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**Research Paper**

# Orthogonality of Some Related Polynomials to Three-parameter Mittag–Leffler Function

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***ABSTRACT:*** *In this paper, the relationship between 3-parameter Mittag-Leffler function and Legendre polynomial is investigated. The determination of orthogonality of some special cases of 3-parameter Mittag-Leffler also is presented by using some properties Legendre polynomial and its generating function.*
***KEYWORDS:*** *Mittag–Leffler function , Legendre Polynomials , Orthogonal Functions.*

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## I. INTRODUCTION

The Mittag–Leffler function was introduced by Gustaf Mittag-Leffler in 1903[1,2]. The fundamental Mittag-Leffler function is a generalization of the expansion of the exponential function [3], it is important function and it has many applications in physics, engineering, biological fields and in another sciences[4-16]. The basic Mittag-Leffler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)} \tag{1}$$

A 2-parameter is generalization of $E_\alpha(x)$ and it is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)} \tag{2}$$

and 3-parameter is generalization of $E_{\alpha,\beta}(x)$ and it is defined by

$$E^{\gamma}{}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\, z^k}{k!\ \Gamma(\gamma)\Gamma(\beta+\alpha k)} \tag{3}$$

All these functions are very important in fractional calculus and mathematical methods, they have relations with some special functions, some of these relations are discussed in this paper.

This paper is structured as follows: Section 2 represents the Legendre polynomials and its basic properties. In section 3 we will show how to represent Legendre function using the 3-parameter Mittag-Leffler function at $\alpha = -2$ , $\beta = 3$ and $\gamma = \frac{-3}{2}$ , we will use this relation to show orthogonality of some special cases 3-parameter Mittag-Leffler function.

## II. LEGENDRE POLYNOMIALS

Legendre polynomial [17,18] is defined by

$$P_n(x) = \sum_{k=0}^{m} \frac{(-1)^k\,(2n-2k)!\ x^{n-2k}}{2^n\ k!\,(n-2k)!\,(n-k)!} \tag{4}$$

where

$$m = \begin{cases} \frac{n}{2} & , \ if\ n \text{ is an even number} \\ \frac{n-1}{2} & , \ if\ n \text{ is an odd number} \end{cases} \tag{5}$$

Legendre polynomial is a solution of the following ordinary differential equation

$$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

$$= \frac{d}{dx}\left[(x^2-1)\frac{dy}{dx}\right] + n(n+1)\,y = 0 \tag{6}$$

Since Legendre polynomial is a solution of (6) , then

$$\frac{d}{dx}\left[(x^2-1)\frac{dP_n(x)}{dx}\right] + n(n+1)\,P_n(x) = 0 \tag{7}$$

$$\frac{d}{dx}\left[(x^2-1)\frac{dP_m(x)}{dx}\right] + m(m+1)\,P_m(x) = 0 \tag{8}$$

We can multiply (7) by $P_m(x)$ and multiply (8) by $P_n(x)$ , we get

$$P_m(x)\frac{d}{dx}\left[(x^2-1)\frac{dP_n(x)}{dx}\right] + n(n+1)\,P_m(x)P_n(x) = 0 \tag{9}$$

$$P_n(x)\frac{d}{dx}\left[(x^2-1)\frac{dP_m(x)}{dx}\right] + m(m+1)\,P_m(x)P_n(x) = 0 \tag{10}$$

by integration with respect to $x$ from $-1$ to $1$ and using integration by parts

$$_{-1}^{\ 1}[\,P_m(x)\,(x^2-1)\frac{dP_n(x)}{dx}\,] - \int_{-1}^{1}(x^2-1)\frac{dP_n(x)}{dx}\frac{dP_m(x)}{dx}\,dx + n(n+1)\int_{-1}^{1}P_n(x)\,P_m(x)\,dx = 0 \tag{11}$$

$$_{-1}^{\ 1}[\,P_n(x)\,(x^2-1)\frac{dP_m(x)}{dx}\,] - \int_{-1}^{1}(x^2-1)\frac{dP_n(x)}{dx}\frac{dP_m(x)}{dx}\,dx + m(m+1)\int_{-1}^{1}P_n(x)\,P_m(x)\,dx = 0 \tag{12}$$

First terms in (11) and (12) are equal to zero, we can subtract$(11)-(12)$, we get

$$[n(n+1) - m(m+1)]\int_{-1}^{1}P_n(x)\,P_m(x)\,dx = 0 \tag{13}$$

If $n \neq m$ then

$$\int_{-1}^{1}P_n(x)\,P_m(x)\,dx = 0 \tag{14}$$

From generating function of Legendre polynomials

$$\frac{1}{\sqrt{1-2zx+z^2}} = \sum_{n=0}^{\infty}P_n(x)\,z^n \tag{15}$$

we can prove that

$$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1} \tag{16}$$

that means the Legendre polynomials are orthogonal in $[-1,1]$

$$\int_{-1}^{1}P_n(x)\,P_m(x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases} \tag{17}$$

## III. THE RELATION BETWEEN 3-PARAMETER MITTAG-LEFFLER FUNCTION AND LEGENDRE POLYNOMIAL

From equation (3) , we can find special case of 3-parameter Mittag-Leffler at $\alpha = -2$ , $\beta = 3$ and $\gamma = \frac{-3}{2}$ let $z = x^2$ we get

$$E^{\frac{-3}{2}}{}_{-2,3}(x^2) = \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{-3}{2}+k\right)\,(x^2)^k}{k!\,\Gamma\left(\frac{-3}{2}\right)\,\Gamma(3-2k)}$$

$$= \frac{\Gamma\left(\frac{-3}{2}+0\right)\,(x^2)^0}{0!\,\Gamma\left(\frac{-3}{2}\right)\,\Gamma(3-2(0))} + \frac{\Gamma\left(\frac{-3}{2}+1\right)\,(x^2)^1}{1!\,\Gamma\left(\frac{-3}{2}\right)\,\Gamma(3-2(1))} + 0 + 0 + \cdots$$

