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Research Paper

Formulas Involving Some Fractional Trigonometric Functions Based on Local Fractional Calculus

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ABSTRACT: *This paper mainly studies the local fractional calculus of fractional trigonometric functions, which is a natural generalization of the classical calculus of trigonometric functions. The concept of fractional analytic function plays an important role in this article. Using a new multiplication of fractional analytic functions and some properties, such as product rule, quotient rule and chain rule, we obtain some formulas involving fractional trigonometric functions. On the other hand, we present several examples to illustrate these formulas.*

KEYWORDS: *Local Fractional Calculus, Fractional Trigonometric Functions, Fractional Analytic Function, New Multiplication*

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I. INTRODUCTION

Fractal was found to be useful in science and engineering [2-4] after Mandelbrot [1] wrote his pioneering work. Recently, new definitions of the local fractional derivative and local fractional integral were applied to describe the non-differentiable problems that occur in fractal engineering [5-8]. Furthermore, many applications of fractal in growth phenomena, turbulence, chaotic systems and other fields have been found [9-12]. Based on Jumarie's fractional calculus [13-14], the local fractional calculus was modified [15]. The fundamental properties of local fractional calculus of one-variable were discussed [16-17].

In this paper, the local fractional calculus of fractional trigonometric functions is studied. It is a generalization of traditional calculus of trigonometric functions. The concept of fractional analytic function plays an important role in this article. We use a new multiplication of fractional analytic functions and some properties such as product rule, quotient rule, chain rule to get several formulas involving fractional trigonometric functions. In Section 2, we introduce the definitions of local fractional derivative and local fractional integral. Also, the definition of fractional analytic function is provided. In Section 3, several formulas of fractional trigonometric functions are proposed. In Section 4, we give some examples to illustrate these formulas.

II. PRELIMINARIES

In the following, some fractional functions and properties are introduced.

Definition 2.1 ([18])**:** Suppose that $0 < \alpha \leq 1$, and x is a real variable. The α-fractional exponential function, cosine function and sine function are defined as follows:

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{1}$$

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{2}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{3}$$

Definition 2.2 ([19])**:** Let x, x_0 and a_n be real numbers, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as a α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}$ on some open interval $(x_0 - r, x_0 + r)$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0, where r is the radius of convergence about x_0. If $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and is α-fractional analytic at every point in open interval (a,b), then we say that f_α is an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.3 ([18])**:** Let $0 < \alpha \leq 1$, $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}. \tag{5}$$

Then we define

$$\begin{aligned} &f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \otimes \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) (x - x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Definition 2.4: Let $\left(f_\alpha(x^\alpha)\right)^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is the n times product of the α-fractional analytic function $f_\alpha(x^\alpha)$. If $g_\alpha(x^\alpha)$ is also an α-fractional analytic function such that $f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $[f_\alpha(x^\alpha)]^{\otimes -1}$.

Definition 2.5 : If $0 < \alpha \leq 1$, then the α-fractional tangent, cotangent, secant, and cosecant functions are defined as:

$$tan_\alpha(x^\alpha) = sin_\alpha(x^\alpha) \otimes [cos_\alpha(x^\alpha)]^{\otimes -1}, \tag{7}$$

$$cot_\alpha(x^\alpha) = cos_\alpha(x^\alpha) \otimes [sin_\alpha(x^\alpha)]^{\otimes -1}, \tag{8}$$

$$sec_\alpha(x^\alpha) = [cos_\alpha(x^\alpha)]^{\otimes -1}, \tag{9}$$

$$csc_\alpha(x^\alpha) = [sin_\alpha(x^\alpha)]^{\otimes -1}. \tag{10}$$

Proposition 2.6 (fractional Euler's formula): *Let* $0 < \alpha \leq 1$, *then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + i sin_\alpha(x^\alpha). \tag{11}$$

Proposition 2.7 (fractional DeMoivre's formula): *If* $0 < \alpha \leq 1$, *and* n *is a positive integer, then*

$$[cos_\alpha(x^\alpha) + i sin_\alpha(x^\alpha)]^{\otimes n} = cos_\alpha(nx^\alpha) + i sin_\alpha(nx^\alpha). \tag{12}$$

Theorem 2.8:

$$[sin_\alpha(x^\alpha)]^{\otimes 2} + [cos_\alpha(x^\alpha)]^{\otimes 2} = 1, \tag{13}$$

$$1 + [tan_\alpha(x^\alpha)]^{\otimes 2} = [sec_\alpha(x^\alpha)]^{\otimes 2}, \tag{14}$$

$$1 + [cot_\alpha(x^\alpha)]^{\otimes 2} = [csc_\alpha(x^\alpha)]^{\otimes 2}. \tag{15}$$

Proof By fractional Euler's formula, we have $E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + i sin_\alpha(x^\alpha)$ and $E_\alpha(-ix^\alpha) = cos_\alpha(x^\alpha) - i sin_\alpha(x^\alpha)$. Therefore,

$$1 = E_\alpha(ix^\alpha) \otimes E_\alpha(-ix^\alpha) = [sin_\alpha(x^\alpha)]^{\otimes 2} + [cos_\alpha(x^\alpha)]^{\otimes 2}.$$

Moreover,

$$\begin{aligned} &1 + [tan_\alpha(x^\alpha)]^{\otimes 2} \\ &= [cos_\alpha(x^\alpha)]^{\otimes -2}([sin_\alpha(x^\alpha)]^{\otimes 2} + [cos_\alpha(x^\alpha)]^{\otimes 2}) \\ &= [cos_\alpha(x^\alpha)]^{\otimes -2} \\ &= [sec_\alpha(x^\alpha)]^{\otimes 2}. \end{aligned}$$

And

$$1+[cot_{\alpha}(x^{\alpha})]^{\otimes 2}$$
$$=[sin_{\alpha}(x^{\alpha})]^{\otimes -2}([sin_{\alpha}(x^{\alpha})]^{\otimes 2}+[cos_{\alpha}(x^{\alpha})]^{\otimes 2})$$
$$=[sin_{\alpha}(x^{\alpha})]^{\otimes -2}$$
$$=[csc_{\alpha}(x^{\alpha})]^{\otimes 2}.$$

Q.e.d.

