Quest Journals
Journal of Research in Applied Mathematics
Volume 7 ~ Issue 11 (2021) pp: 44-50
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

Power Series Solutions of Second Order Ordinary Differential Equation Using Frobenius Method

I.M. ESUABANA*[1], E.O. EKPENYONG[2], E.J. OKON[3]
[1,2]Department of Mathematics, University of Calabar,P.M.B. 1115, Calabar,Cross River State,Nigeria
[3]Department of Statistics, University of Benin, P.M.B. 1154, Ugbowo, Benin City, Edo State, Nigeria

Abstract: *Power series method is an essential method for solving ordinary differential equation (ODE) with variable coefficient. In this paper, we use Frobenius method to obtain power series solutions of second order ODE with coefficient at a singular point $t = 0$ and determined the form of its second linearly independent solution. Some selected problems were solved to distinguish their various roots and concluded that Frobenius method is an efficient method for obtaining power series solution of a second order differential equation.*
Keywords: *Frobenius Method, Ordinary Differential Equation, Ordinary point, Singular point*

Received 08 November, 2021; Revised: 22 November, 2021; Accepted 24 November, 2021

I. INTRODUCTION

Linear ordinary differential equations (ODEs) with constant coefficients can be solved with functions known from calculus. If a linear ODE has variable coefficients like Legendre's and Bessel's ODEs, it must be solved using suitable methods. A formal power series can be a series that is a polynomial with an infinite number of terms. [1], [2], [3], [11] have studied Frobenius method for solving second order ODEs.[11] derived Frobenius series solution of Fuchs second-Order ordinary differential equations via complex integration.

Several approaches or methods could be applied in solving problems when dealing with power series of second order differential equation. It is sufficient to state here that this paper is only centered on Frobenius method as a power series solution to second order differential equation.

II. PRELIMINARIES

A power series is an infinite series of the form;

$$\sum_{n=0}^{\infty} a_n (t - t_0)^n = a_0 + a_1 (t - t_0) + a_2 (t - t_0)^2 + a_3 (t - t_0)^3 + \dots \qquad (1)$$

where t is a variable, $a_0, a_1, a_2, a_3, \dots$ are constants, which are coefficients of the series and t_0 is the center of the series.

If $t_0 = 0$, we obtain a power series in power of t as;

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots \qquad (2)$$

If a differential equation is of the form

$$t^2 y'' + t b(t) y' + c(t) y = 0 \qquad (3)$$

where $b(t)$ and $c(t)$ are analytic function at $t = 0$.

We write (3) in standard form as

$$y'' + \frac{b(t)}{t} y' + \frac{c(t)}{t^2} y = 0 \qquad (4)$$

If $\frac{b(t)}{t}$ and $\frac{c(t)}{t^2}$ are analytic $t = 0$ then the solution of the equation will be analytic at $t = 0$, which can be represented in the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \quad (5)$$

Hence, if either $\frac{b(t)}{t}$ or $\frac{c(t)}{t^2}$ are not analytic $t = 0$, we have a singular point at $t = 0$. Then solution cannot be represented in the series, so we must go to power series expanded method which is called Frobenius method.

Frobenius Method:
If $t = 0$ is a singular point of the ordinary differential (4), then it has at least one solution of the form

$$y(t) = t^k \sum_{n=0}^{\infty} a_n t^n = t^k (a_0 + a_1 t + a_2 t^2 + \ldots),\ a_0 \neq 0 \quad (6)$$

in which k maybe any real or complex number[5][7] [12].
The Frobenius method is effective in solving the Bessel's Equation.