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$$= \frac{1}{\Gamma(3)} + \frac{\frac{-3}{2}\,\Gamma\left(\frac{-3}{2}\right)\ x^2}{\Gamma\left(\frac{-3}{2}\right)\,\Gamma(1)} = \frac{1}{2!} - \frac{3}{2}\,\frac{x^2}{0!}$$

$$= \frac{1}{2} - \frac{3}{2}\,x^2 \qquad (18)$$

And we can find another special case of 3-parameter Mittag-Leffler function at $\alpha = -2$ , $\beta = 1$ and $\gamma = \frac{-1}{2}$

$$E^{\frac{-1}{2}}{}_{-2,1}(x) = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{-1}{2}+k\right)\ (x)^k}{k!\ \Gamma\left(\frac{-1}{2}\right)\ \Gamma(1-2k)}$$

$$= \frac{\Gamma\left(\frac{-1}{2}+0\right)\ (x)^0}{0!\ \Gamma\left(\frac{-1}{2}\right)\ \Gamma(1-2(0))} + 0 + 0 + \cdots$$

$$= \frac{1}{\Gamma(1)} = \frac{1}{0!} = 1 \qquad (19)$$

From $(4)$ we can find $P_2(x)$ by putting $n = 2$ , we get

$$P_2(x) = \sum_{k=0}^{\frac{2}{2}} \frac{(-1)^k\ (\,2(2) - 2k\,)!\ x^{2-2k}}{2^2\ k!\,(2-2k)!\,(2-k)!}$$

$$= \frac{(-1)^0\ (4\ -2(0)\,)!\ x^{2-2(0)}}{4\ \ 0!\,(2-2(0))!\,(2-0)!} + \frac{(-1)^1\ (\,4-2(1)\,)!\ x^{2-2(1)}}{4\ \ 1!\,(2-2(1))!\,(2-1)!}$$

$$= \frac{4!\ x^2}{4\ \ 2!\ \ 2!} - \frac{2!}{4\ \ 0!\ \ 1!} = \frac{4\times3\times2\times1\ \ x^2}{4\ \times\ 2\times1\times\ \ 2\times1} - \frac{2\times1}{4}$$

$$= \frac{3}{2}\,x^2 - \frac{1}{2} \qquad (20)$$

Also we can find $P_1(x)$ from equation $(4)$ by putting, $n = 1$ we get

$$P_1(x) = \sum_{k=0}^{\frac{1-1}{2}} \frac{(-1)^k\ (\,2(1) - 2k\,)!\ x^{1-2k}}{2^1\ k!\,(1-2k)!\,(1-k)!}$$

$$= \frac{(-1)^0\ (\,2-2(0)\,)!\ x^{1-2(0)}}{2\ \ 0!\,(1-2(0))!\,(1-0)!} = \frac{2!}{2}\,x = x \qquad (21)$$

From equations $(18)$ and $(20)$ we can deduce that

$$-E^{\frac{-3}{2}}{}_{-2,3}(x^2)\ =\ P_2(x) \qquad (22)$$

and from equations $(19)$ and $(21)$ we get

$$x\,E^{\frac{-1}{2}}{}_{-2,1}(x)\ =\ P_1(x) \qquad (23)$$

Now we can deduce orthogonality of some special cases of 3-parameter Mittag-Leffler function.
From equation $(14)$ we can substitute $n = 1$ and $m = 2$, we get

$$\int_{-1}^{1} P_1(x)\,P_2(x)\ dx\ = 0 \qquad (24)$$

By using equations $(22)$ and $(23)$ we can rewrite $(24)$ as

$$\int_{-1}^{1} -x\,E^{\frac{-3}{2}}{}_{-2,3}(x^2)\,E^{\frac{-1}{2}}{}_{-2,1}(x)\ \ dx\ = 0 \qquad (25)$$

In equation $(16)$ , if $n = 1$ we get

$$\int_{-1}^{1} [P_1(x)]^2\ dx = \frac{2}{3} \qquad (26)$$

if $n = 2$ we get

$$\int_{-1}^{1} [P_2(x)]^2\ dx = \frac{2}{5} \qquad (27)$$

We can rewrite $(26)$ and $(27)$ as integration of orthogonal special cases of 3-parameter Mittag-Leffler function.
From $(3.5)$ the last equation $(27)$ becomes

$$\int_{-1}^{1} \left[E^{\frac{-3}{2}}{}_{-2,3}(x^2)\right]^2\ dx = \frac{2}{5} \qquad (28)$$

and $(26)$ becomes

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$$\int_{-1}^{1}\left[x^2\, E^{\frac{-1}{2}}{}_{-2,1}(x)\right]^2\ dx = \frac{2}{3} \qquad (29)$$

Then $-E^{\frac{-3}{2}}{}_{-2,3}(x^2)$ and $x\, E^{\frac{-1}{2}}{}_{-2,1}(x)$ are orthogonal polynomials on interval $[-1,1]$ .

## IV. CONCLUSION

The main objective of this paper was to find the relationship between 3-parameter Mittag-Leffler function at $\alpha = -2$, and $\beta = 1\ , 3$ and $\gamma = -\frac{1}{2}\ , -\frac{3}{2}$ and Legendre polynomials $P_1(x), P_2(x)$ respectively, and conclude that the properties of these functions help us to deduce orthogonality of some polynomials related to 3-parameter Mittag-Leffler function.

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