Theorem 2.9: *If* $0<\alpha\leq 1$, *then*

$$sin_{\alpha}(x^{\alpha}+y^{\alpha})=sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})+cos_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha}). \quad (16)$$
$$cos_{\alpha}(x^{\alpha}+y^{\alpha})=cos_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})-sin_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha}). \quad (17)$$

Proof $cos_{\alpha}(x^{\alpha}+y^{\alpha})+isin_{\alpha}(x^{\alpha}+y^{\alpha})$

$$=E_{\alpha}\big(i(x^{\alpha}+y^{\alpha})\big)$$
$$=E_{\alpha}(ix^{\alpha})\otimes E_{\alpha}(iy^{\alpha})$$
$$=[cos_{\alpha}(x^{\alpha})+isin_{\alpha}(x^{\alpha})]\otimes[cos_{\alpha}(y^{\alpha})+isin_{\alpha}(y^{\alpha})]$$
$$=[cos_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})-sin_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha})]+[sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})+cos_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha})]$$

Thus, the desired results hold. Q.e.d.

Theorem 2.10: *Let* $0<\alpha\leq 1$, *then*

$$cos_{\alpha}(2x^{\alpha})=[cos_{\alpha}(x^{\alpha})]^{\otimes 2}-[sin_{\alpha}(x^{\alpha})]^{\otimes 2}=2[cos_{\alpha}(x^{\alpha})]^{\otimes 2}-1=1-2[sin_{\alpha}(x^{\alpha})]^{\otimes 2}. \quad (18)$$

And

$$sin_{\alpha}(2x^{\alpha})=2sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(x^{\alpha}). \quad (19)$$

Proof By fractional DeMoivre's formula, we have

$$[cos_{\alpha}(x^{\alpha})+isin_{\alpha}(x^{\alpha})]^{\otimes 2}=cos_{\alpha}(2x^{\alpha})+isin_{\alpha}(2x^{\alpha}). \quad (20)$$

It follows that

$$[[cos_{\alpha}(x^{\alpha})]^{\otimes 2}-[sin_{\alpha}(x^{\alpha})]^{\otimes 2}]+i[2sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(x^{\alpha})]=cos_{\alpha}(2x^{\alpha})+isin_{\alpha}(2x^{\alpha}). \quad (21)$$

Therefore, we obtain the desired results. Q.e.d.

Theorem 2.11: *If* $0<\alpha\leq 1$, *then*

$$sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})=\frac{1}{2}[sin_{\alpha}(x^{\alpha}+y^{\alpha})+sin_{\alpha}(x^{\alpha}-y^{\alpha})], \quad (22)$$
$$cos_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})=\frac{1}{2}[cos_{\alpha}(x^{\alpha}+y^{\alpha})+cos_{\alpha}(x^{\alpha}-y^{\alpha})], \quad (23)$$
$$sin_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha})=-\frac{1}{2}[cos_{\alpha}(x^{\alpha}+y^{\alpha})-cos_{\alpha}(x^{\alpha}-y^{\alpha})]. \quad (24)$$

Proof $\frac{1}{2}[sin_{\alpha}(x^{\alpha}+y^{\alpha})+sin_{\alpha}(x^{\alpha}-y^{\alpha})]$

$$=\frac{1}{2}[sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})+cos_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha})+sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})-cos_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha})]$$
$$=sin_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha}).$$

On the other hand,

$$\frac{1}{2}[cos_{\alpha}(x^{\alpha}+y^{\alpha})+cos_{\alpha}(x^{\alpha}-y^{\alpha})]$$
$$=\frac{1}{2}[cos_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})-sin_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha})+cos_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha})+sin_{\alpha}(x^{\alpha})\otimes sin_{\alpha}(y^{\alpha})]$$
$$=cos_{\alpha}(x^{\alpha})\otimes cos_{\alpha}(y^{\alpha}).$$

And

$$-\frac{1}{2}[cos_{\alpha}(x^{\alpha}+y^{\alpha})-cos_{\alpha}(x^{\alpha}-y^{\alpha})]$$

$= -\frac{1}{2}[cos_\alpha(x^\alpha) \otimes cos_\alpha(y^\alpha) - sin_\alpha(x^\alpha) \otimes sin_\alpha(y^\alpha) - cos_\alpha(x^\alpha) \otimes cos_\alpha(y^\alpha) - sin_\alpha(x^\alpha) \otimes sin_\alpha(y^\alpha)]$
$= sin_\alpha(x^\alpha) \otimes sin_\alpha(y^\alpha).$

Q.e.d.