Ordinary Point:
The point $t = t_0$ is an ordinary point of the differential equation of the form:

$$P_0(t) y'' + P_1(t) y' + P_2(t) y = Q(t) \quad (7)$$

$P_0(t) \neq 0$, then if $t = t_0$ is an ordinary point, a series solution is set to be,

$$y = \sum_{n=0}^{\infty} a_n (t - t_0)^n \quad (8)$$

where we obtain the solution of the form

$$y(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \ldots + a_n(t - t_0)^n$$

Where the coefficients of a's are to be determined.
Consider a second order differential equation of (8) above, if $t = t_0$ is an ordinary point of the differential equation, where

$$y' = \sum_{n=0}^{\infty} n a_n (t - t_0)^{n-1} \quad (9)$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n (t - t_0)^{n-2} \quad (10)$$

Substituting y, y', y'' into the differential equation (7) and then coefficients of $(t - t_0)^n$, the series equation for a_n are obtained from which all the coefficients can be determined in any of the two coefficients. According to [4],[5],[6] these two coefficients are those two arbitrary constants expected in the general solution of the second order ordinary differential equation and can be determined if any two conditions are given.
If t_0 is not a regular point then it is called a singular point[8] [5].

Regular Point:
The point $t = a$ is a singular point of the D.E;

$$P_0(t) y'' + P_1(t) y' + P_2(t) y = 0 \text{ iff } P_0(t) = 0$$

Suppose $P_0(a) = (t - a) R_1(t), R_1(a) \neq 0$

$$y'' + \frac{p_1(t) y'}{(t - a) R_1(t)} + \frac{p_2(t) y}{(t - a) R_2(t)} = 0$$

The point $t = a$ is a regular singular point of the D.E [12].
If $\lim_{t \to a} \left[\frac{P_1(t)}{R_1}\right] = \alpha$; $\lim_{t \to a} \left[\frac{P_2(t)}{R_2}\right] = \beta$

where α and β are finite.

Indicial equation:

From the equation (3) now $b(t)$ and $c(t)$ are expanding in power series,

$b(t) = b_0 + b_1 t + b_2 t^2 + \cdots$ and

$c(t) = c_0 + c_1 t + c_2 t^2 + \cdots$

Differentiating equation (6), we have

$$y'(t) = \sum_{n=0}^{\infty} (n+k) a_n t^{n+k-1} = t^{k-1}(k a_0 + (k+1) a_1 t)$$

$$y''(t) = \sum_{n=0}^{\infty} (n+k)(n+k-1) a_n t^{n+k-2} = t^{k-2}(k(k-1) a_0 + (k+1) k a_1 t + \ldots)$$

By putting the value of $y(t)$, $y'(t)$ and $y''(t)$ equation (3), we obtain

$$t^k(k(k-1)a_0 \ldots) + (b_0 + b_1 t + \cdots) t^k (k a_0 + \cdots) + (c_0 + c_1 t + \cdots) t^k (a_0 + a_1 t + \cdots) = 0$$

Equating the sum of the coefficients of each power of t^k, t^{k+1}, t^{k+2}, …. to zero.

This gives a structure of equation with the unknown coefficients a_n [10].

The ODE in equation (3) also has a second solution such that they are linearly independent [5] [9].

Its form will be specified by equation (6) in the following cases.

The real case: If the roots of the initial equation are real, then there are the following cases:

Case 1: $k_1 - k_2$ is not an integer; $y_2(t) = t^{k_2}(A_0 + A_1 t + A_2 t^2 + \ldots)$

Case 2: Double roots $k_1 = k_2 = k$; $y_2(t) = y_1(t) \ln t + t^k (A_0 + A_1 t + A_2 t^2 + \ldots)$

Case 3: Roots differing by integer; $y_2(t) = k y_1(t) \ln t + t^{k_2}(A_0 + A_1 t + A_2 t^2 + \ldots)$ [1][8].

III. MAIN RESULTS

We solve some special problems using Frobenius method;

Problem 1

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)y = 0 \tag{11}$$

Solution

Here, $xP(x)$ and $x^2Q(x)$ are analytic (not infinite) at $x = 0$. So, $x = 0$ is regular singular point of this equation.

