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Research Paper

Certain Transformations and Summations of Basic Hypergeometric Series

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Abstract. *In the present work we have established some new transformations and summations of basic hypergeometric series by making the use of WP-Bailey pairs. Using multiple q-integrals and a determinant evaluation, we establish a multivariable extension of Bailey's nonterminating is99 transformation. From this result, we deduce new multivariable terminating 1049 transformations, s& summations and other identities. We also use similar methods to derive new multivariable r+t summations. Some of our results are extended to the case of elliptic hypergeometric series.*
Keywords: *Bailey's lemma; Basic hypergeometric series; Transformation; Summation.*

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I. Introduction

For $|q| < 1, (a;q)_n = (1-a)(1-aq)\ldots\ldots\ldots(1-aq^n-1); n = 1,2\ldots.$

$$(a;q)_0 = 1; \qquad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aqn\,;q)_\infty}\,.$$

where a is real or complex.
A Basic Hypergeometric Series is defined as

$$r\phi_s(a_1,a_2,a_3\ldots.a_r\,;b_1,b_2,b_3\ldots\ldots b_s\,;q,z)$$
$$=\sum_{n=0}^{\infty}\frac{(a_1;q)_n\,(a_2;q)_n\ldots(a_r\,;q)_n}{(q;q)n(b_1;q)n\,(b_2;q)n\ldots.(b_s\,;q)_n}\quad[(-1)^n\,q^{\frac{n(n-1)}{2}}\,]^{1+s-r}\,z^n\,.$$

For $0 < |q| < 1$, the series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b1b2\ldots b_s|/|a_1a_2\ldots a_r|$.
In 1944, Bailey [1] introduced a very useful and simple identity known as Bailey's lemma. The Bailey's lemma states that, if

$$\beta n = \sum_{r=0}^{n}\alpha_r u_{n-r} v_{n+r}$$

$$\gamma_n = \sum_{r=n}^{\infty}\delta_r u_{r-n} v_{n+r}\ ,$$

then under the suitable convergence conditions and if change in the order of summations is allowed

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{n=0}^{\infty}\beta_n\delta_n\ ,$$

where α_r, δ_r, u_r and υ_r are functions of r such that β_n and γ_n exist. The proof of the lemma is trivial.
Taking $u_r = \frac{1}{(q;q)_r}$ *and* $v_r = \frac{1}{(aq;q)_r}$ *in* (1.1), *we have*

$$\beta n = \sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}\quad .$$

The pair of sequence (α_n,β_n) that satisfies (1.4) is called a Bailey pair relative to the parameter a.

The Bailey lemma has been a simple and effective tool in proving Rogers-Ramanujan type of identities and also a verity of transformations of basic hypergeometric series [2]. Slater [3, 4] used Bailey's lemma and gave the long list of 130 identities of Roger-Ramanujan type. After Slater the Bailey lemma have been extensively used to prove Rogers-Ramanujan type of identities and its generalizations [5-8]. Very recently, Warnaar [9] has written a very elegant survey of Bailey lemma. Andrews et al [10-13] exploited very effective the mechanism of Bailey's transform in the form of Bailey pair and Bailey chain. In particular, WP-Bailey pair (α_n,β_n) [14] satisfying

$$\beta_n = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)n-r(aq;q)n+r} \ \alpha r .$$

For k = 0 in (1.5), we get the standard Bailey pair (1.4). The relation (1.5) follows by setting $u_r = \frac{(k/a;q)_r}{(q;q)_r}$ $and\ v_r = \frac{(k;q)_r}{(aq;q)_r}$ in (1.1). The same substitutions in (1.2), gives

$$\gamma_n = \frac{(k;q)_{2n}}{(aq;q)_{2n}} \sum_{r=0}^{\infty} \frac{(k/a;q)_r(kq2n\,;q)_r}{(q;q)_r(aq2n+1\,;q)_r} \ \delta_{r+n}.$$

In the present paper, we have established a number of transformations and summations of basic hypergeometric series by making use of (1.5) and (1.6). Some interesting special cases have also been deduced.