Definition 2.12([19])**:** Let $0 < \alpha \le 1$, $(-1)^\alpha = -1$, $u: [a,b] \to R$ and $x_0 \in (a,b)$. u is called local α-fractional differentiable at x_0 if $\lim_{x \to x_0} \frac{u(x)-u(x_0)}{(x-x_0)^\alpha}$ exists. And the α-fractional derivative of $u(x)$ at x_0 is denoted by

$$u^{(\alpha)}(x_0) = \frac{du}{dx^\alpha}(x_0) = \Gamma(\alpha+1) \cdot \lim_{x \to x_0} \frac{u(x)-u(x_0)}{(x-x_0)^\alpha}, \tag{25}$$

where $\Gamma(\ )$ is the gamma function. If u is local α-fractional differentiable at any point in open interval (a,b), then we say that u is a local α-fractional differentiable function on (a,b). In addition, the n-th order local α-fractional derivative $(u^{(\alpha)})(u^{(\alpha)})\cdots(u^{(\alpha)})(x)$ of $u(x)$, is denoted by $u^{(n\alpha)}(x)$ or $\frac{d^n u}{(dx^\alpha)^n}(x)$, where n is a positive integer, and we define $u^{(0)}(x) = u(x)$.

Proposition 2.13: *Let $0 < \alpha \le 1, (-1)^\alpha = -1$, and r be a real number, If $u, v: [a,b] \to R$ are local α-fractional differentiable at $x \in (a,b)$, then*

$$(u+v)^{(\alpha)}(x) = u^{(\alpha)}(x) + v^{(\alpha)}(x). \tag{26}$$
$$(u-v)^{(\alpha)}(x) = u^{(\alpha)}(x) - v^{(\alpha)}(x). \tag{27}$$
$$(ru)^{(\alpha)}(x) = ru^{(\alpha)}(x). \tag{28}$$

Proposition 2.14 (product rule of fractional analytic functions for local fractional derivative)**:** *Let $0 < \alpha \le 1, (-1)^\alpha = -1$. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional analytic functions, then*

$$(f_\alpha \otimes g_\alpha)^{(\alpha)}(x^\alpha) = f_\alpha^{\ (\alpha)}(x^\alpha) \otimes g_\alpha(x^\alpha) + f_\alpha(x^\alpha) \otimes g_\alpha^{\ (\alpha)}(x^\alpha). \tag{29}$$

Proof $(f_\alpha \otimes g_\alpha)^{(\alpha)}(x^\alpha)$

$$= \Gamma(\alpha+1) \cdot \lim_{\Delta x \to 0} \frac{(f_\alpha \otimes g_\alpha)((x+\Delta x)^\alpha) - (f_\alpha \otimes g_\alpha)(x^\alpha)}{\Delta x^\alpha}$$
$$= \Gamma(\alpha+1) \cdot \lim_{\Delta x \to 0} \left[\frac{f_\alpha((x+\Delta x)^\alpha) - f_\alpha(x^\alpha)}{\Delta x^\alpha} \otimes g_\alpha((x+\Delta x)^\alpha) + f_\alpha(x^\alpha) \otimes \frac{g_\alpha((x+\Delta x)^\alpha) - g_\alpha(x^\alpha)}{\Delta x^\alpha}\right]$$
$$= \Gamma(\alpha+1) \cdot \left[\lim_{\Delta x \to 0} \frac{f_\alpha((x+\Delta x)^\alpha) - f_\alpha(x^\alpha)}{\Delta x^\alpha} \otimes \lim_{\Delta x \to 0} g_\alpha((x+\Delta x)^\alpha) + f_\alpha(x^\alpha) \otimes \lim_{\Delta x \to 0} \frac{g_\alpha((x+\Delta x)^\alpha) - g_\alpha(x^\alpha)}{\Delta x^\alpha}\right]$$
$$= f_\alpha^{\ (\alpha)}(x^\alpha) \otimes g_\alpha(x^\alpha) + f_\alpha(x^\alpha) \otimes g_\alpha^{\ (\alpha)}(x^\alpha).$$

Q.e.d.

Corollary 2.15: *If $0 < \alpha \le 1, (-1)^\alpha = -1$, n is a positive integer, and $f_\alpha(x^\alpha)$ is an α-fractional analytic function, then*

$$([f_\alpha(x^\alpha)]^{\otimes n})^{(\alpha)}(x^\alpha) = n[f_\alpha(x^\alpha)]^{\otimes(n-1)} \otimes f_\alpha^{\ (\alpha)}(x^\alpha). \tag{30}$$

Proposition 2.16 (Leibniz rule of fractional analytic functions for local fractional derivative)**:** *If $0 < \alpha \le 1, (-1)^\alpha = -1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions, then*

$$(f_\alpha \otimes g_\alpha)^{(m\alpha)}(x^\alpha) = \sum_{k=0}^{m} \binom{m}{k} f_\alpha^{\ ((m-k)\alpha)}(x^\alpha) \otimes g_\alpha^{\ (k\alpha)}(x^\alpha). \tag{31}$$

Proposition 2.17 (quotient rule of fractional analytic functions for local fractional derivative)**:** *If $0 < \alpha \le 1, (-1)^\alpha = -1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions, then*

$$(f_\alpha \otimes g_\alpha^{\ \otimes -1})^{(\alpha)}(x^\alpha) = \left[f_\alpha^{\ (\alpha)}(x^\alpha) \otimes g_\alpha(x^\alpha) - f_\alpha(x^\alpha) \otimes g_\alpha^{\ (\alpha)}(x^\alpha)\right] \otimes g_\alpha^{\ \otimes -2}(x^\alpha). \tag{32}$$

