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Research Paper

Positive Solution for a Singular Fourth-Order Boundary Value Problem

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ABSTRACT
This paper investigates the existence of positive solution for a nonlinear fourth- order boundary value problem using a xed point theorem of cones. The nonlinear term may be singular with respect to both the time and space variables. The problem comes from the deformation analysis of an elastic beam in the equilibrium state, whose two ends are simly supported. The results obtained herein generalize and improve some known results including singular and non-singular cases.
KEYWORDS: *positive solutions, fixed point theorem, singular solutions, bending of an elastic beam, cone, boundary value problem, existence.*

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I. INTRODUCTION

It is well known that the bending of an elastic beam can be described with fourth-order boundary value problems. An elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \tag{1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{2}$$

Existence of solutions for problem (1) was established for example by Gupta [1,2], Liu [3], Ma [4], Ma et. al. [5], Ma and Wang [6], Aftabizadeh [7], Yang [8], Del Pino and Manasevich [9], RP Agarwal et.al. [10,11,12] (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions.

In this paper we shall discuss the existence of positive solutions for the fourth-order boundary value problem

$$u^{(4)} + \alpha u'' = f(t, u, u''), \quad 0 < t < 1$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{3}$$

where $\alpha < \pi^2$ is a positive parameter and $f(t, u, v) : (0, 1) \times (0, \infty) \times (-\infty, 0) \longrightarrow (0, \infty)$ is continuous. In fact as we will see below one could consider in Section 2 and 3 $f(t, u, v) \leq f_1(t)\, f_2(t, u, v)$ with $f_2(t, u, v) : (0, 1) \times [0, \infty) \times (-\infty, 0) \longrightarrow (0, \infty)$ continuous and $f_1 \in C\left[(0, 1), R^+\right]$ provided

$$\int_0^1 \int_0^1 K_1(\tau, \tau)\, K_2(\tau, s)\, f_1(s) ds\, d\tau < +\infty;$$

here K_i, $(i = 1, 2)$ is as defined in Section 2.

II. PRELIMINARIES

Let $Y = C[0,1]$ and

$$Y_+ = \{u \in Y : u(t) \geq 0,\ t \in [0,1]\}.$$

It is well known that Y is a Banach space equipped with the norm $\|u\|_0 = \sup_{t\in[0,1]} |u(t)|$. We denote the norm $\|u\|_2$ by

$$\|u\|_2 = \max\left\{\|u\|_0, \|u''\|_0\right\}.$$

It is easy to show that $C^2[0,1]$ is complete with the norm $\|u\|_2$ and $\|u\|_2 \leq \|u\|_0 + \|u''\|_0 \leq 2\|u\|_2$.

Let λ_1 be the first eigenvalue of the problem $u^{(4)} + \alpha u'' = \lambda u,\ u(0) = u(1) = u''(0) = u''(1) = 0$. We known that

$$\frac{\lambda_1}{\pi^4} + \frac{\alpha}{\pi^2} = 1$$

and $\phi_1(t) = \sin \pi t$ is the first eigenfunction.

Suppose that $K_i(t,s),\ \ (i = 1,2)$ is the Green function associated with

$$-u'' + \alpha u = 0, \quad u(0) = u(1) = 0, \tag{4}$$

which is explicitly expressed by

which is explicitly expressed by

$$K_1(t,s) = \begin{cases} t(1-s) & \text{if } 0 \leq t \leq s \leq 1, \\ s(1-t) & \text{if } 0 \leq s \leq t \leq 1, \end{cases} \quad \text{if } \alpha = 0, \tag{5}$$

$$K_2(t,s) = \begin{cases} \frac{sinh\omega t sinh\omega(1-s)}{\omega sinh\omega} & \text{if } 0 \leq t \leq s \leq 1, \\ \frac{sinh\omega s sinh\omega(1-t)}{\omega sinh\omega} & \text{if } 0 \leq s \leq t \leq 1, \end{cases} \quad \text{if } \alpha < 0, \tag{6}$$

$$K_2(t,s) = \begin{cases} \frac{sin\omega t sin\omega(1-s)}{\omega sin\omega} & \text{if } 0 \leq t \leq s \leq 1, \\ \frac{sin\omega s sin\omega(1-t)}{\omega sin\omega} & \text{if } 0 \leq s \leq t \leq 1, \end{cases} \quad \text{if } 0 < \alpha < \pi^2, \tag{7}$$

where $\omega = \sqrt{|\alpha|}$.

