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Research Paper

Regularity and Singularuty of Matrix Polynomial

Bandana Bhunia

***Abstract:** - Here we derived the, some necessary and sufficient conditions for column - &row-regularity & singularity matrix polynomials. The solution behavior of an initial value problem for the system of Differential – Algebraic – Equations is closely related to the properties of regularity & singularity of the matrix polynomials associated with the system. The close relatedness provides us one of the major motivations to study regularity & singularity of matrix polynomials. We shall study from the point of view of the theory of matrices, regularity & singularity of $(m \times n)$ matrix polynomials of degree r,*

$M(\lambda) = \sum_{i=0}^{r} \lambda^i M_i = \lambda^r M_r + \lambda^{r-1} M_{r-1} + \cdots + \lambda M_1 + M_0$ *(1)*

Where $\lambda \in C$ and the matrices $M_i \in C^{m\times n}$, i =0,1,2,⋯, r.Here we call a matrix polynomial $M(\lambda)$ column – singular (or row singular),if rank $(M) < n$ (or $rank(M) < m$). Otherwise it is column regular (or row regular). Using this unimodular matrix polynomial we show that the rank of a matrix polynomial is invariant under left -equivalence, right – equivalence & equivalence transformations respectively. Any two $(m \times n)$ matrix polynomials are said to be left- equivalent, right – equivalent & equivalent or to be connected by a left-equivalence transformation, a right – equivalence transformations & a equivalence transformation, if one of them can be obtained from the other by means of a finite sequence of row elementary, column elementary & row or column elementary operations, respectively.

***KEY – WORD:** - Regularity & singularity of matrix polynomials. Eigen value, eigen vector. Minor & rank of matrix polynomial. Column & row singular, unimodular matrix, equivalent matrix, unimodular matrix.*

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I. INTRODUCTIO:

A polynomial with matrix coefficients is called a matrix polynomial or a polynomial matrix, if we regard it as a matrix whose elements are polynomials. It is well known that matrix polynomials play a vital role in the analytical theory of elementary divisors, that is the theory by which a square matrix can be reduced to some normal form (such as the Smith canonical form and Jordan canonical form) of which important applications have been made to the analysis of differential and difference equation. The motivation for our study of regularity & singularity of matrix polynomials, comes mainly from two sources. One is the study of Differential – Algebraic – Equations, which is due to the close connection between regularity & singularity of matrix polynomials and the properties of the solutions of the system of Differential – Algebraic – Equations which is associated with the matrix polynomial, the other is the study of the polynomial eigen value problems:

$M(\lambda)\ x = 0,\ x \neq 0,\ y^H M(\lambda) = 0,\ y \neq 0$

Where $M(\lambda) = \sum_{i=0}^{r} \lambda^i M_i$ is an $n \times n$ matrix polynomial of degree r, $M_i \in C^{n\times n}$, i =0,1,2,⋯, r, $M_r \neq 0$. The nonzero vector $x \in C^n$ ($y \in C^n$) is the right (left)eigen vector associated with the eigen value λ.

II. MAIN RESULTS:

Definition: - A matrix polynomial $M(\lambda)$ over C (or R) is a polynomial in λ with matrix coefficients:

$M(\lambda) = \sum_{i=0}^{r} \lambda^i M_i = \lambda^r M_r + \lambda^{r-1} M_{r-1}\ M + \cdots + \lambda M_1 + M_0$ (1)

Where $\lambda \in C$ and the matrices $M_i \in C^{m\times n}$ ($\in R^{m\times n}$), i =0,1,2,⋯, r.

If $m = n$, then matrix polynomial $M(\lambda)$ is called square, and number n is called the order of the matrix polynomial, the number r is called the degree of the matrix polynomial, if $M_r \neq 0$.

If $m = 1$ ($or\ n = 1$), then matrix polynomial $M(\lambda)$ is called a row (or column) vector polynomial.

NOTE: - We may represent the matrix polynomial $M(\lambda)$ in the form of a polynomial matrix, that is in the form of an $(m \times n)$ matrix whose elements are polynomial in λ

$M(\lambda) = \left[a_{i,j}^{(r)} \lambda^r + a_{i,j}^{(r-1)} \lambda^{r-1} + \cdots + a_{i,j}^{(0)}\right]_{i,j=1}^{m,n}$

Where r is degree of the matrix polynomial. If $m = 1$ ($or\ n = 1$), then matrix polynomial $M\ (\lambda)$ is called a row (or column) vector polynomial.

