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Research Paper

Second Segment on Homology Computation for A Class of (123) Avoiding Pattern of AUNU Permutation

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Abstract
This paper on second segment on homology computation of the (123) – avoiding pattern of Aunu permutations used the pattern obtained from tree representation of the scheme reported earlier in the works of Ibrahim and Madu (2011).The methodology employed involved a transformation process called mutation on the constructed geometric models on which a homology computation process is applied to determine the cycles and boundries. The results have shown that all circles of the resulting structure are one dimensional indicating that the graphs resulting from this pattern of Aunu scheme can always be drawn on a plane.
Keywords: Aunu pattern,Mutation, Permutation, Triangulation and Tree.

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I. INTRODUCTION

Homology is an important discipline which has found applications in both pure and applied Mathematics areas of study such as in Topology, geometry, group theory and in general algebraic processes. The study of homology represents a functional process of finding interrelationship between different algebraic objects, such as finding relationship between groups and topological structures. This has given rise to concepts such as homology groups which was originally defined in field of algebraic topology.

Mathematics is the heart, search for patterns and for a deep understanding of how and why they occur. It does not matter if the patterns are in naturally occurring phenomena. These patterns may also be found in structures that we created for any number of reasons. Most of us find it more appealing to look for patterns in this kind of image then in the number themselves (Kathleen, 2001). However, the subject of permutation avoidance has been discuss by different writer under a verity of governing conditions and as tools for enumeration of combinational object (Ibrahim and Madu, 2011).

This paper centres on the homology computation of a Tree representation of A_{123} ((123)-avoiding class of Aunu permutation pattern). We will present some geometric interpretation of the figure, betti number for the resulting geometric and determine the simplicial complex of the figure.

II. DEFINATION OF RELATED TERMS

2.1 Permutation

A permutation of a set A is a function $\alpha: A \to A$ that is bijective (i.e both one-to-one and onto).For e.g we can defined a permutation α of the set { 1,2,3} by starting
$\alpha(1) = 2, \alpha(2) = 1, \alpha(3) = 3$.
A slightly more convenient way to represent this permutation is by

$$\alpha \leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

2.2 Permutation pattern

An arrangement of the objects $1, 2, \ldots, n$ is a sequence consisting of these objects arranged in any order. When in addition, a particular order of arrangement is desired, such an arrangement becomes an ordered arrangement governed by a pattern σ and each such permutation $\sigma \in S_n$ naturally results into a certain arrangement of $1, 2, \ldots, n$ given by

$s\ (1)s\ (2)\ldots s\ (n)$

which is called the arrangement associated with a permutation pattern σ of points of a nonempty set

W = $\{1, 2, \ldots, n\}$

2.3 Aunu pattern

The term Aunu pattern refers to a pairing scheme involving pairs of numbers associated by some precedence relation (Ibrahim, 2004a; 2005). The governing conditions for generating these numbers are outline below.

1. The element are paired in order of precedence

$$\lambda_i \lambda_i \ni ipj, ij\epsilon s_n$$

where S_n is a string of length n while $\lambda_i\lambda_i$ are in the i^{th} positions in the permutation pattern generated by the precedence relation p.

2. The precedence parameter (relation ρ) acts on the element to produce pairs that are related in such a way that an element and its first successor, element and its second successor, up to an element and its n^{th}successor. Under the given condition, it is required that the j^{th} partner shifts in position incrementally corresponding to the n^{th} succession so that *j = i+ 1, i+2, …………., i+m .....100 where m≤n*

3. The numeration scheme involves doubt regarding the identity element in the desired pair.

4. Absolute certainly is desired that by the end of the enumeration; the required pair, whichever it is, is achieved. The element displaying such pattern as mentioned, denote such a pattern as Aunu pattern.

2.4. Graph

A pairs G = (V, E) with E ⊆ (V) is called a graph (onV). The elements of V are the vertices of G. and those of the edges of G. The vertex set of a graph G is denoted by V_G and it's edges set by E_GTherefore G = (V_G, E_G). Harju (2011).

2.5. Spanning tree

A spanning tree of a connected graph is a subtree that include all the vertices of that graph. If T is a spanning tree of the graph G. then G – T = $_{det}$T* is co spanning tree.

The edges of a spanning tree are called branches and the edges of the corresponding co spanning tree are called links or chords.

2.6. Homology

Given that an n- dimensional polygon P in a Euclidian space E, P is a simplex if P is then simplest possible polygon of dimension n in E. These simplicies are the point (0 dim), edge (1 dim), triangle (2 dim), tetrahedron (3 dim) and the hypertetrahedron (3+ dim). In any dimension, a simplex K has $\binom{k+1}{i+1}\ i-$faces (Christopher, 2006).