We need the following lemmas.

Lemma 1. $K_i(t,s)$ has the following properties:
(i) $K_i(t,s) > 0, \forall t,s \in (0,1)$;
(ii) $K_i(t,s) \leq C_i K_i(s,s), \forall t,s \in [0,1]$;
(iii) $K_i(t,s) \geq \gamma_i K_i(t,t) K_i(s,s), \forall t,s \in [0,1]$;
(iv) $|K_1(t_1,s) - K_1(t_2,s)| \leq 2|t_1 - t_2|$, for all $t_1, t_2, s \in [0,1]$
where $C_2 = 1,\ \ \gamma_2 = \frac{\omega}{\sinh \omega},\ \ \text{if } \alpha < 0;\ \ C_1 = 1,\ \ \gamma_1 = 1,\ \ \text{if } \alpha = 0;\ \ C_2 = \frac{1}{\sin \omega},\ \ \gamma_2 = \omega \sin \omega,\ \ \text{if } 0 < \alpha < \pi^2$.

Lemma 2 ([14]). Let E be a real Banach space and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subset of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $Q : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to P$ be a completely continuous operator and assume that the conditions

(i) $\|Qu\| > \|u\|$, for $u \in P \cap \partial\Omega_1$

and

(ii) $u \neq \mu Q(u)$, for $\mu \in [0,1)$ and $u \in P \cap \partial\Omega_2$

hold.

Then Q has a fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

Now we consider the existence of a positive solution of (3). The function $u \in C^4(0,1) \cap C^2[0,1]$ is a positive solution of (3), if $u \geq 0$, $t \in [0,1]$, and $u \neq 0$.

Then the solution of (3) can be expressed as

$$u(t) = \int_0^1 \int_0^1 K_1(t,\tau) K_2(\tau,s) f(s,u(s),u''(s))\, ds\, d\tau$$

and the second-order derivative u'' can be expressed by

$$u''(t) = -\int_0^1 K_2(t,s) f(s,u(s),u''(s))\, ds. \tag{8}$$

Set

$$P = \{u \in C^2[0,1] : u(0) = u(1) = 0,\ u(t) \geq K_1(t,t) \|u\|_0 ,$$

$$-u''(t) \geq \frac{\gamma_2}{C_2} K_2(t,t) \left\|u''\right\|_0, t \in [0,1]\}.$$

Note P is a cone in $C^2[0,1]$. For $R > 0$, write $B_R = \{u \in C^2[0,1] : \|u\|_2 < R\}$.
We now define a mapping $T : P \to C^2[0,1]$ by

$$Tu(t) = \int_0^1 \int_0^1 K_1(t,\tau) K_2(\tau,s) f(s,u(s),u''(s))\, ds\, d\tau. \tag{9}$$

Lemma 3. Let $w \in P$. Then the following relations hold:
(a) $(Tw)(t) \geq K_1(t,t) \|Tw\|_0$ for $t \in [0,1]$, and
(b) $-(Tw)''(t) \geq \frac{\gamma_2}{C_2} K_2(t,t) \|(Tw)''\|_0$ for $t \in [0,1]$.

Proof. For simplicity we denote

$$I = \int_0^1 \int_0^1 K_1(\tau,\tau) K_2(\tau,s) h(s) ds d\tau,$$

$$J = \int_0^1 K_2(s,s) h(s) ds,$$

where

$$h(s) = f(s, w(s), w''(s)).$$

From Lemma 1 it is easy to see that

$$K_1(t,t)I \le Tw(t) \le I, \quad t \in [0,1]\,, \tag{10}$$

and

$$\gamma_2 K_2(t,t)J \le -(Tw)''(t) \le C_2 J, \qquad t \in [0,1]\,. \tag{11}$$

Using (10-11), we have

$$\|Tw\|_0 \le I, \quad \left\|-(Tw)''\right\|_0 \le C_2 J,$$

hence

$$(Tw)(t) \ge K_1(t,t)\,\|Tw\|_0\,, \text{ for } t \in [0,1],$$

and

$$-(Tw)''(t) \ge \frac{\gamma_2}{C_2} K_2(t,t)\left\|Tw''\right\|_0\,, \text{ for } t \in [0,1].$$

