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**Research Paper**

# Antiplane Strain Analysis for an Isotopic Cylinder under Traction-Displacement Boundary Conditions

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***ABSTRACT***
*An isotropic linearly elastic cylindrical material under anti-plane deformation, modeled by a traction-displacement boundary value problem, with the only non-vanishing displacement component $w(r,\theta)$ satisfying $w(a,\theta)\neq 0$, is considered. Fracture detection along the axis of symmetry with respect to loading is investigated. The original circular plane of the study, which is of radius $a$, is conformally transformed onto an upper half plane. A solution for the displacement is derived and analyzed for fracture detection. The results obtained extend those known for cylinders of the same geometry, but subject the condition $w(a,\theta)=0$. We showed that the displacement is singular at the origin, and the non-zero stresses $\sigma_{rz}(r,\pi)$, $\sigma_{\theta z}(r,\pi), \sigma_{rz}(r,0)$ tend to tear the material as $r\to a$ and become singular as $r\to 0$*
***KEYWORDS: Traction-displacement, Fracture detection, ModeIII stress Intensity factor, Crack initiation***

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## I. INTRODUCTION

A long homogenous isotropic solid cylinder is described in cylindrical coordinates $(r,\theta,z)$ by $-\infty<z<\infty,\ 0\leq \theta\leq a\ ,\ \ 0\leq\theta\leq 2\pi$. It is subjected to prescribed displacement, $(a,\theta)=\frac{1}{2}(1-cos\theta)w_0,\ \ \pi<\theta<2\pi$ , on the boundary, where $w_0$ is an arbitrary constant and to out-of-plane stress of magnitude $Q$ when $r=a\ ,\ 0\leq\theta\leq\pi$ (see figure 1).

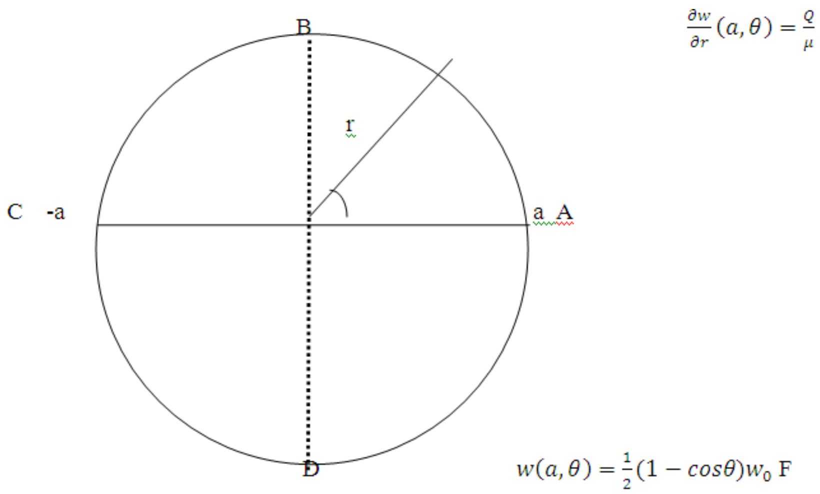


**Figure 1:** Geometry and loading of the problem.

The loading sets up a boundary value problem for Laplace equation in two dimensions of the type studied in [1], but with $w_0\neq 0$ has not been addresses (see for example [2]).Elastic stress analyses have been carried out by several authors. In [3], the investigation involves cracked circular cylinders under anti-plane traction. In [4] the cracked circular cylinder is subjected to tensile load and studied for stress intensity factor. The motivation here is to obtain the displacement field everywhere in the body under the displacement boundary condition $w(a,\theta) \neq 0$, since there are works that involve such general boundary conditions (see for example [5,6,7]) and to investigate the fields along$\theta = 0$ and $\theta = \pi$ , $0 < r \leq a$ for fracture.

