^{n}(x_i-\gamma)}{\sum_{i=1}^{n}(x_i-\gamma)} + \frac{1}{\sigma^2}\sigma^2\sum_{i=1}^{n}(x_i-\gamma)\left[\frac{\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu]}{\sum_{i=1}^{n}(x_i-\gamma)}\right] = 0$$

$$n\sigma^2 + \sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu] = 0 \rightarrow (24)$$

$$n\sigma^2 + \sum_{i=1}^{n}\ln(x_i-\gamma) - n\mu = 0$$

$$n\sigma^2 + \sum_{i=1}^{n}\ln(x_i) - n\ln\gamma - n\mu = 0$$

$$n\sigma^2 + \sum_{i=1}^{n}\ln(x_i) - n\mu = n\ln\gamma \rightarrow (25)$$

$$\ln\gamma = \sigma^2 + \frac{1}{n}\sum_{i=1}^{n}\ln(x_i) - \mu$$

$$\hat{\gamma} = e^{\left[\sigma^2+\frac{1}{n}\sum_{i=1}^{n}\ln(x_i)-\mu\right]} \rightarrow (26)$$

Substituting the scale ($\hat{\mu}$) and shape ($\hat{\sigma}$) parameters into equation (26) we have

$$\hat{\gamma} = \exp\left(n^{-\frac{1}{2}}\sum_{i=1}^{n}[\ln(x_i-\gamma)-\mu] + n^{-1}\sum_{i=1}^{n}\ln(x_i)-\mu\right) \rightarrow (27)$$

$$\hat{\gamma} = \exp\left(n^{-\frac{1}{2}}\sum_{i=1}^{n}\left[\ln(x_i-\gamma)-n^{-1}\sum_{i}^{n}\ln(x_i-\gamma)\right] + n^{-1}\sum_{i=1}^{n}\ln(x_i) - n^{-1}\sum_{i}^{n}\ln(x_i-\gamma)\right) \rightarrow (28)$$

Equation (28) does not exist in a closed form because $\hat{\mu}$ and $\hat{\sigma}$ estimates contain $\hat{\gamma}$ hence the estimation of $\hat{\gamma}$ can only be obtained through numerical solution.

2.3 Weibull PDF Estimation

$$Log\ f(x;\alpha,\beta,\mu)=\prod_{i=1}^{n}\frac{\alpha}{\beta}\left(x_i-\mu\right)^{\alpha-1}\exp\left[-\frac{\left(x_i-\mu\right)^{\alpha}}{\beta}\right] \rightarrow (29)$$

$$=\prod_{i=1}^{n}\frac{\alpha}{\beta}\left(x_i-\mu\right)^{\alpha-1}\exp\left[-\frac{\left(x_i-\mu\right)^{\alpha}}{\beta}\right] \rightarrow (30)$$

Taking the log likelihood function of the equation (30), we have:

$$LogL\left((x;\gamma,\beta,\alpha)\right)=-n\log\left(\frac{\alpha}{\beta}\right)-\frac{1}{\beta}\sum_{i=1}^{n}\left(x_i-\mu\right)^{\alpha}+\left(\alpha-1\right)\sum_{i=1}^{n}\log\left(x_i-\mu\right) \rightarrow (31)$$

Differentiating equation (31) with respect to β and setting the derivative to zero, we have:

$$\frac{d\log l}{d\beta}=0\Rightarrow-\frac{n}{\beta}+\frac{1}{\beta^2}\sum_{i=1}^{n}\left(x_i-\mu\right)^{\alpha}=0 \rightarrow (32)$$

Making change of formula

$$\frac{1}{\beta^2}\sum_{i=1}^{n}\left(x_i-\mu\right)^{\alpha}=\frac{n}{\beta}$$

$$\frac{\sum_{i=1}^{n}\left(x_i-\mu\right)^{\alpha}}{\beta^2}=\frac{n}{\beta}$$

$$\sum_{i=1}^{n}\left(x_i-\mu\right)^{\alpha}\beta=n\beta^2$$

$$\frac{\sum_{i=1}^{n}\left(x_i-\mu\right)^{\alpha}\beta}{n\beta}=\frac{n\beta^2}{n\beta}$$

$$\hat{\beta}=\frac{\sum_{i=1}^{n}\left(x_i-\mu\right)^{\alpha}}{n} \rightarrow (33)$$

Differentiating equation (31) with respect to γ and setting the derivative to zero, we have:

$$\frac{d\log l}{d\alpha}=0\Rightarrow\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(x_i-\mu\right)-\frac{1}{\beta}\sum_{i=1}^{n}\log\left(x_i-\mu\right)^{\alpha}\log\left(x_i-\mu\right)=0 \rightarrow (34)$$