A face of a simplex K is a simplex K such that $\iota \subset K$ and dim ι < dim K . This isdenoted by ι < K if ι has dimension I, then ι is called an ι -face of K (Christopher, 2006).

An n-dimensional simplicial complex K is a collection of simplices of dimension ≤ n such that:

i) if ι is a face of a simplex $k \in$ K then $\mathrm{l} \in K$.

ii) For K, $\mathrm{l} \in K, K \cap \mathrm{l} < K\ and\ K \cap \mathrm{l} < \mathrm{l}$ (Christopher, 2006).

Remark

A simplicial complex can also be referred to as a triangulation of a space (Christopher, 2006).

III. METHODOLOGY

This section provides method of construction of a graph model using tree representing of (123)-avoiding class of Aunu permutation patterns, method of computing it homology and method of finding betti numbers.

3.1 Formulation of Connectivity Relation on Vertices of the Tree Graph

Let m, n, $\epsilon\ \mathbb{Z}^+$and define S = { m $\epsilon\ \mathbb{Z}^+ | m = [1 + (n+1)] mod p\}$ where p is an nth prime ≥5it follow that Sdefine the vertices of the tree representation with a root p + 1, i.e S = V(T) where T stands for the tree representation of (123) avoiding class of Aunu permutation patterns.

We need to conduct a mutation of the graph T in order to obtain a connected graph representation for the computation of boundaries. This can be achieved via the following Algorithm which describes the fundamental procedure in the mutation scheme.

3.2 Algorithm for Mutation of the Second Segment of the Tree Representation of the Class of (123)-Avoiding Aunu Permutation Pattern

Step 1

Obtain a formula for generating individual nodes in the tree

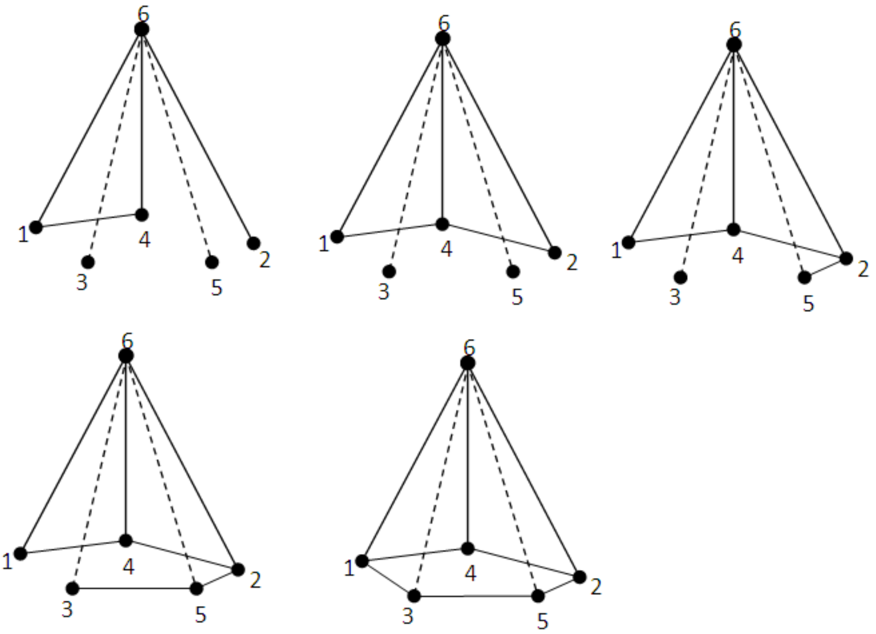
Step 2

Define a relation $\varphi on S$ the vertices set of T such that consecutive pairs of vertices in T are joined whenever $\varphi(a) = [1 + (a + 2)]mod p = b$; then (a,b) defines a path in T for all $a, b \in S$ and for all $p \geq 5$

Step 3

End when a complete cycle is formed i.e when $\varphi(a) = [1 + (a + 2)]mod p = 1$. end if

3.3 Construction of the mutation graph



3.4 Construction of Homology Groups

The construction begin with an object such as a topological space X, on which first defines a chain complex c(x) encoding information about X. A chain complex is a sequence of abelin groups or modules C_o, C_1, C_2, --- connected by homomorphism $\partial_n : C_n \rightarrow C_{n-1}$ which are called boundary operators that is,

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_o \xrightarrow{\partial_o} 0$$