We define a WP-Bailey Unit Bailey pair as

$$\alpha_n = \frac{(a, q\sqrt{a}, -q\sqrt{a}, a/k;q)_n}{(q,\sqrt{a},-\sqrt{a},kq;q)_n} \ (k/a)_n \ ,$$

$$\beta_n = \begin{matrix} 1, n = 0, \\ 0, n > 0. \end{matrix}$$

The trivial WP-Bailey pair is defined as

$$\beta_n = \frac{(k,k/a;q)_n}{(q,aq;q)_n} \ ,$$

$$\alpha_n = \begin{matrix} 1, n = 0, \\ 0, n > 0. \end{matrix}$$

A WP-Bailey pair due to Singh [15] is

$$\alpha_n = \frac{(a, q\sqrt{a}, -q\sqrt{a}, y, z, a^2 q/kyz;q)n}{(q,\sqrt{a},-\sqrt{a},aq/y,aq/z,kyz/a;q)n} \ (k/a)^n \ ,$$

$$\beta_n = \frac{(ky/a,kz/a,k,aq/yz;q)_n}{(q,aq/y,aq/z,kyz/a;q)_n} .$$

In our analysis we shall also require the following known results,

$$4\phi_3(a,-q\sqrt{a},b,c;-\sqrt{a},aq/b,aq/c;q,q\sqrt{a}/bc) = \frac{(aq,q\sqrt{a}/b,q\sqrt{a}/c,aq/bc;q)_\infty}{(aq/b,aq/c,q\sqrt{a},q\sqrt{a}/bc;q)_\infty}$$

$$3\phi_2(a,\lambda q,b;\lambda,q\lambda^2\ /b;q,\lambda^2\ /ab^2\,) = \frac{1-\lambda\ +\lambda/b(1-\lambda/a)(\lambda^2\,/b^{\,2}\,,q\lambda^2\,/ab;q)_\infty}{(1-\lambda)(1+\lambda/b)\ (q\lambda^2\,/b,\lambda^2\,/ab^2\,;q)_\infty} \ ,$$

$|\lambda^2\,/ab^2\,| < 1.$

$$2\phi1(a,b;aq/b;q,-q/b) = \frac{(-q;q)\infty(aq,aq^2/b^{\,2}\,;q^2\,)_\infty}{(-q/b,aq/b;q)_\infty} \ .$$

$$4\phi3(a,q\sqrt{a},-q\sqrt{a},b;\ \sqrt{a},-\sqrt{a},aq/b;q,1/b^{\,2}\,q) = \frac{(a/b^2\ ,1/bq;q)_\infty}{(aq/b,1/b^2\,q;q)_\infty} \ .$$

$$8\phi7(a,q\sqrt{a},-q\sqrt{a},\sqrt{(a/b)},-\sqrt{(a/b)},\sqrt{(aq/b)},-\sqrt{(aq/b)},b;\sqrt{a},-\sqrt{a},q\sqrt{(ab)},-q\sqrt{(ab)},\sqrt{(abq)},-\sqrt{(abq)},aq/b;q;bq) = \frac{(aq,b^2\,q;q)\infty}{(bq,abq;q)\infty} \ ,$$

$|bq| < 1.$

II. Result and Discussion

If (α_n,β_n) is a WP-Bailey pair, then under suitable convergence conditions, the following relations are true

$$\sum_{n=0}^{\infty} \frac{(-q\sqrt{k},c;q)_n}{(-\sqrt{k},kq/c;q)_n} \left(\frac{aq}{c\sqrt{k}}\right)^n \beta_n =$$

$$\frac{(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)\infty}{(aq/c\sqrt{k},aq,kq/c,q\sqrt{k};q)\infty} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{(kq;q)_{2n}} \frac{(-q\sqrt{k},q\sqrt{k},c;q)_n}{(-\sqrt{k},aq/\sqrt{k},aq/c;q)_n} \left(\frac{aq}{c\sqrt{k}}\right)^n \alpha_n.$$

