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Research Paper

Ideal Elements in Ordered Semigroups

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Abstract: *The aim of this paper is to study the structures of some or- dered semigroups with ideal elements. The ideal elements play a vital role in studying the structure of ordered semigroups. We introduce the notion of ideal elements, interior ideal elements, quasi ideal elements, bi-ideal el- ements, quasi interior ideal elements and weak interior ideal elements of ordered semigroups. We study the properties and relations of ideal elements and characterize the ordered semigroups, regular ordered semigroups and simple ordered semigroups using ideal elements.*

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I. Introduction

Semigroup, as the basic algebraic structure was used in the areas of theoret- ical computer science as well as in the solutions of graph theory. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, which was generalized by Noether for associative rings. The notion of a one sided ideal of any algebraic structure is a generalization of an ideal. The no- tion of the bi-ideal in semigroups is a special case of (m,n) ideals. In 1952, the concept of bi-ideals for semigroup was introduced by Lazos and Szasz[3, 4]. The quasi ideals are generalization of left ideals and right ideals whereas the bi-ideals are generalization of quasi ideals. Steinfeld[10] rst introduced the notion of quasi ideals for semigroups and then for rings. Rao[6, 7, 8, 9] introduced bi-quasi-interior ideals bi-quasi-ideals and bi-interior ideals in semigroups.

II. Preliminaries

In this section, we will recall some of the definitions, which are necessary for this paper.

Definition 2.1. A semigroup M is called an ordered semigroup if it admits a compatible relation $\leq$. i.e. $\leq$ is a partial ordering on M satisfies the following conditions. If $a \leq b$ and $c \leq d$ then

(i) $ac \leq bd$

(ii) $ca \leq db$, for all $a, b, c, d \in M$

Definition 2.2. A non-empty subset A of an ordered semigroup M is called:

(i) a subsemigroup of M, if $AA \subseteq A$,

(ii) a quasi ideal of M, if A is a subsemigroup of M and $AM \cap MA \subseteq A$,

(iii) a bi-ideal of M, if A is a subsemigroup of M and $AMA \subseteq A$,

(iv) an interior ideal of M, if A is a subsemigroup of M and $MAM \subseteq A$,

(v) a left (right) ideal of M, if A is a subsemigroup of M and $MA \subseteq A(AM \subseteq A)$,

(vi) an ideal, if $AM \subseteq A$ and $MA \subseteq A$,

(vii) a bi-quasi-interior ideal of M, if A is a subsemigroup of M and $AMAMA \subseteq A$,

(viii) a bi-interior ideal of M, if A is a subsemigroup of M and $MAM \cap AMA \subseteq A$,

(ix) a left (right) bi-quasi ideal of M, if A is a subsemigroup of M and $MA \cap AMA \subseteq A$ $(AM \cap AMA \subseteq A)$,

(x) a bi-quasi ideal of M, if A is a subsemigroup of M, $MA \cap AMA \subseteq A$ *and* $AM \cap AMA \subseteq A$,

(xi) a left (right) quasi-interior ideal of M, if A is a subsemigroup of M and $MAMA \subseteq A$ $(AMAM \subseteq A)$.

(xii) a quasi-interior ideal of M, if A is a subsemigroup of M and $MAMA \subseteq A$ *and* $AMAM \subseteq A$.

III. Ideal elements in ordered semigroups

In this section, we introduce the notion of ideal elements, interior ideal elements, bi-ideal elements, quasi interior ideal elements and weak interior ideal elements of ordered semigroups. We study the properties and relations of ideal elements.

Definition 3.1. An element a of an ordered semigroup M is called a subsemigroup element if $aa \leq a$.

Definition 3.2. An element a of an ordered semigroup M is called a left(right) ideal element of M, if $xa \leq a(ax \leq a)$, for all $x \in M$.

Definition 3.3. An element of an ordered semigroup M is called an ideal element of M, if it is both a left ideal element and a right ideal element of M.