## II. FORMULATION AND TRANSFORMATION OF GOVERNING EQUATIONS

The loading puts the solid in a state of antiplane shear for which the resultingcomponents of displacement are all zero except $w(r,\theta)$, the one in the $z$ −direction, which equilibrium conditions require that it satisfies the Laplace equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) w(r,\theta) = 0\,,\ \ 0 \leq r \leq a\,,\ \ 0 \leq \theta \leq 2\pi \qquad (1)$$

Equation (1) must be solved together with the mixed boundary conditions:

$$w(a,\theta) = \frac{1}{2}(1 - cos\theta)w_0,\ \ 0 \leq \theta \leq 2\pi \qquad (2)$$

and $\frac{\partial w}{\partial r}(a,\theta) = \frac{Q}{\mu}$ , $0 < \theta < \pi$ (3)

where $\mu$ is the scalar modulus.

The nonzero polar stresses are $\sigma_{rz}(r,\theta) = \mu \frac{\partial w}{\partial r}(r,\theta)$ $and$ $\sigma_{\theta z}(r,\theta) = \frac{\mu}{r}\frac{\partial w}{\partial \theta}(r,\theta)$

The original region is then transformed onto the upper half plane (see figure 2) with the assistance of the conformal mapping function:

$$\zeta(z) = \frac{i(a+z)}{a-z}\ ,\ z = re^{i\theta}.$$

$$= u + iv = \rho e^{i\theta} \qquad (4)$$

Then , $u(r,\theta) = -\frac{2arsin\theta}{a^2 - 2arcos\theta + r^2}$ , $v(r,\theta) = \frac{a^2 - r^2}{a^2 - 2arcos\theta + r^2}$ ,

$$tan\phi(r,\theta) = \frac{a^2 - r^2}{-2arsin\theta}\ ,\ \ \rho^2(r,\theta) = u^2(r,\theta) + v^2(r,\theta). \qquad (5)$$

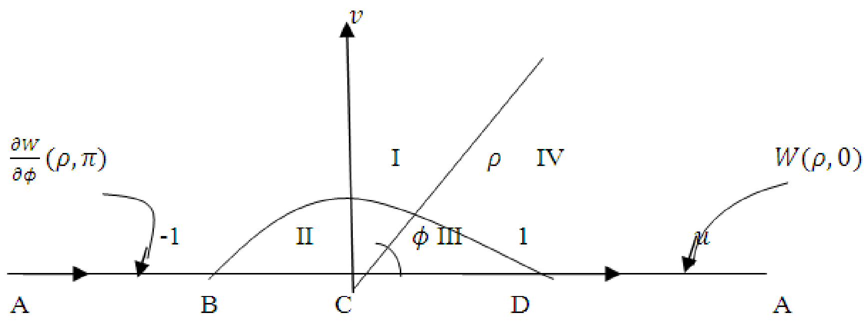


Figure 2. Sketch of The Upper Half $\rho\phi$- plane and Corresponding Quadrants

The conformity condition, $W(\rho,\phi) = w(r,\theta)$, chain rule and the boundary conditions are transformed to get

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right) w(\rho,\phi) = 0\,,\ \ 0 \leq \rho < \infty\,,\ \ 0 \leq \phi \leq \pi \qquad (6)$$

$$W(\rho,0) = \frac{w_0}{1+\rho^2} \qquad (7a)$$

$$\frac{\partial W}{\partial \phi}(\rho,\pi) = \frac{2aQ}{\mu}\frac{\rho}{1+\rho^2} \qquad (7b)$$

where $W(\rho,\phi)$ is the upper half plane displacement.

## III. SOLUTION OF THE TRANSFORMED PROBLEM

When the Mellin transform of $W(\rho,\phi)$ defined by

$$\overline{W}(s,\phi) = \int_0^\infty \rho^{s-1} W(\rho,\phi)d\rho\ \ ,\ \ 0 < Res < 1 \qquad (8)$$

is applied to equations (6) and (7) the result is

$$\left(\frac{d^2}{d\phi^2} + s^2\right)\overline{W}(s,\phi) = 0\ ,\ \ 0 < Res < 1 \qquad (9)$$

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subjected to

$$lim_{\substack{\rho\to 0\\ \rho\to\infty}}\left[\rho^{s+1}\frac{\partial}{\partial\rho}W(\rho,\phi)+\rho^{s}W(\rho,\phi)\right]=0 \qquad (10)$$

Utilizing equations (7) and (10), we obtain the asymptotic behaviours and the strip of regularity in equation (8).