$$\sum_{n=0}^{\infty}\frac{(q\,n+1\sqrt{ak};q)n}{(q\,n\sqrt{ak};q)n}\left(\frac{a^2}{k^2}\right)^n\beta_n=$$
$$\frac{(a/k,a^2q/k;q)_\infty}{(a\,2/k^2,aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(k,kqn\,,q\,n+1\sqrt{ak};q)_n}{(q\,n\sqrt{ak},a\,2q/k,a\,2q\,n+1/k;q)_n}\;\frac{(1-\sqrt{akq}^{\,2n}+\sqrt{a}/(\sqrt{k}-a^{3/2}q^{2n}))}{(1-q^{2n}\sqrt{(ak)})(1+\sqrt{a}/\sqrt{k})}\left(\frac{a^2}{k^2}\right)^n\alpha_n.$$
$$\sum_{n=0}^{\infty}\frac{(q\sqrt{k},-q\sqrt{k};q)_n}{(\sqrt{k},-\sqrt{k};q)_n}\left(\frac{a^2}{k^2q}\right)^n\beta_n=$$
$$\frac{(a^2/k,a/kq;q)\infty}{(aq,a^2/k^2q;q)\infty}\sum_{n=0}^{\infty}\frac{(k,kq^n,q\sqrt{k},-q\sqrt{k};q)_n}{(a^2/k,a^2q\,n/k,\sqrt{k},-\sqrt{k};q)_n}\left(\frac{a^2}{qk^2}\right)^n\alpha_n.$$
$$\sum_{n=0}^{\infty}\frac{(q\sqrt{k},-q\sqrt{k},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq};q)_n}{(\sqrt{k},-\sqrt{k},-kq\sqrt{(1/a)},kq\sqrt{(1/a)},k\sqrt{(q/a)},-k\sqrt{(q/a)};q)_n}\left(\frac{kq}{a}\right)^n\beta_n=$$
$$\frac{(kq,\;k^2q/a^2;q)_\infty}{(kq/a,k^2q/a;q)_\infty}\sum_{n=0}^{\infty}\frac{\left(k,kq^n,q\sqrt{k},-q\sqrt{k},\sqrt{a},-\sqrt{a},\sqrt{(aq)},-\sqrt{(aq)};q\right)_n}{\left(\sqrt{k},-\sqrt{k},kq\sqrt{(1/a)},-kq\sqrt{(1/a)},k\sqrt{(q/a)},-k\sqrt{(q/a)};q\right)_n}$$
$$\frac{(k^2q/a,k^2q^{(n+1)}/a;q)_n}{(aq,aq^{n+1},kq,kq^{n+1};q)_n}\left(\frac{kq}{a}\right)^n\alpha_n$$
$$\sum_{n=0}^{\infty}\left(\frac{-aq}{k}\right)^n\beta_n=\frac{(kq,a^2q^2/k;q^2)_\infty(-q;q)_\infty}{(-aq/k,aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(k,kq^n;q)_n}{(kq,a^2q^2/k;q^2)_{2n}}\left(\frac{-aq}{k}\right)^n\alpha_n.$$

Proof 2.1. Substituting $a=kq^{2n}, b=k/a\ and\ c=cq^n$ in (1.10), we have

$$4\phi3(kq^{2n},-q^{n+1}\sqrt{k},k/a,cq^n;-q^n\sqrt{k},aq^{2n+1},kq^{n+1}/c;q,aq/c\sqrt{k})$$
$$=\frac{(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)_\infty(aq;q)_{2n}(kq/c,q\sqrt{k};q)_n}{(aq/c\sqrt{k},aq,kq/c,q\sqrt{k};q)_\infty(kq;q)_{2n}(aq/\sqrt{k},aq/c;q)_n}.$$

Putting $\delta_r=\frac{(c,-q\sqrt{k};q)r}{(-\sqrt{k},kq/c;q)r}\left(\frac{aq}{c\sqrt{k}}\right)^r$ in (1.6) and making the use of (2.6), we get

$$\gamma_n=\frac{(k;q)_{2n}(q\sqrt{k},-q\sqrt{k},c;q)_n(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)_\infty}{(kq;q)_{2n}(-\sqrt{k},aq/\sqrt{k},aq/c;q)_n(aq,aq/c\sqrt{k},kq/c,q\sqrt{k};q)_\infty}\left(\frac{aq}{c\sqrt{k}}\right)^n.$$

Substituting $\delta_n\ and\ \gamma_n$ as above in (1.3), we get (2.1).