Proof $(f_\alpha \otimes g_\alpha^{\ \otimes -1})^{(\alpha)}(x^\alpha)$

$$= \Gamma(\alpha+1) \cdot \lim_{\Delta x \to 0} \frac{(f_\alpha \otimes g_\alpha^{\ \otimes -1})((x+\Delta x)^\alpha) - (f_\alpha \otimes g_\alpha^{\ \otimes -1})(x^\alpha)}{\Delta x^\alpha}$$

$= \Gamma(\alpha+1) \cdot \lim_{\Delta x \to 0} \frac{[f_\alpha((x+\Delta x)^\alpha) \otimes g_\alpha(x^\alpha) - f_\alpha(x^\alpha) \otimes g_\alpha((x+\Delta x)^\alpha)] \otimes [g_\alpha((x+\Delta x)^\alpha) \otimes g_\alpha(x^\alpha)]^{\otimes -1}}{\Delta x^\alpha}$

$= \Gamma(\alpha+1) \cdot \lim_{\Delta x \to 0} \frac{\left[[f_\alpha((x+\Delta x)^\alpha) - f_\alpha(x^\alpha)] \otimes g_\alpha(x^\alpha) - f_\alpha(x^\alpha) \otimes [g_\alpha((x+\Delta x)^\alpha) - g_\alpha(x^\alpha)]\right] \otimes [g_\alpha((x+\Delta x)^\alpha) \otimes g_\alpha(x^\alpha)]^{\otimes -1}}{\Delta x^\alpha}$

$= \Gamma(\alpha+1) \cdot \lim_{\Delta x \to 0} \left(\left[\left[\frac{f_\alpha((x+\Delta x)^\alpha) - f_\alpha(x^\alpha)}{\Delta x^\alpha}\right] \otimes g_\alpha(x^\alpha) - f_\alpha(x^\alpha) \otimes \frac{g_\alpha((x+\Delta x)^\alpha) - g_\alpha(x^\alpha)}{\Delta x^\alpha}\right] \otimes [g_\alpha((x+\Delta x)^\alpha) \otimes g_\alpha(x^\alpha)]^{\otimes -1}\right)$

$= \left[f_\alpha{}^{(\alpha)}(x^\alpha) \otimes g_\alpha(x^\alpha) - f_\alpha(x^\alpha) \otimes g_\alpha{}^{(\alpha)}(x^\alpha)\right] \otimes g_\alpha{}^{\otimes -2}(x^\alpha).$

Q.e.d.

Remark 2.18: The followings are local fractional differentials of fractional analytic functions. Let f_α and g_α be α-fractional analytic functions, then

$$df_\alpha = f_\alpha^{(\alpha)} dx^\alpha, \tag{33}$$

$$d(f_\alpha + g_\alpha) = df_\alpha + dg_\alpha, \tag{34}$$

$$d(f_\alpha - g_\alpha) = df_\alpha - dg_\alpha, \tag{35}$$

$$d(pf_\alpha) = pdf_\alpha, \tag{36}$$

where p is a constant. And

$$d(f_\alpha \otimes g_\alpha) = df_\alpha \otimes g_\alpha + f_\alpha \otimes dg_\alpha, \tag{37}$$

$$d(f_\alpha \otimes g_\alpha{}^{\otimes -1}) = [df_\alpha \otimes g_\alpha - f_\alpha \otimes dg_\alpha] \otimes g_\alpha{}^{\otimes -2}. \tag{38}$$

Theorem 2.19 (chain rule of fractional analytic functions for local fractional derivative): *Suppose that* $0 < \alpha \le 1$, $(-1)^\alpha = -1$. *Let* $F(z) = \sum_{n=0}^\infty a_n z^n$ *and* $g_\alpha(x^\alpha)$ *be an* α*-fractional analytic function. If* $F_{\otimes\alpha}(g_\alpha(x^\alpha)) = \sum_{n=0}^\infty a_n \left(g_\alpha(x^\alpha)\right)^{\otimes n}$ *and* $F'_{\otimes\alpha}(g_\alpha(x^\alpha)) = \sum_{n=1}^\infty n a_n \left(g_\alpha(x^\alpha)\right)^{\otimes(n-1)}$ *exist, then*

$$\left[F_{\otimes\alpha}(g_\alpha(x^\alpha))\right]^{(\alpha)}(x^\alpha) = F'_{\otimes\alpha}(g_\alpha(x^\alpha)) \otimes [g_\alpha(x^\alpha)]^{(\alpha)}(x^\alpha). \tag{39}$$

Proof By Corollary 2.15, we obtain

$$\begin{aligned}
&\left[F_{\otimes\alpha}(g_\alpha(x^\alpha))\right]^{(\alpha)}(x^\alpha) \\
&= \Gamma(\alpha+1) \cdot \lim_{\Delta x \to 0} \frac{F_{\otimes\alpha}(g_\alpha((x+\Delta x)^\alpha)) - F_{\otimes\alpha}(g_\alpha(x^\alpha))}{\Delta x^\alpha} \\
&= \Gamma(\alpha+1) \cdot \lim_{\Delta x \to 0} \frac{\sum_{n=0}^\infty a_n (g_\alpha((x+\Delta x)^\alpha))^{\otimes n} - \sum_{n=0}^\infty a_n (g_\alpha(x^\alpha))^{\otimes n}}{\Delta x^\alpha} \\
&= \Gamma(\alpha+1) \cdot \sum_{n=1}^\infty a_n \left[\lim_{\Delta x \to 0} \frac{(g_\alpha((x+\Delta x)^\alpha))^{\otimes n} - (g_\alpha(x^\alpha))^{\otimes n}}{\Delta x^\alpha}\right] \\
&= \sum_{n=1}^\infty n a_n \left(g_\alpha(x^\alpha)\right)^{\otimes(n-1)} \otimes [g_\alpha(x^\alpha)]^{(\alpha)}(x^\alpha) \\
&= F'_{\otimes\alpha}(g_\alpha(x^\alpha)) \otimes [g_\alpha(x^\alpha)]^{(\alpha)}(x^\alpha).
\end{aligned}$$

Q.e.d.