Definition 3.4. Let M be an ordered semigroup. An element a of M is said to be bi-ideal element of M if $aa \leq a$, $axa \leq a$, for all $x \in M$.

Definition 3.5. An element a of an ordered semigroup M is called a quasi ideal element of M, if $aa \leq a$, there exist elements $x, y \in M$, such that $xa = ay \leq a$.

Definition 3.6. Let M be an ordered semigroup. An element a of M is said to be quasi interior ideal element of M if $aa \leq a$, $axay \leq a$ and $xaya \leq a$, for all $x, y \in M$.

Definition 3.7. A bi-ideal element b is said to be minimal if for every bi-ideal element a of ordered semigroup M, $a \leq b \Rightarrow a = b$.

Definition 3.8. Let M be an ordered semigroup. An element a of M is said to be interior ideal element if $aa \leq a$, $xay \leq a$, for all $x, y \in M$.

Definition 3.9. An element a of an ordered semigroup M is called a left(right) weak interior ideal of M if $aa \leq a$, $xaa \leq a(aax \leq a)$, for all $x \in M$.

Definition 3.10. An element of an ordered semigroup M is called a weak interior ideal element if $aa \leq a$, $aax \leq a$ and $xaa \leq a$, for all $x \in M$.

Definition 3.11. An element a of an ordered semigroup M is called a regular if there exist elements $x \in M$, such that $a \leq axa$.

In the following theorem, we mention some important properties of ideal elements and we omit the proofs since proofs are straight forward.

Theorem 3.12. *Let M be an ordered semigroup. Then the following hold.*

(1) *If a is an ideal element of an ordered semigroup M, then a is an interior ideal element of M..*

(2) *If a is a left ideal element of an ordered semigroup M, then a is a quasi ideal element of M.*

(3) *If a is an ideal element of an ordered semigroup M, then a is a bi-ideal element of M.*

(4) *If a is a maximal element of M then a is an ideal element of M.*

(5) *If a is a minimal element of an ordered semigroup M, then a is a not an ideal element of M.*

(6) *Every left ideal element of an ordered semigroup M is a left quasi interior element of M.*

(7) *If a is an ideal element of an ordered semigroup$M$, then a is a left weak interior ideal element of M.*

(8) *If a is a bi-ideal element of an ordered semigroup M,then a is an interior ideal element of M.*

Theorem 3.13. *If a is an ideal element of an ordered semigroup M, then a is an interior ideal element of M..*

Proof. Suppose a is an ideal element of the semigroup M.
Then $ax \leq a$ and $ya \leq a$, for all $x, y \in M$.
That implies $xay \leq xa \leq a$.
Hence a is an interior ideal element of M. □

Theorem 3.14. *If a is an interior ideal element and idempotent element of ordered semigroup M, then a is an ideal element of M.*

Proof. Suppose a is an interior ideal element and a is an idempotent element of the ordered semigroup M.
Then $xay \leq a$, for all $x, y \in M$.
That implies $xaa \leq a$ and hence $xa \leq a$, for all $x \in M$.
Similarly we can prove that $ax \leq a$.
Hence a is an ideal element of M. □

Theorem 3.15. *Every interior ideal element of an ordered semigroup M is a quasi interior ideal element.*

Proof. Suppose a is an interior ideal element of M.
Then $xay \leq a$, for $x, y \in M$.
That implies $axay \leq a$ and $xaya \leq a$.
Hence every interior ideal element is a quasi interior ideal element. □

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Theorem 3.16. *If a is a quasi interior ideal element of a regular ordered semigroup M then a is an ideal element of M.*

Proof. Suppose a is a quasi interior ideal element of M.
Then $axay \leq a$, $xaya \leq a$, for all $x, y \in M$ and $a \leq aba$, for some $b \in M$.
Suppose $x \in M$.
Then $ax \leq abax \leq a$ and $xa \leq xaba \leq a$.
Hence a is an ideal element of M. □