Proof 2.2 Setting $a=k/a, b=kq^{2n}\ and\ \lambda=q^{2n}\sqrt{ak}$ in (1.11), we get

$$3\phi2(k/a,q^{2n+1}\sqrt{ak},kq^{2n};q^{2n}\sqrt{ak},aq^{2n+1};q,a^2/k^2)=\frac{(a/k,a^2q/k;q)_\infty(aq;q)_{2n}}{(a^2/k^2,aq;q)_\infty(a^2q/k;q)_{2n}}$$
$$\times\frac{(1-q^{2n}\sqrt{ak}+\sqrt{a}/(\sqrt{k}-a^{3/2}q^{2n}))}{(1-q^{2n}\sqrt{ak})(1+\sqrt{a}/\sqrt{k})},$$

$a^2/k^2|<1$.

Choosing $\delta r=\frac{(q^{n+1}\sqrt{(ak)};q)_r}{(q^n\sqrt{(ak)};q)_r}\left(\frac{a^2}{k^2}\right)^r$ in (1.6) and substituting in (2.7), we have

$$\gamma_n=\frac{(1-q^{2n}\sqrt{ak}+\sqrt{a}/(\sqrt{k}-a^{3/2}q^{2n}))}{(1-q^{2n}\sqrt{(ak)})(1+\sqrt{a}/\sqrt{k})}\frac{(a/k,a^2q/k;q)_\infty(k,kq^n,q^{n+1}\sqrt{(ak)};q)_n}{(aq,a^2/k^2;q)_\infty(q^n\sqrt{(ak)},a^2q/k,a^2q^{n+1}/k;q)_n}\left(\frac{a^2}{k^2}\right)^n.$$

using $\delta_n\ and\ \gamma_n$ in (1.3), we obtain (2.2).

Applications

By using (1.7) in (2.1) and taking $n\to\infty$, we get

$$8\phi7(k,q\sqrt{k},-q\sqrt{k},c,a,q\sqrt{a},-q\sqrt{a},a/k;\;kq,-\sqrt{k},aq/\sqrt{k}\sqrt{a},-\sqrt{a},kq,aq/c;q,q\sqrt{k}/c)$$
$$=\frac{(q\sqrt{k},kq/c,aq,aq/c\sqrt{k};q)_\infty}{(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)_\infty}.$$

Again by making the use of (1.9) in (2.1) and taking $n\to\infty$, we obtain

$$10\phi9(k,-q\sqrt{k},c,q\sqrt{k},a,q\sqrt{a},-q\sqrt{a},y,z,a^2q/kyz;\;kq,-\sqrt{k},aq/\sqrt{k},aq/c,\sqrt{a},-\sqrt{a},aq/y,aq/z,kyz/a;q,q\sqrt{k}/c)$$
$$=\frac{(aq/c\sqrt{k},aq,kq/c,q\sqrt{k};q)\infty}{(kq,aq/\sqrt{k},q\sqrt{k}/c,aq/c;q)\infty}\;6\phi_5(-q\sqrt{k},c,ky/a,kz/a,k,aq/yz;-\sqrt{k},kq/c,aq/y,aq/z,kyz/a;q,aq/c\sqrt{k}).$$

On using (1.8) in (2.2) and taking $n\to\infty$, we get

$$2\phi1(k,k/a;aq;q,a^2/k^2) = \frac{(a/k,a^2q/k;q)_\infty}{(aq,a^2/k^2;q)_\infty}\frac{(1-\sqrt{ak}+\sqrt{a}/(\sqrt{k}-a^{3/2})}{(1-\sqrt{ak})(1+\sqrt{a}/\sqrt{k})}.$$