The followings are the local α-fractional derivatives of six elementary α-fractional trigonometric functions.

Proposition 2.20: *Suppose that* $0 < \alpha \le 1$. *Then*

$$[sin_\alpha(x^\alpha)]^{(\alpha)}(x) = cos_\alpha(x^\alpha), \tag{40}$$

$$[cos_\alpha(x^\alpha)]^{(\alpha)}(x) = -sin_\alpha(x^\alpha), \tag{41}$$

$$[tan_\alpha(x^\alpha)]^{(\alpha)}(x) = [sec_\alpha(x^\alpha)]^{\otimes 2}, \tag{42}$$

$$[cot_\alpha(x^\alpha)]^{(\alpha)}(x) = -[csc_\alpha(x^\alpha)]^{\otimes 2}, \tag{43}$$

$$[sec_\alpha(x^\alpha)]^{(\alpha)}(x) = sec_\alpha(x^\alpha) \otimes tan_\alpha(x^\alpha), \tag{44}$$

$$[csc_\alpha(x^\alpha)]^{(\alpha)}(x) = -csc_\alpha(x^\alpha) \otimes cot_\alpha(x^\alpha). \tag{45}$$

Remark 2.21: The local fractional differentials of fractional trigonometric functions are

$$dsin_\alpha(x^\alpha) = cos_\alpha(x^\alpha)dx^\alpha, \tag{46}$$

$$dcos_\alpha(x^\alpha) = -sin_\alpha(x^\alpha)dx^\alpha, \tag{47}$$

$$dtan_\alpha(x^\alpha) = [sec_\alpha(x^\alpha)]^{\otimes 2}dx^\alpha, \tag{48}$$

$$dcot_\alpha(x^\alpha) = -[csc_\alpha(x^\alpha)]^{\otimes 2}dx^\alpha, \tag{49}$$

$$dsec_\alpha(x^\alpha) = sec_\alpha(x^\alpha) \otimes tan_\alpha(x^\alpha)dx^\alpha, \tag{50}$$

$$dcsc_\alpha(x^\alpha) = -csc_\alpha(x^\alpha) \otimes cot_\alpha(x^\alpha)dx^\alpha. \tag{51}$$

Definition 2.22 ([19])**:** Let $0 < \alpha \le 1$, and $u: [a, b] \to R$. If

$$\lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} u(\xi_k)\,(x_k - x_{k-1})^\alpha \tag{52}$$

exists, then we say that u is a α-fractional Riemann integrable function on $[a, b]$. Where the partitions of the interval $[a, b]$ are $[x_{k-1}, x_k]$, $k = 1, \cdots, n$, and $x_0 = a, x_n = b$, $\xi_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$, $\|\Delta\| = \max_{k=1,\cdots,n}\{\Delta x_k\}$. The limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} u(\xi_k)\,(x_k - x_{k-1})^\alpha = \int_a^b u(x)\,dx^\alpha, \tag{53}$$

which is called the α-fractional Riemann integral (or local α-fractional integral) of u on $[a, b]$.

Theorem 2.23 (fundamental theorem of fractional analytic functions for local fractional calculus)**:** *Let* $0 < \alpha \le 1$, $(-1)^\alpha = -1$. *If* $f_\alpha: [a, b] \to R$ *and* $F_\alpha: [a, b] \to R$ *are* α*-fractional analytic functions satisfies* $F_\alpha{}^{(\alpha)}(x^\alpha) = f_\alpha(x^\alpha)$ *for all* $x \in (a, b)$, *then*

$$\int_a^b f_\alpha(x^\alpha)\,dx^\alpha = \Gamma(\alpha+1)\big(F_\alpha(b^\alpha) - F_\alpha(a^\alpha)\big). \tag{54}$$

In the following, we introduce the indefinite α-fractional integrals involving some α-fractional trigonometric functions.

Proposition 2.24: *If* $0 < \alpha \le 1$, *then*

$$\int sin_\alpha(x^\alpha)\,dx^\alpha = -\Gamma(\alpha+1)cos_\alpha(x^\alpha) + C, \tag{55}$$

$$\int cos_\alpha(x^\alpha)\,dx^\alpha = \Gamma(\alpha+1)sin_\alpha(x^\alpha) + C, \tag{56}$$

$$\int [sec_\alpha(x^\alpha)]^{\otimes 2}\,dx^\alpha = \Gamma(\alpha+1)tan_\alpha(x^\alpha) + C, \tag{57}$$

$$\int [csc_\alpha(x^\alpha)]^{\otimes 2}\,dx^\alpha = -\Gamma(\alpha+1)cot_\alpha(x^\alpha) + C, \tag{58}$$

$$\int sec_\alpha(x^\alpha) \otimes tan_\alpha(x^\alpha)\,dx^\alpha = \Gamma(\alpha+1)sec_\alpha(x^\alpha) + C, \tag{59}$$

$$\int csc_\alpha(x^\alpha) \otimes cot_\alpha(x^\alpha)\,dx^\alpha = -\Gamma(\alpha+1)csc_\alpha(x^\alpha) + C. \tag{60}$$

III. MAIN RESULTS

At first, we find indefinite fractional integrals involving some fractional trigonometric functions.