In order to introduce the concepts of regularity & singularity of matrix polynomials, we required the ideas of the minor and rank of a matrix polynomial which are natural generalizations of those of the minor and rank of a matrix, as follows:

Definition: - Let $M\ (\lambda)$ be an $m \times n$ rectangular matrix polynomial. A minor of order s (1≤ s≤ $\min(m,n)$ of $M\ (\lambda)$ is defined to be the determinant of a $s \times s$ sub matrix polynomial of $M\ (\lambda)$, obtained from $M\ (\lambda)$ by striking out $(m - s\ )$ rows & $(n - s)$ columns. If the retained rows & columns are given by subscripts,

$1\leq\ i_1 < i_2 < \cdots < i_s \leq m$,

$1\leq\ j_1 < j_2 < \cdots < j_s \leq n$, respectively, then the corresponding $s - th$ order minor is denoted by,

$M\ (\lambda) \begin{pmatrix} i_1 & i_2 & \cdots & i_s \\ j_1 & j_2 & \cdots & j_s \end{pmatrix}$ = det $\left[a_{i_l,j_l}(\lambda)\right]_{l=1}^{s}$.

Definition: - An integer v is said to be the rank of the matrix polynomial if it is the rank of it's largest minor that is not identically equal to zero obviously, $v \leq \min(m,n)$.

Example: -

1. For $M_1(\lambda) = \lambda \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda+1 & 2\lambda \\ 3\lambda & 0 \\ \lambda & \lambda \end{bmatrix}$. Then rank ($M_1(\lambda)$ = 2.

2. For $M_2(\lambda)=\lambda^2 \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 2 \\ -2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2(\lambda+1)(\lambda-1) & 2(\lambda+1) \\ (\lambda-1)^2 & \lambda-1 \\ \lambda(\lambda-1) & \lambda \end{bmatrix}$.

   Then rank ($M_2(\lambda)$) = 1.

3. For $M_3(\lambda) = \lambda \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2\lambda+1 \\ 0 & \lambda+2 \end{bmatrix}$. Then rank ($M_3(\lambda)$ = 1.

4. For $M_4(\lambda) = \lambda^2 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\lambda^2 & 2\lambda \\ \lambda & 1 \end{bmatrix}$. Then rank ($M_4(\lambda)$ = 1.

Definition: - A matrix $M \in C^{m\times n}$ is said to be column regular, or to have full column rank, if rank (M) = n,otherwise, it is said to be column singular or column rank deficient. And matrix $M \in C^{m\times n}$ is said to be row regular, or to have full row rank, if rank (M) = m,otherwise, it is said to be row singular or row rank deficient.

Property: - A matrix $M \in C^{m\times n}$ is column – regular (or column – singular) if and only if its conjugate transpose $M^H \in C^{n\times m}$ (or transpose $M^T \in C^{n\times m}$) is row – regular (or row – singular).

Definition: - Let $M\ (\lambda\ ) = \sum_{i=0}^{r} \lambda^i M_i$ is an $m \times n$ matrix polynomial of degree r, where $m\ ,n, r \in N_0$ $M_{\ i} \in C^{m\times n}$, i =0,1,2,⋯, r, let v be the rank of $M\ (\lambda\ )$,then

1. $v < n$, then $M\ (\lambda\ )$ is said to be column singular, otherwise, if $v = n$ then matrix polynomial is said to be column regular.
2. $v < m$, then $M\ (\lambda\ )$ is said to be row singular, otherwise, if $v = m$ then matrix polynomial is said to be row regular.
3. $v = m = n$, then $M\ (\lambda\ )$ is said to be regular, otherwise, if $v < m = n$ then matrix polynomial is said to be singular.

Note: - In order to be consistent with the concepts of the regularity and singularity of matrix pencils, we always call a non – square matrix polynomial is singular though by previous definition it may be column – regular or row – regular.