The behaviours are

$W(\rho,\phi)=0(1)\ as\ \rho\to 0$

$\quad = 0(\rho^{-1})\ \ as\ \ \rho\to 0$

The Mellin transform of equation (7) gives $\bar{W}(s,0)=w_0\beta(s)$ (11)

$\frac{\partial\bar{w}}{\partial\theta}(s,\pi)=\frac{2aQ}{\mu}\beta(s+1)$ (12)

where formula 3.241 2[8] produces

$\beta(s)=\int_0^\infty\frac{\rho^{s-1}}{1+\rho^2}d\rho=\frac{\pi}{2}cosec\frac{\pi}{2}s\ ,\qquad 0<Res\le 2$

Equations (11) and (12) together with the solution of equation (9) in the form

$\bar{W}(s,\phi)=A(s)sin\phi s+B(s)cos\phi s$

lead to

$B(s)=w_0\beta(s)$ and $A(s)=\frac{2aQ}{\mu scos\pi s}\ \beta(s+1)+B(s)\frac{sin\pi s}{cos\pi s}$ and

$$\bar{W}(s,\phi)=\frac{\pi}{2}\left(\frac{2aQsin\phi s}{\mu scos\pi scos\ \frac{\pi}{2}s}+\frac{w_0cos\,(\pi-\phi)s}{cos\pi s\ \ sin\frac{\pi}{2}s}\right) \qquad (13)$$

The inversion formula for Mellin transform produces the displacement as

$$W(\rho,\phi)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{W}(s,\phi)\rho^{-s}ds\quad ,\qquad 0<\sigma<\frac{1}{2} \qquad (14)$$

Equations (13) and (14) lead to $W(\rho,\phi)=\frac{\pi aQ}{\mu}W_1(\rho,\phi)+\frac{\pi w_0}{2}W_2(\rho,\phi)$ (15)

where $W_1(\rho,\phi)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\sin\,\phi s\rho^{-s}}{s\ cos\pi s\ \ cos\frac{\pi}{2}s}ds$ (16)

$$W_2(\rho,\phi)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\cos\,(\pi-\phi)s}{s\ cos\pi s\ \ sin\frac{\pi}{2}s}ds \qquad (17)$$

Evaluation of $W_1(\rho,\phi)$ is associated with simple poles at

$s=\pm(2n-1)\ \ and\ s=\pm(n-\frac{1}{2})$, $n$=1,2,3,... Cauchy's residue theorem leads to $W_1(\rho,\phi)=$

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\rho^{2n-1}\sin(2n-1)\,\phi-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-\frac{1}{2})\alpha_n}\rho^{n-\frac{1}{2}}\sin(n-\tfrac{1}{2})\phi\ ,\ \rho>1 \qquad (18)$$

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\rho^{2n-1}\sin(2n-1)\,\phi-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-\frac{1}{2})\alpha_n}\rho^{n-\frac{1}{2}}\sin(n-\tfrac{1}{2})\phi\ ,\ \rho>1$$

where $\alpha_n=cos\frac{\pi}{2}(n-\frac{1}{2})$

$W_2(\rho,\phi)$ has its integrand with simple poles at $s=0, s=\pm 2n, and\ s=\pm(n-\frac{1}{2})$ ,$n$=1,2,3,...

The residue theorem leads to

$$W_2(\rho,\phi)=\frac{2}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\rho^{2n}\ \cos 2n\,\phi-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha_n}\rho^{n-\frac{1}{2}}\sin(n-\tfrac{1}{2})\phi\ ,\ \rho<1$$

$$=\ -\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\rho^{-2n}\cos 2n\,\phi+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha_n}\rho^{n-\frac{1}{2}}\sin(n-\tfrac{1}{2})\phi\ ,\ \rho>1 \qquad (19)$$

The upper half plane displacement is obtained from equations (15) , (18) and (19).