Throughout this paper, we assume additionaly that there exists an $a > 0$ and an continuous function $f_1(t,w) : (0,1) \times (0,a] \to [0,+\infty)$ such that the function $f(t,u,v)$ satisfies

$$(H1) \;\; f(t,u,v) \le f_1(t, u+|v|), \;\; \text{for all} \;\; u+|v| \le a, \qquad t \in (0,1),$$

$$u \in (0,a], \;\; \text{and } v \in [-a, 0).$$

(H2) The function $f(t,u,v)$ is nonincreasing in $u \le a$ and nondecreasing in $v \ge -a$ for each fixed $t \in [0,1]$ i.e. if $-a \le v_2 \le v_1 < 0$ and $0 < u_1 \le u_2 \le a$ then $f(t,u_1,v_1) \ge f(t,u_2,v_2)$.

(H3) The function $f_1(t,w)$ is nonincreasing in $w \le a$ for each fixed $t \in [0,1]$, i.e. if $0 < w_1 \le w_2$ then $f_1(t,w_1) \ge f_1(t,w_2)$ and each fixed $0 < r \le a$

$$0 < \int_0^1 f_1(t, r(K_1(t,t) + \frac{\gamma}{C}K_2(t,t)))ds < \infty.$$

(H4) There exists a nonnegative measurable function p defined $(0,1)$ and a nonnegative continuous function q defined on $[a,+\infty)$ such that $f(t,u,v) \le p(t)q(u+|v|)$ if $u+|v| > a$, where p and q satisfy $0 < \int_0^1 K_2(s,s)p(s)ds < +\infty$ and $\lim_{w\to+\infty} \frac{q(w)}{w} < \lambda_1$.

Let us introduce the following notations

$$D_1 = \int_0^1 K_1(s,s)\,ds,$$

$$D_2 = \int_0^1 K_2(s,s)p(s)\,ds,$$

$$D_r = \int_0^1 K_2(s,s)f_1(s, r\frac{\gamma_2}{C_2}K_2(s,s))\,ds.$$

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Lemma 4. Let $(H1)$, $(H2)$, and $(H3)$ hold. Then for all $u \in P \cap \overline{B}_R/B_r$ where $r < a < R$ the following hold

$$(Tu)(t) \le D_1(D_r + \sup_{w\in[a,2R]} q(w)D_2),$$

and

$$-(Tu)''(t) \le D_r + \sup_{w\in[a,2R]} q(w)D_2.$$

Proof. Let $u \in P \cap \overline{B}_R/B_r$, then by Lemma 6, $\|u\|_0 \le \|u''\|_0$ and by Corollary 7, $\|u\|_2 = \|u''\|_0$. Thus $r \le \|u''\|_0 \le R$. Also, since $u \in P$ we have $-u''(t) \ge \frac{\gamma_2}{C_2}K_2(t,t)\|u''\|_0$, $u(t) \ge K_1(t,t)\|u\|_0$, $t \in [0,1]$. So, we have $u(t) + |u''(t)| \ge K_1(t,t)\|u\|_0 + \frac{\gamma_2}{C_2}K_2(t,t)\|u''\|_0 \ge \frac{\gamma_2}{C_2}K_2(t,t)r$, $t \in [0,1]$.

By Lemma 1. and $(H1)-(H3)$ we have

$$Tu(t) = \int_0^1 \int\limits_{u(s)+|u''(s)|\le a} K_1(t,\tau)K_2(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau +$$

$$+\int_0^1 \int\limits_{u(s)+|u''(s)|\ge a} K_1(t,\tau)K_2(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau$$

$$\le \int_0^1 \int\limits_{u(s)+|u''(s)|\le a} K_1(t,\tau)K_2(\tau,s)\, f_1(s,u(s)+|u''(s)|)\, ds\, d\tau +$$

$$+\int_0^1 \int\limits_{u(s)+|u''(s)|\ge a} K_1(t,\tau)K_2(\tau,s)p(s)\, q(u(s)+|u''(s)|)\, ds\, d\tau$$