Theorem 3.17. *Let M be a ordered regular semigroup and a be an element of M. Then a is an interior ideal element if and only if a is an ideal element of M.*

Proof. Assume that a be an interior ideal element of M.
Then $xay \leq a$, for all $x, y \in M$.
We have that $a \leq axa$, for some $x \in M$.
Then $ax \leq axax \leq a$, by the definition of interior ideal element of M.
Similarly we can prove that $xa \leq a$.
Hence a is an ideal element of M. Conversely, assume that a is an ideal element of M. Suppose $y, z \in M$.
Then $yaz \leq az \leq a$. Hence a is an interior ideal element of the ordered semigroup M. □

Theorem 3.18. *Let M be an ordered regular semigroup. If a is a bi-ideal element of M and a commutes with every element of M then a is an ideal element.*

Proof. Let M be an ordered semigroup, a be a bi-ideal element of M and a commutes with every element of M.
Then $a \leq axa$, for some $x \in M$.
That implies $a \leq axa \leq a$.
Therefore $axa = a$.
Suppose $y \in M$.
Then $ay = axay = axya \leq a$, for all $y \in M$.
Hence a is an ideal element of M. □

Theorem 3.19. *If a and b are minimal left ideal elements of an ordered semigroup M, then ab is a minimal left ideal element of ordered semigroup M.*

Proof. Suppose a and b are minimal left ideals of the ordered semigroup M.
Then $xa \leq a$ and $xb \leq b$, for all $x \in M$.
That implies $xab \leq ab$, for all $x \in M$.
Therefore ab is the left ideal element of M.
Suppose c is any left ideal element of M and $c \leq ab$.
Then $c \leq ab \leq a$

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Therefore $ab = c$.
Hence ab is the minimal left ideal of the ordered semigroup M. □

Theorem 3.20. *Let M be an ordered semigroup. If a is a weak interior ideal element and idempotent element of M then a is an interior ideal element of M.*

Proof. Let a be an interior ideal and idempotent element of M.
Then $aa = a$.
Suppose $x, y \in M$
Now $xay = xaaaay \leq aa = a$. □

Theorem 3.21. *Let M be a simple ordered semigroup. Then every element of M is an ideal element of M.*

Proof. Let $I = \{a \mid ax \leq a \text{ and } xa \leq a, \text{ for all } x \in M\}$ and $a, b \in I$.
Now $abx = a(bx) \leq ab$ and $x(ab) = (xa)b \leq ab$.
Therefore $ab \in I$. Hence I is a subsemigroup of M.
Suppose $a \in M,\ b \in I$.
Then $(ab)x = a(bx) \leq ab$ and $x(ab) = (xa)b \leq ba = ab$.
Suppose that $x \in M,\ a \in I,\ \in$ and $x \leq a$,
Then $yx \leq ya \leq a$ and $xy \leq ay \leq a$. Hence $x \in I$.
Therefore I is an ideal of the ordered semigroup. Hence $I = M$. □

Corollary 3.22. *Let M be a left (right) simple ordered semigroup. Then every element of M is a left (right) ideal element of M.*

Theorem 3.23. *If a is an ideal element of an ordered semigroup M then a is a left weak interior ideal element of M.*

Proof. Suppose a is an ideal element of M.
Then $ax \leq a$ and $xa \leq a$, for all $x \in M$.
That implies $xaa \leq aa \leq a$ and $ax \leq a$.
Therefore $aax \leq aa \leq a$.
Hence ideal element of M is a weak interior ideal element of M. □

Corollary 3.24. *Let a be an interior ideal element of an ordered semigroup M. Then a is a weak interior ideal element of M.*

IV. Conclusion:

In this paper, we studied the structures of some ordered semigroups with the generalization of ideal elements. The ideal elements play an important and necessary role in studying the structure of ordered semigroups. Interior ideal elements, quasi ideal elements, bi-ideal elements and weak interior ideal elements of ordered semigroups were studied. We characterized the ordered semigroups, regular ordered semigroups and simple ordered semigroups using ideal elements.

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