By making the use of (1.7) in (2.3) and then taking $n \to \infty$, we obtain

$$7\phi6(a,q\sqrt{a},-q\sqrt{a},a/k,k,q\sqrt{k},-q\sqrt{k};\sqrt{a},-\sqrt{a},kq,a^2/k,\sqrt{k},-\sqrt{k};q,a/kq) = \frac{(aq,a^2/k^2q;q)_\infty}{(a^2/k,a/kq;q)_\infty}.$$

In (2.3) using (1.9) and then taking $\to \infty$, we get the following transformation

$$9\phi8(k,q\sqrt{k},-q\sqrt{k},a,q\sqrt{a},-q\sqrt{a},y,z,a2q/kyz;a^2/k,\sqrt{k},-\sqrt{k},\sqrt{a},-\sqrt{a},aq/y,aq/z,kyz/a;q,a/kq)$$
$$= \frac{(aq,a^2/k^2q;q)_\infty}{(a^2/k,a/kq;q)_\infty}\ 6\phi5(q\sqrt{k},-q\sqrt{k},ky/a,kz/a,k,aq/yz;\sqrt{k},-\sqrt{k},aq/y,aq/z,kyz/a;q,a^2k^2q).$$

Again in (2.4) making the use of (1.7) and taking $n \to \infty$, we obtain the following summation

$$12\phi11(k,q\sqrt{k},-q\sqrt{k},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},k^2q/a,a,q\sqrt{a},-q\sqrt{a},a/k;\sqrt{k},-\sqrt{k}\,kq\sqrt{1/a},-kq\sqrt{1/a},k\sqrt{q/a},-k\sqrt{q/a},\sqrt{a},-\sqrt{a},kq,aq,kq;q;k^2q/a^2) = \frac{(kq/a,k^2q/a;q)_\infty}{(kq,k^2q/a^2;q)_\infty}.$$

Now use (1.9) in (2.4) and taking $n \to \infty$, we have

$$14\phi13(k,q\sqrt{k},-q\sqrt{k},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},k^2q/a,a,q\sqrt{a},-q\sqrt{a},y,z,a^2q/kyz;\sqrt{k},-\sqrt{k},kq\sqrt{1/a},-kq\sqrt{1/a},k\sqrt{q/a},-k\sqrt{q/a},aq,kq,\sqrt{a},-\sqrt{a},aq/y,aq/z,kyz/a;q;k^2q/a^2)$$
$$= \frac{(kq/a,k^2q/a;q)_\infty}{(kq,k^2q/a^2;q)_\infty}\ 10\phi9(q\sqrt{k},-q\sqrt{k},\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},ky/a,kz/a,k,aq/yz;\sqrt{k},-\sqrt{k},kq\sqrt{1/a},-kq\sqrt{1/a},k\sqrt{q/a},-k\sqrt{q/a},aq/y,aq/z,kyz/a;q;kq/a).$$

By using (1.7) in (2.5), we have

$$\sum_{n=0}^{\infty}\frac{(k,kq^n,a,q\sqrt{a},-q\sqrt{a},a/k;q)_n}{(q,\sqrt{a},-\sqrt{a},kq;q)_n(kq,a^2q^2/k;q^2)_{2n}}(-q)^n = \frac{(-aq/k,aq;q)_\infty}{(-q;q)_\infty(kq,a^2q^2/k;q^2)_\infty}.$$

and again in (2.5) using (1.9), we get

$$4\phi3(ky/a,kz/a,k,aq/yz;aq/y,aq/z,kyz/a;q,-aq/k)$$
$$= \frac{(-q;q)\infty(kq,a2q2/k;q2)\infty}{(-aq/k,aq;q)\infty}\sum_{n=0}^{\infty}\frac{(k,kq^n,a,q\sqrt{a},-q\sqrt{a},y,z,a^2q/kyz;q)_n}{(kq,a^2q^2/k;q^2)_{2n}(q,\sqrt{a},-\sqrt{a},aq/y,aq/z,kyz/a;q)_n}(-q)^n.$$

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