While 0 donates the trivial group and $C_i \equiv 0$ for i < 0, it is also required that the composition of any two consecutive boundary operators be trivial. That is for all n,

$$\partial_n o\, \partial_{n+1} = 0_{n+1,n-1,}$$

i.e the constant map sending every element on C_{n+1} to the group identify in C_{n-1} that the boundary of a boundary is trivial implies $im(\partial_{n+1}) \leq \ker(\partial_n)$, where in (∂_{n+1}) donate the image of the boundary operator and $\ker(\partial_n)$ its kernel. Element of $\beta_n(x) = im(\partial_{n+1})$ are called boundary and elements of cycles. Since each chain group C_n is abelian all its subgroups are normal then because $im(\partial_{n+1})$ and $\ker(\partial_n)$ are both subgroups of C_n, $im(\partial_{n+1})$ is normal subgroup of $\ker(\partial_n)$. Then one can create the quotient group

$$H_n(X) = \frac{\ker(\partial_n)}{im(\partial_{n+1})}$$

$$= \frac{\square_n(X)}{\beta_n(X)}$$

Called the n-th homology group of X

3.5 The Betti Number of A Group Space X

The formula for finding betti numbers is given

as $b_n = \beta_n(X^2) = b_o - b_1 + b_2 - b_3 + b_4 - b_5 - - -$

Where

b_o is the number of connected components
b_1 is the number of one-dimensional or “circular” holes
b_2 is the number of two- dimensional “vilds” or “cavities”

IV. RESULT AND DISCUSSION

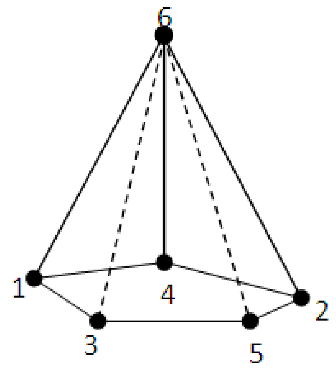


Figure 4.1

$X = \{(614)(462)(256)(635)(613)\}$

These are the maximum simplices we can have
We now write the chain group as

$$\begin{array}{cccc} C_3 \xrightarrow{\partial_3} & C_2 \xrightarrow{\partial_2} & C_1 \xrightarrow{\partial_1} & C_0 \xrightarrow{\partial_0} 0 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & \mathbb{Z}^5 & \mathbb{Z}^{10} & \mathbb{Z}^6 \\ & \cong & \downarrow & \cong \\ & \langle(614)(462)(256)(635)(613)\rangle & \langle(61)(14)(46)(62)(24) & \langle 1,4,2,5,3,6\rangle \\ & & (25)(56)(63)(65)(13)\rangle & \end{array}$$

Now as our tetrahedron is hallow (empty) we cannot have 3-dimension simplices which implies $C_3=0$. We are now set to complete the homology by computing the boundary and cycles at each stage. The general formula is given as:-

$$H_n = \mathbb{Z}_n / \beta_n \quad \text{where,}$$

$$\mathbb{Z}_n = \ker \delta_n = (cycle)$$

$$\beta_n = im\,\delta_{n+1} = (boundary)$$

We now start the computation with first homology which is the zero homology.

$$H_o = {Z_o}/{\beta_o}$$

$$Z_o = \ker \delta_o = C_o = \langle 1, 4, 2, 5, 3, 6 \rangle$$

$$\beta_o = \langle (1-6)(4-1)(6-4)(2-6)(4-2)(5-2)(6-5)(3-6)(5-6)(3-1) \rangle$$

$$Z_o/\beta_o = \frac{\langle 1, 4, 2, 5, 3, 6 \rangle}{\langle (1-6)(4-1)(6-4)(2-6)(4-2)(5-2)(6-5)(3-6)(5-6)(3-1) \rangle}$$

$$= \mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}$$

$$H_1 = {Z_1}/{\beta_1}$$

$$Z_1 = \ker \delta_1 = \alpha(1-6) + \beta(4-1) + \gamma(6-4) + \delta(2-6) + \Sigma(4-2)$$
$$+\eta(5-2) + \rho(6-5) + \varphi(3-6) + \omega(5-6) + \mu(3-1)$$

These are the typical element in C_1. Then, we now seek to know what happen to this expression under the boundary δ .

$$= \alpha(1-6) + \beta(4-1) + \gamma(6-4) + \delta(6-2) + \Sigma(4-2) + \eta(5-2) + \rho(6-5) + \varphi(3-6) + \omega(5-6) + \mu(3-1)$$

(by definition of boundary final-initial).