Property 1: - If $M\ (\lambda\ ) = \sum_{i=0}^{r} \lambda^i M_i$ is an $n \times n$ matrix polynomial of degree r, where $n, r \in N_0$, $M_{\ i} \in C^{n\times n}$, i =0,1,2,⋯, r. Then $M\ (\lambda\ )$ is singular if and only if $\forall\ \lambda \in C$,
det ($M\ (\lambda)$) = det($\lambda^r M_r + \lambda^{r-1} M_{r-1} + \cdots + \lambda M_1 + M_0$) = 0.

Property 2: -A matrix $M \in C^{m\times n}$ is column singular if and only if its column vectors are linearly dependent in C^m, or in other words ∃ a nonzero vector $x \in C^n$ such that Mx=0

Along the same lines, our main idea is to prove that a matrix polynomial $M\ (\lambda\ )$ is column – singular if and only if ∃ a vector polynomial $x\ (\lambda)$, which is not identically equal to zero, such that $M\ (\lambda\ )\ x\ (\lambda)$ = 0. It is asserted that the equation ($M_1\lambda + M_0$) $x\ (\lambda)$ = 0 has a nonzero vector polynomial $x\ (\lambda)$ as its solution for any given singular matrix pencil $M_1\lambda + M_0$.

Definition: - An $n \times n$ square matrix polynomial $M(\lambda)$ with nonzero constant determinant is referred to as unimodular matrix polynomial.

Now from the equivalence of elementary operations & operations with elementary matrix polynomial that $M(\lambda)$ & $P(\lambda)$ are

1.Left equivalent 2. Right equivalent & 3. Equivalent,

If and only if there are elementary matrix polynomial $F_1(\lambda), F_2(\lambda), \cdots F_k(\lambda), F_{k+1}(\lambda) \cdots F_s(\lambda)$ such that

1. $P(\lambda) = R(\lambda)\ M(\lambda)$ 2. $P(\lambda) = M(\lambda)\ S(\lambda)$ & 3. $P(\lambda) = R(\lambda)\ M(\lambda)\ S(\lambda)$, respectively, where $R(\lambda) = F_k(\lambda) \cdots F_1(\lambda)$ & $S(\lambda) = F_{k+1}(\lambda) \cdots F_s(\lambda)$

Properties: - The rank of the matrix polynomial is invariant under equivalence transformation.

Proof: - Let the matrix polynomial $M(\lambda)$ & $P(\lambda)$ be equivalent. Then there are unimodular matrix $R(\lambda)$ & $S(\lambda)$ such that

$P(\lambda) = R(\lambda)\ M(\lambda)\ S(\lambda)$.

Apply the Binet– Cauchy formula twice to this equation to express a minor $p(\lambda)$ of order 's' of $P(\lambda)$ in terms of minor $m_s(\lambda)$ of M(λ) of the same order as follows

$p(\lambda) = \sum_s r_s(\lambda)\ m_s(\lambda)\ s_s(\lambda)$. (1)

Where the $r_s(\lambda)$ & $s_s(\lambda)$ denote the appropriate minor of order 's' of the matrix polynomials $R(\lambda)$ & $S(\lambda)$. Respectively. If now $p(\lambda)$ is a nonzero minor of $P(\lambda)$ of the greatest order v (that is, the rank of $P(\lambda)$ is v), then it follows from equation (1) that at least one minor $m_s(\lambda)$ (of order v) is nonzero polynomial and hence the rank of $P(\lambda)$ does not exceed the rank of $M(\lambda)$.

∴ rank($P(\lambda)) \leq$ rank $(M(\lambda))$

(2)

However, applying the same argument to the equation

$M(\lambda) = [R(\lambda)]^{-1}\ P(\lambda)\ [S(\lambda)]^{-1}$, we obtain

rank $(M(\lambda) \leq$ rank $(P(\lambda))$ (3)

Hence from (2) & (3) we get,

rank $(P(\lambda) =$ rank $(M(\lambda))$

Thus, the ranks of matrix polynomials are invariant under equivalence transformation.

Properties: - The matrix polynomial $M(\lambda)$ & $P(\lambda)$ are equivalent denoted by $M \sim P$ then

(i). $M \sim M$,

(ii). $M \sim P => P \sim M$,

(iii). $M \sim P, P \sim Q => M \sim Q$.

Example: - Show that any unimodular matrix is equivalent to identity matrix.