Theorem 3.1: *If* $0 < \alpha \le 1$, p, q *are real numbers and* $p + q \ne 0, p - q \ne 0$. *Then*

$$\int sin_\alpha(px^\alpha) \otimes cos_\alpha(qx^\alpha)\,dx^\alpha = -\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{p+q}cos_\alpha((p+q)x^\alpha) + \frac{1}{p-q}cos_\alpha((p-q)x^\alpha)\right] + C. \tag{61}$$

$$\int cos_\alpha(px^\alpha) \otimes cos_\alpha(qx^\alpha)\,dx^\alpha = \frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{p+q}sin_\alpha((p+q)x^\alpha) + \frac{1}{p-q}sin_\alpha((p-q)x^\alpha)\right] + C. \tag{62}$$

$$\int sin_\alpha(px^\alpha) \otimes sin_\alpha(qx^\alpha)\,dx^\alpha = -\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{p+q}sin_\alpha((p+q)x^\alpha) - \frac{1}{p-q}sin_\alpha((p-q)x^\alpha)\right] + C. \tag{63}$$

Proof

$$\int sin_\alpha(px^\alpha) \otimes cos_\alpha(qx^\alpha)\,dx^\alpha$$

$$= \frac{1}{2}\int [sin_\alpha((p+q)x^\alpha) + sin_\alpha((p-q)x^\alpha)]\,dx^\alpha$$

$$= -\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{p+q}cos_\alpha((p+q)x^\alpha) + \frac{1}{p-q}cos_\alpha((p-q)x^\alpha)\right] + C.$$

$$\int cos_\alpha(px^\alpha) \otimes cos_\alpha(qx^\alpha)\, dx^\alpha$$
$$= \frac{1}{2}\int [cos_\alpha((p+q)x^\alpha) + cos_\alpha((p-q)x^\alpha)]\, dx^\alpha$$
$$= \frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{p+q} sin_\alpha((p+q)x^\alpha) + \frac{1}{p-q} sin_\alpha((p-q)x^\alpha)\right] + C.$$
$$\int sin_\alpha(px^\alpha) \otimes sin_\alpha(qx^\alpha)\, dx^\alpha$$
$$= -\frac{1}{2}\int [cos_\alpha((p+q)x^\alpha) - cos_\alpha((p-q)x^\alpha)]\, dx^\alpha$$
$$= -\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{p+q} sin_\alpha((p+q)x^\alpha) - \frac{1}{p-q} sin_\alpha((p-q)x^\alpha)\right] + C.$$

Q.e.d.

Theorem 3.2: *Let* $0 < \alpha \le 1$, m, n *be positive integers. Suppose that the indefinite* α*-fractional integral* $\int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha$ *exists.*

Case 1. If m *is odd, then*

$$\int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha = -\int [1 - [cos_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{m-1}{2}\right)} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dcos_\alpha(x^\alpha). \quad (64)$$

Case 2. If n *is odd, then*

$$\int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha = \int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [1 - [sin_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{n-1}{2}\right)}\, dsin_\alpha(x^\alpha). \quad (65)$$

Case 3. If m, n *are even, then*

$$\int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha = \int [1 - [cos_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{m}{2}\right)} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha. \quad (66)$$

Proof *Case 1.* If m is odd and n is even, we have

$$\int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha$$
$$= -\int [sin_\alpha(x^\alpha)]^{\otimes(m-1)} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dcos_\alpha(x^\alpha)$$
$$= -\int [1 - [cos_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{m-1}{2}\right)} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dcos_\alpha(x^\alpha).$$

Case 2. If m is even and n is odd, then

$$\int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [cos_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha$$
$$= \int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [cos_\alpha(x^\alpha)]^{\otimes(n-1)}\, dsin_\alpha(x^\alpha)$$
$$= \int [sin_\alpha(x^\alpha)]^{\otimes m} \otimes [1 - [sin_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{n-1}{2}\right)}\, dsin_\alpha(x^\alpha).$$

Case 3 is obvious. Q.e.d.

Theorem 3.3: *If* $0 < \alpha \le 1$, *and* m, n *are positive integers. Suppose that the* α*-fractional indefinite integral* $\int [sec_\alpha(x^\alpha)]^{\otimes m}[tan_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha$ *exists.*

Case 1. If m, n *are odd, then*

$$\int [sec_\alpha(x^\alpha)]^{\otimes m}[tan_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha = \int [sec_\alpha(x^\alpha)]^{\otimes(m-1)} \otimes [[sec_\alpha(x^\alpha)]^{\otimes 2} - 1]^{\otimes\left(\frac{n-1}{2}\right)}\, dsec_\alpha(x^\alpha). \quad (67)$$

Case 2. If m *is even, then*

$$\int [sec_\alpha(x^\alpha)]^{\otimes m}[tan_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha = \int [1 + [tan_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{m-2}{2}\right)}[tan_\alpha(x^\alpha)]^{\otimes n}\, dtan_\alpha(x^\alpha). \quad (68)$$