$$\le \int_0^1 \int\limits_{u(s)+|u''(s)|\le a} K_1(t,\tau)K_2(\tau,s)\, f_1(s,\frac{\gamma_2}{C_2}K_2(s,s)r)\, ds\, d\tau +$$

$$+\sup_{w\in[a,2R]} q(w)\int_0^1 \int\limits_{u(s)+|u''(s)|\ge a} K_1(t,\tau)K_2(\tau,s)p(s)\, ds\, d\tau$$

$$\le \int_0^1 K_1(\tau,\tau)[\int_0^1 K_2(s,s)\, f_1(s,\frac{\gamma_2}{C_2}K_2(s,s)r)\, ds]\, d\tau +$$

$$+\sup_{w\in[a,2R]} q(w)\int_0^1\int_0^1 K_1(\tau,\tau)K_2(s,s)\, p(s)\, ds\, d\tau$$

$$\le D_1(D_r + \sup_{w\in[a,2R]} q(w)D_2).$$

and similarly, we also have

$$-(Tu)''(t) \le D_r + \sup_{w\in[a,2R]} q(w)D_2.$$

Lemma 5. $T(P) \subset P$ and $T : P \cap (\overline{B}_R/B_r) \to P$ is completely continuous.

Proof. Let $u \in P$, then we define mapping $T : P \cap (\overline{B}_R/B_r) \to C^2[0,1]$ by (9). Then for any $u \in P$, it is clear that

$$(Tu)''(t) = -\int_0^1 K_2(t,s) f(s,u(s),u''(s)) ds \leq 0. \tag{12}$$

By Lemma 3,

$$Tu(t) \geq K_1(t,t) \|Tu\|_0, \quad t \in [0,1],$$

and

$$-(Tu)''(t) \geq \frac{\gamma_2}{C_2} K_2(t,t) \left\|(Tu)''\right\|_0 \quad t \in [0,1].$$

Hence $T(P) \subset P$.

Let $V \subset P \cap (\overline{B}_R/B_r)$ be a bounded set. Then there exists a $d > 0$, such that $\sup\{\|u\|_2 : u \in V\} = d$.

First we prove $T(V)$ is bounded. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$, we have $u(t) + |u''(t)| \leq \|u\|_0 + \|u''\|_0 \leq 2d$, for all $t \in [0,1]$. Let $M_d = \sup\{q(w) : w \in [a, 2d]\}$. Now, from Lemma 4 we have for any $u \in V$ and $t \in [0,1]$ that

$$|Tu(t)| = \left| \int_0^1 \int_0^1 K_1(t,\tau) K_2(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau \right|$$

$$\leq D_1 \left(\int_0^1 K_2(s,s)\, f_1(s, r\frac{\gamma_2}{C_2} K_2(s,s))\, ds + \sup_{w \in [a,2d]} q(w) \int_0^1 K_2(s,s)\, p(s)\, ds \right).$$

$$\leq D_1(D_r + M_d D_2).$$

We have a similar type inequality for $|(Tu)''(t)|$.

Therefore $T(V)$ is bounded.

Next we prove that $T(V)$ is equicontinuous. Now from Lemma 4 we have for any $u \in V$ and any $t_1, t_2 \in [0,1]$ that

$$|(Tu)(t_1) - (Tu)(t_2)| \leq$$

$$\leq \int_0^1 \int_0^1 |K_1(t_1,\tau) - K_1(t_2,\tau)|\, K_2(\tau,s) f(s,u(s),u''(s))\, ds\, d\tau$$

$$\leq 2|t_1 - t_2|\, D_1 \left(\int_0^1 K_2(s,s)\, f_1(s, r\frac{\gamma_2}{C_2} K_2(s,s))\, ds + M_d \int_0^1 K_2(s,s)\, p(s)\, ds \right).$$

We have a similar type inequality for $|(Tu)''(t_1) - (Tu)''(t_2)|$.

Therefore $T(V)$ is equicontinuous.