## IV. DISCUSSION OF THE RESULTS

The fields for which $\rho<1$ in the upper half plane refer to those of the semi-circular region of the left half $r\theta$-plane while those for $\rho>1$ correspond to those of the semicircle in the right half $r\theta$-plane. The line of symmetry , $\phi=\frac{\pi}{2}$ , $0\le\rho<\infty$ in the $\rho\phi$-plane, through $u(r,\theta)=0$ (equation (5)) , corresponds to $\theta=0,\ 0\le r\le$ a ($\rho>1$), $\theta=\pi$ , $0\le r\le$ a ($\rho<1$) and can be studied for fracture initiation.

Using equation(5) in the form $\rho^2(r,\theta)=\frac{a^2+2arcos\theta\ +r^2}{a^2-2arcos\theta\ +r^2}$ and noting that $a\le 2r+a$ implies $\frac{a-r}{a+r}\le 1$ we find that

$\rho(r,0)=\frac{a+r}{a-r}\ge 1$ , $\rho(r,\pi)=\frac{a-r}{a+r}\le 1$ , $\rho\left(r,\pm\frac{\pi}{2}\right)=1$ (20)

From equations (5) and (20)

$tan\phi(r,0)=\infty$ implies $\phi(r,0)=\frac{\pi}{2}$ , $\rho>1$ (21a)

$tan\phi(r,\pi)=\infty$ implies $\phi(r,\pi)=\frac{\pi}{2}$ , $\rho<1$ (21b)

We may use equations (20) and (21b) to investigate fracture along

$\theta = \pi\,, 0 < r \le a\,,\ \frac{a-r}{a+r} < 1$ these equations lead to

$$W_1(\rho,\tfrac{\pi}{2}) = -\frac{2}{\pi}\left(\sum_{n=1}^{\infty}\frac{\rho^{2n-1}}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\rho^{n-\frac{1}{2}}}{n-\frac{1}{2}}\right) = -\frac{1}{\pi}\left\{ln\left(\frac{1+\rho}{1-\rho}\right) - ln\left(\frac{1+\rho^{\frac{1}{2}}}{1-\rho^{\frac{1}{2}}}\right)\right\} \tag{22}$$

$$W_2(\rho,\tfrac{\pi}{2}) = -\frac{2}{\pi}\left(\sum_{n=0}^{\infty}\rho^n - \frac{1}{2}\sum_{n=1}^{\infty}\rho^{n-\frac{1}{2}}\right) = -\frac{2}{\pi}\left(\frac{1-\frac{1}{2}\rho^{\frac{1}{2}}(1+\rho)}{1-\rho}\right) \tag{23}$$

The series are summed with the aid of the relations

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}\ ,|z| < 1, \quad \sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1} = \frac{1}{2}ln\left(\frac{1+z}{1-z}\right),\quad |z| < 1$$

and /or the use of entries [8] 2.2481 and [9] 7.4(60,61 and 120).

Equations (15),(22) , and (23) yield

$$W\left(\rho,\tfrac{\pi}{2}\right) = -\frac{aQ}{\mu}\left[ln\left(\frac{1+\rho}{1-\rho}\right) + ln\left(\frac{1+\rho^{\frac{1}{2}}}{1-\rho^{\frac{1}{2}}}\right)\right] + w_0\left[\frac{1-\frac{1}{2}\rho^{\frac{1}{2}}(1+\rho)}{1-\rho^2}\right]\ ,\ \rho < 1 \tag{24}$$

Equations (20b) and (24) lead to

$$w(r,\pi) = \frac{2aQ}{\mu}ln\left(\frac{a+(a^2-r^2)^{\frac{1}{2}}}{a}\right) + \frac{w_0}{4ar}\left[(a+r)^2 - a(a^2-r^2)^{\frac{1}{2}}\right]\ ,\quad 0 < r \le a \tag{25}$$