Case 3. If n *is even, then*

$$\int [sec_\alpha(x^\alpha)]^{\otimes m}[tan_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha = \int [sec_\alpha(x^\alpha)]^{\otimes m}[[sec_\alpha(x^\alpha)]^{\otimes 2} - 1]^{\otimes\left(\frac{n}{2}\right)}\, dx^\alpha. \quad (69)$$

Proof *Case 1.* If m, n are odd, we obtain

$$\int [sec_\alpha(x^\alpha)]^{\otimes m}[tan_\alpha(x^\alpha)]^{\otimes n}\, dx^\alpha$$
$$= \int [sec_\alpha(x^\alpha)]^{\otimes(m-1)}[tan_\alpha(x^\alpha)]^{\otimes(n-1)}\, dsec_\alpha(x^\alpha)$$

$$= \int [sec_\alpha(x^\alpha)]^{\otimes(m-1)} \otimes [[sec_\alpha(x^\alpha)]^{\otimes 2} - 1]^{\otimes\left(\frac{n-1}{2}\right)} \, dsec_\alpha(x^\alpha).$$

Case 2. If m is even, we have

$$\int [sec_\alpha(x^\alpha)]^{\otimes m} [tan_\alpha(x^\alpha)]^{\otimes n} \, dx^\alpha$$
$$= \int [sec_\alpha(x^\alpha)]^{\otimes(m-2)} [tan_\alpha(x^\alpha)]^{\otimes n} \, dtan_\alpha(x^\alpha)$$
$$= \int [1 + [tan_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{m-2}{2}\right)} [tan_\alpha(x^\alpha)]^{\otimes n} \, dtan_\alpha(x^\alpha).$$

Case 3 is obvious. Q.e.d.

Theorem 3.4: *Let* $0 < \alpha \le 1$, m, n *be positive integers. Assume that the* α*-fractional indefinite integral* $\int [csc_\alpha(x^\alpha)]^{\otimes m} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dx^\alpha$ *exists.*

Case 1. If m, n *are odd, then*

$$\int [csc_\alpha(x^\alpha)]^{\otimes m} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dx^\alpha = -\int [csc_\alpha(x^\alpha)]^{\otimes(m-1)} \otimes [[csc_\alpha(x^\alpha)]^{\otimes 2} - 1]^{\otimes\left(\frac{n-1}{2}\right)} \, dcsc_\alpha(x^\alpha). \quad (70)$$

Case 2. If m *is even, then*

$$\int [csc_\alpha(x^\alpha)]^{\otimes m} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dx^\alpha = -\int [1 + [cot_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{m-2}{2}\right)} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dcot_\alpha(x^\alpha). \quad (71)$$

Case 3. If n *is even, then*

$$\int [csc_\alpha(x^\alpha)]^{\otimes m} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dx^\alpha = \int [csc_\alpha(x^\alpha)]^{\otimes m} \otimes [[csc_\alpha(x^\alpha)]^{\otimes 2} - 1]^{\otimes\left(\frac{n}{2}\right)} \, dx^\alpha. \quad (72)$$

Proof *Case 1.* If m, n are odd, we have

$$\int [csc_\alpha(x^\alpha)]^{\otimes m} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dx^\alpha$$
$$= -\int [csc_\alpha(x^\alpha)]^{\otimes(m-1)} \otimes [cot_\alpha(x^\alpha)]^{\otimes(n-1)} \, dcsc_\alpha(x^\alpha)$$
$$= -\int [csc_\alpha(x^\alpha)]^{\otimes(m-1)} \otimes [[csc_\alpha(x^\alpha)]^{\otimes 2} - 1]^{\otimes\left(\frac{n-1}{2}\right)} \, dcsc_\alpha(x^\alpha).$$

Case 2. If m is even, then

$$\int [csc_\alpha(x^\alpha)]^{\otimes m} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dx^\alpha$$
$$= -\int [csc_\alpha(x^\alpha)]^{\otimes(m-2)} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dcot_\alpha(x^\alpha)$$
$$= -\int [1 + [cot_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{m-2}{2}\right)} \otimes [cot_\alpha(x^\alpha)]^{\otimes n} \, dcot_\alpha(x^\alpha).$$

Case 3 is obvious. Q.e.d.

IV. EXAMPLES

Example 4.1: By product rule and chain rule, we obtain the following local α-fractional derivative

$$([sin_\alpha(x^\alpha)]^{\otimes 3} \otimes [cos_\alpha(x^\alpha)]^{\otimes 5})^{(\alpha)}$$
$$= ([sin_\alpha(x^\alpha)]^{\otimes 3})^{(\alpha)} \otimes [cos_\alpha(x^\alpha)]^{\otimes 5} + [sin_\alpha(x^\alpha)]^{\otimes 3} \otimes ([cos_\alpha(x^\alpha)]^{\otimes 5})^{(\alpha)}$$
$$= \left(3[sin_\alpha(x^\alpha)]^{\otimes 2} \otimes cos_\alpha(x^\alpha)\right) \otimes [cos_\alpha(x^\alpha)]^{\otimes 5} + [sin_\alpha(x^\alpha)]^{\otimes 3} \otimes \left(-5[cos_\alpha(x^\alpha)]^{\otimes 4} \otimes sin_\alpha(x^\alpha)\right)$$
$$= 3[sin_\alpha(x^\alpha)]^{\otimes 2} \otimes [cos_\alpha(x^\alpha)]^{\otimes 6} - 5[sin_\alpha(x^\alpha)]^{\otimes 4} \otimes [cos_\alpha(x^\alpha)]^{\otimes 4}. \quad (73)$$