Next we prove that T is continuous. Suppose $u_n, u \in P \cap (\overline{B}_R/B_r)$ and $\|u_n - u\|_2 \to 0$ which implies that $u_n(t) \to u(t), u_n''(t) \to u''(t)$ uniformly on $[0,1]$. Similarly for $f(t,u,v) \leq f_1(t, u+|v|)$, $f_1(t, u_n(t)+|u_n''(t)|) \to f_1(t, u(t)+|u''(t)|)$ and $q(u_n(t)+|u_n''(t)|) \to q(u(t)+|u''(t)|)$ uniformly on $[0,1]$. The assertion follows from the estimate

$$|Tu_n(t) - Tu(t)| \leq$$

$$\leq \int_0^1 \int_0^1 K_1(t,\tau)\, K_2(\tau,s)\ \mid f(s,u_n(s)|u_n''(s)|) - f(s,u(s),|u''(s)|) \mid ds\, d\tau,$$

and the similar estimate for $|(Tu_n)''(t) - (Tu)''(t)|$.
by an application of the standard theorem on the convergence of integrals. The Ascoli-Arzela theorem guarantees that $T: P \to P$ is completely continuous.

Lemma 6. If $u(0) = u(1) = 0$ and $u \in C^2[0,1]$, then $\|u\|_0 \leq \|u''\|_0$, and so, $\|u\|_2 = \|u''\|_0$.
Proof. Since $u(0) = u(1)$, there is a $\alpha \in (0,1)$ such that $u'(\alpha) = 0$, and so $u'(t) = \int_\alpha^t u''(s)ds,\ t \in [0,1]$. Hence $|u'(t)| \leq \int_\alpha^t |u''(s)|\, ds \leq \int_0^1 |u''(s)|\, ds \leq \|u''\|_0,\ t \in [0,1]$. Thus $\|u'\|_0 \leq \|u''\|_0$. Since $u(0) = 0$, we have $u(t) = \int_0^t u'(s)ds,\ t \in [0,1]$, and so $|u(t)| \leq \int_0^1 |u'(s)|\, ds \leq \|u'\|_0$. Thus $\|u\|_0 \leq \|u'\|_0 \leq \|u''\|_0$. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$ and $\|u\|_0 \leq \|u''\|_0$, we obtain that $\|u\|_2 = \|u''\|_0$.

Corollary 7. Let $r > 0$ and let $u \in \partial B_r \cap P$. Then $\|u\|_2 = \|u''\|_0 = r$.

3. Main results

Theorem 1. Let $(H1)$,$(H2)$, $(H3)$ and $(H4)$ hold. Then problem (3) has at least one positive solution.
Proof.
We recall that $f(t,u,v): (0,1) \times (0,\infty) \times (-\infty, 0) \longrightarrow (0,\infty)$ is continuous and positive, so using $(H1 - H3)$ and Lemma 4, we have

$$0 < \int_0^1 \int_0^1 K_1(t,\tau) K_2(\tau,s)\, f(s,u(s),u''(s))\, ds\, d\tau$$

$$\leq D_1\Big(\int_0^1 K_2(s,s)\, f_1(s, \frac{\gamma_2}{C_2} K_2(s,s) r)\, ds + \sup_{w \in [a,2R]} q(w) \int_0^1 K_2(s,s)\, p(s)\, ds\Big) < +\infty.$$

By assumptions $(H1 - H3)$, for each $\eta \in (0, \frac{1}{2})$ sufficiently small, since

$$a\eta^2 K_1(t,t) \leq aK_1(t,t), \text{ and } a\eta^2 \frac{\gamma_2}{C_2} K_2(t,t) \leq a\frac{\gamma_2}{C_2} K_2(t,t), \text{ for all } t \in [\eta, 1-\eta], \tag{13}$$

we have

$$0 < \int_\eta^{1-\eta} \int_\eta^{1-\eta} K_1(\tau,\tau) K_2(\tau,s) f(s, a\eta^2 K_1(s,s), -a\eta^2 \frac{\gamma_2}{C_2} K_2(s,s)) ds d\tau$$

$$\leq \int_\eta^{1-\eta} \int_\eta^{1-\eta} K_1(\tau,\tau) K_2(\tau,s) f(s, aK_1(s,s), -a\frac{\gamma_2}{C_2} K_2(s,s)) ds d\tau.$$