Differentiation of equation (25) yields the nonzero polar stress $\sigma_{rz}(r,\pi)$ in the form

$$\sigma_{rz}(r,\pi) = \frac{1}{r(a^2-r^2)^{\frac{1}{2}}}\left\{2aQ\left[(a^2-r^2)^{\frac{1}{2}} - a\right] + \frac{\mu w_0}{4a}\left[\frac{a}{r}(a^2-r^2)^2(a^2-r^2)^{\frac{1}{2}} + (a+r)(a^2-r^2)^{\frac{1}{2}} + ar\right]\right\}\ ,$$

$$0 < r < a \tag{26}$$

The other nonzero polar stress along $\theta = \pi\ ,\ 0 < r < a,\ \frac{a-r}{a+r} < 1$ is deduced by use of equations (18) , (19) and chain rule as $\sigma_{r\theta}(r,\pi) = \frac{\mu}{r}\frac{\partial W}{\partial\phi}\left(\rho,\frac{\pi}{2}\right)\frac{\partial\phi}{\partial\theta}(r,\pi)\ ,\ \rho = \rho(r,\pi) < 1$ Differentiating and simplifying equations (18) and (19) and applying (15) yields $\frac{\partial W}{\partial\phi}\left(\rho,\frac{\pi}{2}\right) = \frac{\rho^{\frac{1}{2}}}{1+\rho}\left\{\frac{aQ}{r\mu} + \frac{w_0}{4a}\left(\frac{1-\rho}{1+\rho}\right)\right\}$ Consequently ,

$$\sigma_{r\theta}(r,\pi) = -\frac{\mu r}{(a^2-r^2)^{\frac{1}{2}}}\left(\frac{aQ}{r\mu} + \frac{w_0}{4a}\right)\ ,\quad 0 < r < a \tag{27}$$

Equations (20a) and (20b) are used to investigate fracture along the ray

$\theta = 0\ ,0 < r \le a\,,\ \frac{a+r}{a-r} > 1\,,\ \rho = \rho(r,0) > 1.$

For this case, we use equation (15) and

$$W_1(\rho,\tfrac{\pi}{2}) = -\frac{2}{\pi}\left(\sum_{n=1}^{\infty}\frac{\rho^{-(2n-1)}}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\rho^{-(n-\frac{1}{2})}}{n-\frac{1}{2}}\right) = -\frac{1}{\pi}\left[ln\left(\frac{1+\rho}{1-\rho}\right) - ln\left(\frac{1+\rho^{\frac{1}{2}}}{1-\rho^{\frac{1}{2}}}\right)\right]$$

$$W_2(\rho,\tfrac{\pi}{2}) = -\frac{2}{\pi}\left(\sum_{n=0}^{\infty}\rho^{-2n} + \frac{1}{2}\sum_{n=1}^{\infty}\rho^{-(n-\frac{1}{2})}\right) = -\frac{2}{\pi}\left[\frac{1+\frac{1}{2}\rho^{\frac{1}{2}}(1+\rho)}{\rho^2-1}\right]$$

to derive $W\left(\rho,\frac{\pi}{2}\right) = -\frac{aQ}{\mu}\left[ln\left(\frac{\rho+1}{\rho-1}\right) - ln\left(\frac{\rho^{\frac{1}{2}}+1}{\rho^{\frac{1}{2}}-1}\right)\right] - w_0\left[\frac{1+\frac{1}{2}\rho^{\frac{1}{2}}(1+\rho)}{\rho^2-1}\right],\ \rho > 1$

Hence , $w(r,0) = \frac{2aQ}{\mu}\left[ln\left(\frac{a+(a^2-r^2)^{\frac{1}{2}}}{a}\right) - \frac{1}{2}ln\left(\frac{a}{r}\right)\right] - \frac{w_0}{4ar}\left[(a-r)^2 + a(a^2-r^2)^{\frac{1}{2}}\right]\ ,\ 0 < r \le a$ (28)

And

$$\sigma_{rz}(r,0) =$$

$$\frac{1}{r^2(a^2-r^2)^{\frac{1}{2}}}\left\{\frac{aQ}{\mu}\left[\frac{(a^2-r^2)^{\frac{1}{2}}-a}{(a^2-r^2)^{\frac{1}{2}}}\right] + \frac{\mu w_0}{4a}\left[(a-r)^2(a^2-r^2)^{\frac{1}{2}} + a(a^2-r^2) + 2r(a-r)(a^2-r^2)^{\frac{1}{2}} + r^2a\right]\right\},$$

$$o < r < a \tag{29}$$

Similarly, $\sigma_{\theta z}(r,0) = \frac{\mu}{r}\left(\frac{2ar}{a^2-r^2}\right)\frac{\partial W}{\partial\phi}\left(\rho,\frac{\pi}{2}\right)\ ,\ \rho = \rho(r,0) > 1$

$$= \frac{a\mu}{(a^2-r^2)^{\frac{1}{2}}}\left[-\frac{a}{r}\frac{Q}{\mu} + \frac{rw_0}{8a^2}\right]\ ,\quad 0 < r < a \tag{30}$$