= 0(by hypothesis)

$$6(-\alpha+\gamma-\delta+\rho-\varphi) + 1(\alpha-\beta-\mu) + 2(\eta-\rho+\omega) + 3(\beta-\gamma+\Sigma) + 4(\varphi-\omega+\mu) + 5\left(\delta - \Sigma - \eta\right) = 0$$

⇒ six equation in ten unknown

$$\begin{array}{cccccccccc} \alpha & \beta & \gamma & \delta & \varepsilon & \eta & \rho & \varphi & \omega & \mu \end{array}$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{cccccccccc} \alpha & \beta & \gamma & \delta & \varepsilon & \eta & \rho & \varphi & \omega & \mu \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} R_1 \to -R_1 \; and \quad R_2 - R_1 \to R_2$$

$$\begin{array}{cccccccccc} \alpha & \beta & \gamma & \delta & \varepsilon & \eta & \rho & \varphi & \omega & \mu \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 & 1 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} R_2 \to -R_2 \; and \quad R_3 - R_2 \to R_3$$

$$\begin{array}{cccccccccc} \alpha & \beta & \gamma & \delta & \varepsilon & \eta & \rho & \varphi & \omega & \mu \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} R_3 \to -R_3 \; and \quad R_4 - R_3 \to R_4$$

$$\begin{array}{cccccccccc} \alpha & \beta & \gamma & \delta & \varepsilon & \eta & \rho & \varphi & \omega & \mu \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} R_4 \to -R_4 \; and \quad R_5 - R_4 \to R_5$$

$$\begin{matrix} \alpha & \beta & \gamma & \delta & \varepsilon & \eta & \rho & \varphi & \omega & \mu \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} R_5 \to -R_5 \; and \quad R_6 - R_5 \to R_6$$

$$\begin{matrix} \alpha & \beta & \gamma & \delta & \varepsilon & \eta & \rho & \varphi & \omega & \mu \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} R_3 \to R_3 - R_2 \,, -R_3 \to R_3 \,, R_5 \to R_5 - R_4 \; and \; R_5 \to -R_5$$

↓ ↓ ↓ ↓ ↓

r s t k l

$\alpha = r + s + t + k$

$\beta = r + s + t + k + l$

$\gamma = r$

$\delta = s + k + l$

$\varepsilon = s$

$\eta = t$

$\rho = t$

$\varphi = k + l$

$\omega = k$

$\mu = l$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \\ \eta \\ \rho \\ \varphi \\ \omega \\ \mu \end{pmatrix} = r \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + l \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus, all solutions are multiple of

$\alpha + \beta + \gamma , \alpha + \beta + \delta + \varepsilon , \alpha + \beta + \eta + \rho , \alpha + \beta + \delta + \varphi + \omega , \beta + \delta + \varphi + \mu$

Therefore, the results have shown that the cycles

$\square_1 = \square_5$

$\beta_1 = im \delta_{1+1}$

$= (\alpha + \beta + \gamma , -\gamma + \delta + \varepsilon , \rho - \varepsilon + \eta , -\rho + \varphi + \omega , -\omega + \mu - \alpha) \neq 0$

Since, the boundary is not equal to cycle $(\square_1 \neq \beta)$ that is, the graph have 1-dimension hole's

We are now set to compute 2-dimensional homology (C_2) which is simply

$H_2 = \square_2 / \beta_2$

$= \square_2 = im \delta_3$

To do this we ask geometrical questions that what combination of geometric cycles (faces) give us zero (see graph). There are five faces in the graph.

Observe that some of faces out of (614),(462),(256),(635),(613), have no opposite orientation but all the edges divides them therefore not all the coefficients of the faces are equal so few of them cancel out this procedure also applies to every edge. So, the combination that give us required boundary.

$\delta \left((614) + (462) + (256) + (635) + (613) \right) = 0$

A B C D E

As can also be checked algebraically.

This therefore, is a generators of the cycles i.e

$\square_2 = \langle A + B + C + D + E \rangle \square \square$

Now, it also follows that $\beta_2 = 0$ since we have no 3-dimensional cells thus all these things coming from C_3 are 0

$\Rightarrow \beta_2 = 0$

$\therefore H_2 = \square_2 / \beta_2 = Z$

In general we've

$H_o(x^2) = Z$

$H_1\left(x^2\right) = Z$

$H_2\left(x^2\right) = Z$

All higher homology are zero i.e

$H_n\left(\chi^2\right) = 0 , n \geq 3$

Betti numbers of the space X

The betti numbers are the rank which is b_n = rank H_n of the space

The formula for finding betti number given as $b_n = B(x^2) = b_0 - b_1 + b_2 - b_3 + b_4 - b_5$ -- where

The betti number of the first graph, haven computed its Homology as $H_o(x^2) = \square$, $H_1(x^2) = \square$, and $H_2(x^2) = \square$ is given as

$B(X^2) = b_0 - b_1 + b_2 - b_3$ ----

= 1 -1 +1 ---

=1

V. CONCLUSION

This research paper provided some computational framework for study and analysis of permutation pattern and pattern avoidance. In particular we have established a geometric approach to study of pattern trees emanating from a class of (123)-avoiding permutation pattern of Aunu numbers via a process of graph mutation subject to some precedence condition.

Our results have provided a platform for further theoretic formulation in both algebraic geometry and in linear algebra.

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