Differentiating equation (31) with respect to α and setting the derivative to zero, we have:

$$\frac{d\log l}{d\mu}=0\Rightarrow-\frac{1}{\beta}\alpha\sum_{i=1}^{n}\left(x_i-\mu\right)\left(-1\right)+\left(\alpha-1\right)\sum_{i=1}^{n}\log\left(x_i-\mu\right)^{-1}\left(-1\right)=0 \rightarrow (35)$$

$$=\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(x_i-\mu\right)-\left(\alpha-1\right)\sum_{i=1}^{n}\log\left(x_i-\mu\right)^{-1}=0 \rightarrow (36)$$

$$=\frac{\alpha}{\beta}\sum_{i=1}^{n}\left(x_i-\mu\right)-\left(\alpha-1\right)\sum_{i=1}^{n}\log\frac{1}{\left(x_i-\mu\right)}=0 \rightarrow (37)$$

Equations (34) and (37) have no close form solutions using maximum likelihood estimation method. Hence, it will be obtained through numerical solution.

III. MODEL SELECTION CRITERIA

In this section, the goodness of fit of each estimated model is presented. Two measures for the best model, that is, information criteria are used. These measures are Akaike Information Criteria (AIC) and Bayes Criteria (BIC).

$$\text{AIC} = 2K - 2\,ln(LL) \quad \rightarrow (38)$$

$$\text{BIC} = k\,ln\left(n\right) - 2\,ln(LL) \quad \rightarrow (39)$$

where, k is the number of parameters in the model and LL is the maximized value of the likelihood function for the model.

IV. FINDINGS AND DISCUSSION

In this section, the results from fitting each probability distribution to the data are presented and discussed.

Table 1: Summary Statistics for the Rainfall Data

Statistic	
Mean	96.40139815
Standard Error	0.726840625
Median	96.7386054
Standard Deviation	7.794497343
Sample Variance	60.75418883
Kurtosis	0.200587948
Skewness	-0.375245916
Range	38.8265213
Minimum	73.01479005
Maximum	111.8413114
Sum	11086.16079
Count	1380
Confidence Level (95.0%)	1.43986568

Table 1 above presents the summary statistics for the rainfall data and it can be observed that the mean rainfall intensity for Nigeria is 96.4014(mm) with a slight increase in the median at 96.7386(mm). Other statistics presented are the standard deviation, range, sample variance and confident level. Also, it is observed that the Nigeria monthly rainfall intensity is skewed to the left with a value of − 0.37524 and its kurtosis value is 0.20058.

Table 2: Maximum Likelihood Parameter Estimates Results

Distribution	Parameter Estimate
3 Parameter Gamma	threshold = -10.213 scale = 0.579615 shape = 183.9402
3 Parameter Lognormal	threshold = -119.78 scale = 5.375466 shape = 0.036357
3 Parameter Weibull	threshold = 57.46 scale = 42.0418 shape = 5.811701

Table 2 presents the estimated parameter of the three distributions that is, threshold (location), scale and shape estimates. It can be observed that the Weibull distribution function produced the threshold and scale parameter estimates of 57.46 and 42.0418 respectively compared to Lognormal (− 119.78 and 5.3754 respectively) and Gamma (-10.213 and 0.5796 respectively) distribution functions. While the Gamma distribution function produced the largest shape parameter estimate of 183.9401 compared to the Lognormal (0.036357) and Weibull (5.811701) distribution functions.

From the results in Table 3 below shows that weibull has the smallest AIC and BIC with the value of 18.40866 and 34.09817 respectively followed by lognormal distribution and gamma distribution. We can therefore conclude that the best model for modelling monthly rainfall intensity in Nigeria followed weibull distribution.

Table 3: Model selection criteria of the three fitted distributions

Distribution	Log likelihood	AIC	BIC	Rank
3 Parameter Gamma	-412.13	830.26	845.95	3
3 Parameter Lognormal	-207.6509	421.3018	436.9913	2
3 Parameter Weibull	-6.204328	18.40866	34.09817	1

V. CONCLUSION

Based on the results obtained and interpreted from the analysis performed, it was observed that the maximum likelihood estimates for the 3-parameter gamma, lognormal and Weibull models produced estimates that are not significantly different from each other for the fitted Nigeria average monthly rainfall intensity. Furthermore, using the information criteria that is, AIC and BIC to selection the best model in fitted monthly rainfall in Nigeria, the three parameter weibull distribution performed better than the other two distributions under study.

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