Example 4.2: Using quotient rule yields the local α-fractional derivative

$$(tan_\alpha(x^\alpha) \otimes [sin_\alpha(x^\alpha) + sec_\alpha(x^\alpha) + 2]^{\otimes -1})^{(\alpha)}$$
$$= ([sec_\alpha(x^\alpha)]^{\otimes 2} \otimes [sin_\alpha(x^\alpha) + sec_\alpha(x^\alpha) + 2] - tan_\alpha(x^\alpha) \otimes [cos_\alpha(x^\alpha) + sec_\alpha(x^\alpha) \otimes tan_\alpha(x^\alpha)]) \otimes$$
$$[sin_\alpha(x^\alpha) + sec_\alpha(x^\alpha) + 2]^{\otimes -2}. \quad (74)$$

Example 4.3:

$$\int cos_\alpha(2x^\alpha) \otimes cos_\alpha(7x^\alpha) \, dx^\alpha$$
$$= \frac{1}{2} \int [cos_\alpha(9x^\alpha) + cos_\alpha(5x^\alpha)] \, dx^\alpha$$
$$= \frac{\Gamma(\alpha+1)}{2} \left[\frac{1}{9} sin_\alpha(9x^\alpha) + \frac{1}{5} sin_\alpha(5x^\alpha)\right] + C$$

$$= \Gamma(\alpha+1)\left[\frac{1}{18}sin_\alpha(9x^\alpha) + \frac{1}{10}sin_\alpha(5x^\alpha)\right] + C. \quad (75)$$

Example 4.4: The indefinite α-fractional integral

$$\int [sin_\alpha(x^\alpha)]^{\otimes 5} \otimes [cos_\alpha(x^\alpha)]^{\otimes 4}\, dx^\alpha$$

$$= -\int ([cos_\alpha(x^\alpha)]^{\otimes 4} - 2[cos_\alpha(x^\alpha)]^{\otimes 2} + 1) \otimes [cos_\alpha(x^\alpha)]^{\otimes 4}\, dcos_\alpha(x^\alpha)$$

$$= -\int ([cos_\alpha(x^\alpha)]^{\otimes 8} - 2[cos_\alpha(x^\alpha)]^{\otimes 6} + [cos_\alpha(x^\alpha)]^{\otimes 4})\, dcos_\alpha(x^\alpha)$$

$$= -\Gamma(\alpha+1)\left(\frac{1}{9}[cos_\alpha(x^\alpha)]^{\otimes 9} - \frac{2}{7}[cos_\alpha(x^\alpha)]^{\otimes 7} + \frac{1}{5}[cos_\alpha(x^\alpha)]^{\otimes 5}\right) + C. \quad (76)$$

Example 4.5: $\int [sec_\alpha(x^\alpha)]^{\otimes 3}[tan_\alpha(x^\alpha)]^{\otimes 7}\, dx^\alpha$

$$= \int [sec_\alpha(x^\alpha)]^{\otimes 2} \otimes [[sec_\alpha(x^\alpha)]^{\otimes 2} - 1]^{\otimes 3}\, dsec_\alpha(x^\alpha)$$

$$= \int ([sec_\alpha(x^\alpha)]^{\otimes 8} - 3[sec_\alpha(x^\alpha)]^{\otimes 6} + 3[sec_\alpha(x^\alpha)]^{\otimes 4} - [sec_\alpha(x^\alpha)]^{\otimes 2})\, dsec_\alpha(x^\alpha)$$

$$= \Gamma(\alpha+1)\left(\frac{1}{9}[sec_\alpha(x^\alpha)]^{\otimes 9} - \frac{3}{7}[cos_\alpha(x^\alpha)]^{\otimes 7} + \frac{3}{5}[sec_\alpha(x^\alpha)]^{\otimes 5} - \frac{1}{3}[sec_\alpha(x^\alpha)]^{\otimes 3}\right) + C. \quad (77)$$

Example 4.6:

$$\int [csc_\alpha(x^\alpha)]^{\otimes 4} \otimes [cot_\alpha(x^\alpha)]^{\otimes 9}\, dx^\alpha$$

$$= -\int [1 + [cot_\alpha(x^\alpha)]^{\otimes 2}] \otimes [cot_\alpha(x^\alpha)]^{\otimes 9}\, dcot_\alpha(x^\alpha)$$

$$= -\int ([cot_\alpha(x^\alpha)]^{\otimes 11} + [cot_\alpha(x^\alpha)]^{\otimes 9})\, dcot_\alpha(x^\alpha)$$

$$= -\Gamma(\alpha+1)\left(\frac{1}{12}[cot_\alpha(x^\alpha)]^{\otimes 12} + \frac{1}{10}[cot_\alpha(x^\alpha)]^{\otimes 10}\right) + C. \quad (78)$$

V. CONCLUSION

From the above discussion, we know that the fractional analytic function plays an important role in this paper. The local fractional calculus involving fractional trigonometric functions is the subject of this article. We obtain some local fractional differential and integral formulas involving fractional trigonometric functions by using a new multiplication and some important properties of fractional analytic functions. These formulas help us to understand the importance of fractional analytic functions in local fractional calculus. In the future, we will extend the methods used in this article to study the engineering mathematics and applied science problems based on local fractional calculus.

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