Solution – Let $M(\lambda)$ be the unimodular matrix, so its determinant value is nonzero. Therefor inverse of unimodular matrix $M(\lambda)$ exists. So

$M(\lambda)\ [M(\lambda)]^{-1} = I$

Or $I \,.\, M(\lambda)\ [M(\lambda)]^{-1} = I$

∴ $R(\lambda)\ M(\lambda)\ S(\lambda) = I$.

Where $R(\lambda) = I,\ S(\lambda) = [M(\lambda)]^{-1}$, which shows that $M(\lambda)$ is equivalent to identity matrix.

Proposition: - An arbitrary $m \times n$ rectangular matrix polynomial $M(\lambda) = \left[a_{i,j}(\lambda)\right]_{i,j=1}^{m,n}$ of degree r can be transformed through row elementary operations in to an $m \times n$ upper triangular matrix polynomial $P(\lambda)$ that is described in the following form (1) & (2), where the polynomial $e_{1,j}(\lambda)$, $e_{2,j}(\lambda)$, $\cdots$, $e_{j-1,j}(\lambda)$ are of degree less than degree of $e_{j,j}(\lambda)$, provided $e_{j,j}(\lambda)$ is not identically equal to zero and all are identically equal to zero if $e_{j,j}(\lambda)$ = constant $\neq 0$, j = 2, 3, ⋯, min (m, n).

$$\begin{bmatrix} e_{1,1}(\lambda) & e_{1,2}(\lambda) & \cdots & e_{1,m}(\lambda) & \cdots & e_{1,n}(\lambda) \\ 0 & e_{2,2}(\lambda) & \cdots & e_{2,m}(\lambda) & \cdots & e_{2,n}(\lambda) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_{m,m}(\lambda) & \cdots & e_{m,n}(\lambda) \end{bmatrix} \text{(for } m \leq n) \qquad (1).$$

And
$$\begin{bmatrix} e_{1,1}(\lambda) & e_{1,2}(\lambda) & \cdots & e_{1,n}(\lambda) \\ 0 & e_{2,2}(\lambda) & \cdots & e_{2,n}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e_{n,n}(\lambda) \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & o & \cdots & 0 \end{bmatrix} \text{(for } m \geq n) \qquad (2)$$

Proof: -- To begin with, we shall examine what comparatively simple form we can obtain for a rectangular polynomial matrix $M(\lambda)$ by means of row elementary operations only.

Let us assume that the 1st column of $M(\lambda)$ contains elements not identically equal to zero. Among them we choose a polynomial of least degree and by a permutation of the rows, we make it into the elements $a_{1,1}(\lambda)$. then we divide $a_{i,1}(\lambda)$ by $a_{1,1}(\lambda)$; we denote quotient and remainder by $q_{i,1}(\lambda)$ and $r_{i,1}(\lambda)$, $i = 2, 3, \cdots, m$, reapectively.

$a_{i,1}(\lambda) = a_{1,1}(\lambda)\, q_{i,1}(\lambda) + r_{i,1}(\lambda)$, $(i = 2, 3, \cdots, m)$

Now we subtract the first row multiplicated by) $q_{i,1}(\lambda)$,$(i = 2, 3, \cdots, m)$, from the $i -$ th row.If not all the remainder $r_{i,1}(\lambda)$ are identically equal to zero, then we choose one of them ,that is not equal to zero and is of least degree and put into the place of $a_{1,1}(\lambda)$ by a permutation of the rows . as the results of all these operations the degree of the polynomial $a_{1,1}(\lambda)$ is reduced.

Now we repeat this process , since the degree of the polynomial $a_{1,1}(\lambda)$ is finite , this must come to an end at some stage ,that is at this stage all the elements $a_{2,1}(\lambda)$, $a_{3,1}(\lambda)$,⋯, $a_{m,1}(\lambda)$ turn out to be identically equal to zero.