Fix $\eta \in (0, \frac{1}{2})$ as above. Let $r > 0$ be such that

$$r < \min\{a, \int_\eta^{1-\eta} \int_\eta^{1-\eta} K_1(\frac{1}{2},\tau) K_2(\tau,s) f(s, a\eta^2 K_1(s,s), -a\eta^2 \frac{\gamma_2}{C_2} K_2(s,s)) ds d\tau,$$

$$\int_0^1 K_2(\frac{1}{2}, s) f(s, a\eta^2 K_1(s,s), -a\eta^2 \frac{\gamma_2}{C_2} K_2(s,s)) ds, a\eta^2 K_1(\eta,\eta), a\eta^2 \frac{\gamma_2}{C_2} K_2(\eta,\eta)\}.$$

It is easy to see that

$$r < a\eta^2 K_1(\eta,\eta) \leq a\eta^2 K_1(t,t), \text{ and } r < a\eta^2 \frac{\gamma_2}{C_2} K_2(\eta,\eta) \leq a\eta^2 \frac{\gamma_2}{C_2} K_2(t,t), \forall t \in (\eta, 1-\eta). \tag{14}$$

We now show that

$$\|Tu\|_2 > \|u\|_2, \quad \forall u \in P \cap \partial B_r.$$

To see this, let $u \in P \cap \partial B_r$, then by Corollary 7, $\|u\|_2 = \|u''\|_0 = r$ and $u(0) = u(1) = 0$. Also since $\|u\|_0 \leq \|u''\|_0$ we have $u(t) \leq \|u\|_0 \leq r < a$, $|u''(t)| \leq \|u''\|_0 = r < a$, $\forall t \in [0,1]$. Thus $0 \leq u(t) + |u''(t)| \leq 2\,r$, $\forall t \in [0,1]$.

Thus, by Lemma 4, and $(H1-H2)$ we have

$$(Tu)(\frac{1}{2}) \geq \int_\eta^{1-\eta}\int_\eta^{1-\eta} K_1(\frac{1}{2},\tau)K_2(\tau,s)f(s,u(s),u''(s))dsd\tau$$

$$\geq \int_\eta^{1-\eta}\int_\eta^{1-\eta} K_1(\frac{1}{2},\tau)K_2(\tau,s)f(s,r,-r)dsd\tau$$

$$\geq \int_\eta^{1-\eta}\int_\eta^{1-\eta} K_1(\tau,\tau)K_2(\tau,s)f(s,a\eta^2K_1(s,s),-a\eta^2\frac{\gamma_2}{C_2}K_2(s,s))dsd\tau > r = \left\|u''\right\|_0.$$

Consequently,

$$\|Tu\|_0 > \|u\|_2\,, \qquad \forall u \in \partial B_r \cap P. \tag{15}$$

Similarly we also have

$$(Tu)''(t) = -\int_0^1 K_2(t,s)f(s,u(s),u''(s))ds.$$

Hence

$$\left|(Tu)''(\frac{1}{2})\right| \geq \int_\eta^{1-\eta} K_2(\frac{1}{2},s)f(t,u(s),u''(s))ds$$

$$\geq \int_\eta^{1-\eta} K_2(\frac{1}{2},s)f(s,r,-r)dsd\tau$$

$$\geq \int_\eta^{1-\eta} K_2(\frac{1}{2},s)f(s,a\eta^2K_1(s,s),-a\eta^2\frac{\gamma_2}{C_2}K_2(s,s)))dsd\tau > r = \left\|u''\right\|_0,$$

$\forall u \in \partial B_r \cap P,\quad t \in [0,1]$.

Consequently,

$$\left\|(Tu)''\right\|_0 > \|u\|_2\,, \qquad \forall u \in \partial B_r \cap P. \tag{16}$$

Using (15) and (16) we have

$$\|Tu\|_2 > \|u\|_2\,, \qquad \forall u \in \partial B_r \cap P. \tag{17}$$

On the other hand, let $R > a$ be chosen large enough later.