## V. CONCLUSION

The solution of the upper half plane problem given by equations (15), (18) and (19) satisfies equations (6) and (7). Equations (25) and (27) show that the displacement $w(r,\pi)$ and $w(r,0)$ satisfy the boundary condition given in equation (2).
Using

$\sigma_{rz}(a,\theta) = \mu \frac{\partial w}{\partial \phi}(\rho,\pi)\frac{\partial \phi}{\partial r}(a,\theta),\ \rho \geq 0\ , 0 < \theta < \pi$ , we get $\sigma_{rz}(a,\theta) = Q\ ,\ 0 < \theta < \pi$

From equations (26), (27), (29) and (30), it follows that as $r \to a^+, a - r \to 0$ and

$$\sigma_{rz}(r,\pi) = \left(\frac{a}{2}\right)^{\frac{1}{2}} \frac{\mu}{(a-r)^{\frac{1}{2}}}\left(\frac{w_0}{8a} - \frac{Q}{\mu}\right)\ \ as\ r \to a^+ \tag{31}$$

$$\sigma_{\theta z}(r,\pi) = -\left(\frac{a}{2}\right)^{\frac{1}{2}} \frac{\mu}{(a-r)^{\frac{1}{2}}}\left(\frac{Q}{\mu} + \frac{w_0}{4a}\right)\ \ as\ r \to a^+ \tag{32}$$

$$\sigma_{rz}(r,0) = \left(\frac{a}{2}\right)^{\frac{1}{2}} \frac{\mu}{(a-r)^{\frac{1}{2}}}\left(\frac{w_0}{8a} - \frac{Q}{\mu}\right)\ \ as\ r \to a^+ \tag{33}$$

$$\sigma_{\theta z}(r,0) = \left(\frac{a}{2}\right)^{\frac{1}{2}} \frac{\mu}{(a-r)^{\frac{1}{2}}}\left(\frac{w_0}{8a} - \frac{Q}{\mu}\right)\ \ as\ r \to a^+ \tag{34}$$

The form of the displacement $w(r,\pi)$ and $w(r,0)$ given by equations (25) and (28) indicate likelihood of singularity of the displacement towards the origin along the rays $\theta = 0$ and $\theta = \pi$.

Equations (31), (32), (33) and (34) indicate a tendency to locally tear [2] the material near the points $(a,\pi)$ and $(a,0)$ , meaning that cracking may commence there.

In figure 3, the relation linking the non-dimensional quantities $\frac{\sigma_{rz}(r,\pi)}{Q} = \left[2\left(1 - \frac{r}{a}\right)\right]^{-\frac{1}{2}}\left(1 + \frac{\zeta\eta}{4}\right)$
For $w_0 = \zeta a\ ,\ 0 \leq \zeta < 1$ and $Q = \eta\mu\ ,\ 0 < \eta < \infty$ is represented graphically for various values of $\zeta\eta$. These indicate that the tearing stress is linearly dependent on the applied stress

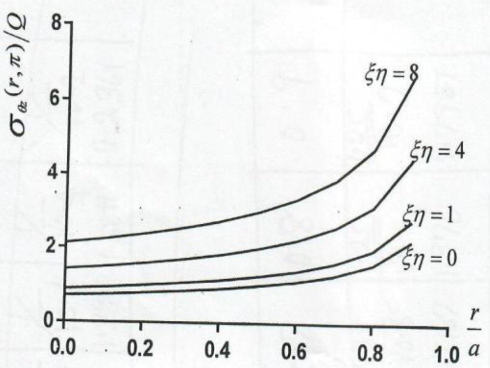


Figure 3 The Dependence of $\sigma_{\theta z}(r,\pi)/Q$ on $r/a$
For Various Values of $\xi\eta$

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