Next, we take the elements $a_{2,2}(\lambda)$ and apply the same procedure to rows numbered $2, 3, \cdots, m$. Achieving $a_{3,2}(\lambda) = \cdots = a_{m,2}(\lambda) = 0$. Containing still further, we finally reduce the matrix $M(\lambda)$ to the following form

$$\begin{bmatrix} e_{1,1}(\lambda) & e_{1,2}(\lambda) & \cdots & e_{1,m}(\lambda) & \cdots & e_{1,n}(\lambda) \\ 0 & e_{2,2}(\lambda) & \cdots & e_{2,m}(\lambda) & \cdots & e_{2,n}(\lambda) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_{m,m}(\lambda) & \cdots & e_{m,n}(\lambda) \end{bmatrix} \text{(for } m \le n) \text{ . And}$$

$$\begin{bmatrix} e_{1,1}(\lambda) & e_{1,2}(\lambda) & \cdots & e_{1,n}(\lambda) \\ 0 & e_{2,2}(\lambda) & \cdots & e_{2,n}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e_{n,n}(\lambda) \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & o & \cdots & 0 \end{bmatrix} \text{(for } m \ge n).$$

If the polynomial $e_{2,2}(\lambda)$ is not identically equal to zero, then by applying a row- elementary operation of the second type we can make the degree of the element $e_{1,2}(\lambda)$ less than d the degree of the element t$e_{2,2}(\lambda)$. (if $e_{2,2}(\lambda)$ is of degree zero, then $e_{1,2}(\lambda)$ becomes identically equal to zero). In the same way, if $e_{3,3}(\lambda) = 0$, then by row elementary operation of the 2nd type we make the degree of elements $e_{1,3}(\lambda)$, $e_{2,3}(\lambda)$ less than the degree of $e_{3,3}(\lambda)$ etc.

Proposition: -- An $n \times n$ square matrix polynomial $M(\lambda)$ is unimodular can be represented in the form of a finite number of elementary matrices.

Proof: -- By previous proposition, the matrix $M(\lambda)$ can be brought into the form

$$\begin{bmatrix} e_{1,1}(\lambda) & e_{1,2}(\lambda) & \cdots & e_{1,n}(\lambda) \\ 0 & e_{2,2}(\lambda) & \cdots & e_{2,n}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e_{n,n}(\lambda) \end{bmatrix} \qquad (1)$$

By row elementary operations, where n is the order of $M(\lambda)$. Since in the application of elementary operations to a square matrix polynomial the determinant of the matrix is only multiplied by constant non zero factor, the determinant of the matrix (1), like of $M(\lambda)$, does not depend on λ and is different from 0. That is

$e_{1,1}(\lambda), e_{2,2}(\lambda), \cdots, e_{n,n}(\lambda)$ = constant $\neq 0$.

Hence $e_{j,j}(\lambda)$ = constant $\neq 0$. $(i = 2, 3, \cdots, m)$.

Again, also by previous proposition, the matrix (1) has a diagonal form $\left[b_j\, \delta_{i,j}\right]_{i=1}^{n}$ and can therefore be reduced to the unit matrix F by means of row-elementary operations of type-1. But then conversely, the unit matrix F can be transformed into $M(\lambda)$ by means of the row elementary operations whose matrix are $S_1, S_2, \cdots, S_s$. Therefore

$M\ (\lambda) = S_s S_{s-1} \cdots S_1 F \ = \ S_s S_{s-1} \cdots S_1.$

Proposition: -- Let $M\ (\lambda)\ be\ a$n $m \times n$ rectangular matrix polynomial of degree r, and let $M\ (\lambda)$ be left-equivalent to an $m \times n$ matrix polynomial $P\ (\lambda)$, if

$$P\ (\lambda) = \begin{bmatrix} P_{1,1}(\lambda) & P_{1,2}(\lambda) \\ 0 & P_{2,2}(\lambda) \end{bmatrix} \qquad (1)$$

Where the polynomial matrix $P_{1,1}(\lambda)$ is of dimension $s \times s$, $s \in N$, $1 \le s \le \min(m,n)$, and det $(P_{1,1}(\lambda))$ is not identically to zero, then
deg (det $(P_{1,1}(\lambda)) \le sr$.

Proof: --Since the matrix polynomial $M\ (\lambda)$ is left- equivalent to $P\ (\lambda)$ then there exists a unimodular matrix polynomial $R(\lambda)$, such that

$M\ (\lambda) = R\ (\lambda)\ P\ (\lambda)$ (2).