Note that using by $(H4)$ $\limsup_{w\to+\infty}\frac{q(w)}{w} < \lambda_1$, we have $\limsup_{v\to-\infty}\frac{f(t,u,v)}{|v|} < \lambda_1$ uniformly on $[0,1]$ and $u \in (0,+\infty)$,

Since $\limsup_{v\to-\infty}\frac{f(t,u,v)}{|v|} < \lambda_1$ uniformly on $[0,1]$ and $u \in (0,+\infty)$, there exist $0 < \delta < 1$ and $H > a$ such that

$f(t,u,v) \le (\lambda_1 - \delta)|v|, \quad \forall t \in [0,1], \quad u \in (0,+\infty), \quad |v| \ge H.$

If $u(t)+|u''(t)| \le a$ then $f(t,u(t),u''(t)) \le f_1(t, r(K_1(t,t)+\frac{\gamma}{C}K_2(t,t)))$. Moreover, if $u(t)+|u''(t)| \ge a$ then $f(t,u(t),u''(t)) \le \max_{t\in[0,1]} \sup_{w\in[a,2R]} p(t)q(w)$, where $w = u(t)+|u''(t)|$. Let $C = \max_{t\in[0,1]} p(t) \sup_{w\in[a,2R]} q(w)$.

It is easy to see that

$$f(t,u(t),u''(t)) \le f_1(t, r(K_1(t,t)+\frac{\gamma}{C}K_2(t,t))) + C + (\lambda_1-\delta)|u''(t)|,$$

$\forall t \in [0,1], \quad u \in P, \quad \|u''\|_0 \ge H.$

Next we show that if R is large enought, then $\mu Tu \ne u$ for any $u \in P \cap \partial B_R$ and $0 \le \mu < 1$. If this is not true, then there exists $u_0 \in P \cap \partial B_R$ and $0 \le \mu_0 < 1$ such that $\mu_0 T u_0 = u_0$. Thus $\|u_0''\|_0 = R > a$ and $-u_0''(t) \ge \frac{\gamma_2}{C_2}K_2(t,t)R$. Note that $u_0(t)$ satisfies

$$u_0^{(4)} + \alpha u_0'' = f(t,u_0,u_0''), \quad 0<t<1 \tag{18}$$

and the boundary condition

$$u_0(0) = u_0(1) = u_0''(0) = u_0''(1) = 0. \tag{19}$$

Multiply Eq. (18) by $\phi_1(t) = \sin(\pi t)$ and integrate from 0 to 1, using integration by parts in the left side, to obtain

$$(\pi^4 - \alpha\pi^2)\int_0^1 u_0(t)\phi_1(t)dt = \mu_0 \int_0^1 u_0(t)\phi_1(t) f(t,u_0(t),u_0''(t))dt,$$

i.e.

$$\lambda_1 \int_0^1 u_0(t)\phi_1(t)dt = \mu_0 \int_0^1 u_0(t)\phi_1(t) f(t,u_0(t),u_0''(t))dt$$

$$\le \frac{\lambda_1-\delta}{\pi^2}\int_0^1 (-u_0'')(t)\phi_1(t)dt + C\int_0^1 \phi_1(t)dt + \int_0^1 f_1(t, r(K_1(t,t)+\frac{\gamma}{C}K_2(t,t)))\phi_1(t)dt.$$

Consequently, using integration by parts in the left side, we obtain that

$$\frac{1}{\pi^2}\int_0^1 (u_0'')(t)\phi_1(t)dt \le \frac{1}{\delta}[C\int_0^1 \phi_1(t)dt + \int_0^1 f_1(t, r(K_1(t,t)+\frac{\gamma}{C}K_2(t,t)))\phi_1(t)dt]. \tag{20}$$

We also have

$$\frac{1}{\pi^2}\int_0^1 (-u_0'')(t)\phi_1(t)dt \ge \frac{1}{\pi^2}\|u_0''\|_0 \int_0^1 \frac{\gamma_2}{C_2}K_2(t,t)\phi_1(t)dt$$

and this together with (21) yields

$$\|u_0''\|_0 \le \frac{C\int_0^1 \phi_1(t)dt + \int_0^1 f_1(t, r(K_1(t,t)+\frac{\gamma}{C}K_2(t,t)))\phi_1(t)dt}{\delta \int_0^1 \frac{\gamma_2}{C_2}K_2(t,t)\phi_1(t)dt} =: \overline{R}. \tag{21}$$

Let $R > \max\{\overline{R}, H\}$. Then for any $u \in P \cap \partial B_R$ and $0 \le \mu < 1$, we have $\mu Tu \ne u$. Hence all the hypotheses of Lemma 3.1 hold. Then T has a fixed point in $P \cap (\overline{B}_R \backslash B_r)$.

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