Let $M\ (\lambda) = [M_1(\lambda)\ M_1(\lambda)]$, where the matrix polynomial $M_1(\lambda)$ is of dimension $m \times s$. Then from (1) & (2), we have

$$M_1(\lambda) = R\ (\lambda) \begin{bmatrix} P_{1,1}(\lambda) \\ 0 \end{bmatrix}, \qquad (3)$$

Such as, $M_1(\lambda)$ is left equivalent to $[P_{1,1}(\lambda), 0]^T$. Since det $(P_{1,1}(\lambda))$ is not identically equal to zero, then rank ($[P_{1,1}(\lambda), 0]^T$ = rank $(P_1(\lambda)\ ) = s$. (4)

As $M_1(\lambda)$ is left equivalent to $[P_{1,1}(\lambda), 0]^T$ and we know that the rank of a matrix polynomial is invariant under left equivalence transformation, then we have

rank $(M_1(\lambda)\ ) = s$. (5)

Again, as we know an integer s is said to be rank of a matrix polynomial if it is order of its largest minor that is not identically zero, obviously $s \le \min(m,n)$,then there exists a permutation matrix F of dimension $m \times m$, such that the leading principal submatrix $\widehat{M}_{1,1}(\lambda)$ of $F\ M_1(\lambda)$has full rank s,where $\widehat{M}_{1,1}(\lambda)$ is of dimension $s \times s$. Hence by (3), we get

$$F\ M_1(\lambda) = \begin{bmatrix} \widehat{M}_{1,1}(\lambda) \\ \widehat{M}_{2,1}(\lambda) \end{bmatrix} = F\ R(\lambda) \begin{bmatrix} P_{1,1}(\lambda) \\ 0 \end{bmatrix} \qquad (6)$$

Where $\widehat{M}_{2,1}(\lambda)$ is of dimension $(m-s) \times s$. We rewrite (6) in the form

$$\begin{bmatrix} \widehat{M}_{1,1}(\lambda) \\ \widehat{M}_{2,1}(\lambda) \end{bmatrix} = \begin{bmatrix} \hat{R}_{1,1}(\lambda) & \hat{R}_{1,2}(\lambda) \\ \hat{R}_{2,1}(\lambda) & \hat{R}_{2,2}(\lambda) \end{bmatrix} \begin{bmatrix} P_{1,1}(\lambda) \\ 0 \end{bmatrix} \qquad (7)$$

Where $F\ R(\lambda) = \begin{bmatrix} \hat{R}_{1,1}(\lambda) & \hat{R}_{1,2}(\lambda) \\ \hat{R}_{2,1}(\lambda) & \hat{R}_{2,2}(\lambda) \end{bmatrix}$, and the matrix polynomial $\hat{R}_{1,1}(\lambda)$ is of dimension $s \times s$;then we have,

$\widehat{M}_{1,1}(\lambda) = \hat{R}_{1,1}(\lambda)\ P_{1,1}(\lambda)$ (8)

And therefore, it follows that,

det ($\widehat{M}_{1,1}(\lambda)$) = det ($\hat{R}_{1,1}(\lambda)$) det ($P_{1,1}(\lambda)$) (9)

Note that $\widehat{M}_{1,1}(\lambda)$ has full rank, or in other words det $(\widehat{M}_{1,1}(\lambda))$ is a polynomial in λ which is not identically equal to zero. By (9) it follows that det $(\hat{R}_{1,1}(\lambda))$ is not identically equal to zero, and therefore also

$0 \le$ deg (det $(P_{1,1}(\lambda))) \le$ deg (det $(\widehat{M}_{1,1}(\lambda))$) (10)

Since $F\ M_1(\lambda)$ is obtained from $M_1(\lambda)$ through interchanges of rows, every entry of the submatrix polynomial $\widehat{M}_{1,1}(\lambda)$ of $F\ M_1(\lambda)$ is either a polynomial in λ with its degree less than or equal to 1 or zero. Thus, we have,

deg (det $(\widehat{M}_{1,1}(\lambda))) \le\ sr$ (11),

hence from (10) & (11), we get

deg (det $(P_{1,1}(\lambda)) \le sr$.

III. CONCLUSION:

We have discussed the regularity and singularity of matrix polynomials. Several sufficient and necessary conditions for the column- and row- regularity and singularity of rectangular matrix polynomials have been presented. Such conditions have laid a theoretical foundation for the subsequent related investigations. We have presented a canonical form under equivalence transformations for 2×2 singular quadratic matrix polynomials.

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