Let $y = \sum a_k x^{m+k}$

$$\frac{dy}{dx} = \sum a_k (m+k) x^{m+k-1}$$

$$\frac{d^2y}{dx^2} = \sum a_k (m+k-1)(m+k-1) x^{m+k-2}$$

Substituting the values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y in (11), we get

$$x^2 \sum a_k (m+k)(m+k-1) x^{m+k-2} + \sum a_k (m+k) x^{m+k-1} + (x^2 - 4) \sum a_k x^{m+k} = 0$$

$$\Rightarrow \sum a_k [(m+k)(m+k-1) + (m+k) - 4] x^{m+k} + \sum a_k x^{m+k+2} = 0$$

$$\Rightarrow \sum a_k (m+k+2)(m+k-2) x^{m+k} + \sum a_k x^{m+k+2} = 0 \tag{12}$$

The coefficient of lowest degree term x^m in (12) is obtained by putting $k = 0$ in first summation only and equating it to zero. Then the indicial equation is

$$a_0(m+2)(m-2) = 0 \Rightarrow \quad m = 2, -2$$

The coefficient of next lowest term x^{m+1}in (12) is obtained by putting $k = 1$ in first summation only and equating it to zero.

$$a_1(m+3)(m-1) = 0 \Rightarrow a_1 = 0$$

Equating to zero the coefficient of x^{m+k+2}, we get

$$a_{k+2}(m+k+4)(m+k)+a_k=0 \Rightarrow a_{k+2}=-\frac{a_k}{(m+k+4)(m+k)}$$

$$a_1=a_3=a_5=\cdots=0$$

$$a_2=-\frac{a_0}{m(m+4)}$$

$$a_4=-\frac{a_2}{(m+2)(m+6)}=\frac{a_0}{m(m+2)(m+4)(m+6)}$$

$$a_6=-\frac{a_4}{(m+4)(m+8)}=-\frac{a_0}{m(m+2)(m+4)^2(m+6)(m+8)}$$

Hence,

$$y=a_0x^m=\left[1-\frac{x^2}{m(m+4)}+\frac{x^4}{m(m+2)(m+4)(m+6)}-\frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)}+\ldots\right] \quad (13)$$

Putting $m=2$ in (13), we get

$$y_1=a_0x^2=\left[1-\frac{x^2}{2\cdot 6}+\frac{x^4}{2\cdot 4\cdot 6\cdot 8}-\frac{x^6}{2\cdot 4\cdot 6^2\cdot 8\cdot 10}+\ldots\right] \quad (14)$$

For $m=-2$

Coefficient of $x^4, x^6,$ etc. in (14) becomes infinite on putting $m=-2$. To overcome this difficulty, we put $a_0=b_0(m+2)$ in (14) and we get

$$y=bx^m=\left[(m+2)-\frac{(m+2)x^2}{m(m+4)}+\frac{x^4}{m(m+4)(m+6)}-\frac{x^6}{m(m+4)^2(m+6)(m+8)}+\cdots\right] \quad (15)$$

On differentiating (15) w.r.t $'m'$, we get

$$\frac{\partial y}{\partial m}=b_0(x^m.\log x)\left[(m+2)-\frac{(m+2)x^2}{m(m+4)}+\frac{x^4}{m(m+4)(m+6)}-\frac{x^6}{m(m+4)^2(m+6)(m+8)}+\ldots\right]$$

$$+b_0x^m\left[1-\frac{(m+2)x^2}{m(m+4)}\left(\frac{1}{m+2}-\frac{1}{m}-\frac{1}{m+4}\right)-\frac{x^4}{m(m+4)(m+6)}\left(\frac{1}{m}-\frac{1}{m+4}-\frac{1}{m+6}\right)+\ldots\right]$$

On replacing m by -2, we get

$$\left(\frac{\partial y}{\partial m}\right)_{m=-2}=(b_0x^{-2}\log x)\left[0-0+\frac{x^4}{(-2)(2)(4)}-\frac{x^6}{(-2)(2)^2(4)(6)}+\ldots\right]+b_0x^{-2}\left[1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4}\left(\frac{1}{4}\right)+\ldots\right]$$

$$\Rightarrow y_2=b_0x^{-2}\log x\left(-\frac{1}{2^2\cdot 4}+\frac{x^2}{2^3\cdot 4\cdot 6}-\frac{x^4}{2^3\cdot 4^2\cdot 6\cdot 8}+\cdots\right)+b_0x^{-2}\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}+\cdots\right)$$

General solution is $y=c_1y_1+c_2y_2$

$$y=c_1x^2\left(1-\frac{1}{2\cdot 6}+\frac{x^4}{2\cdot 4\cdot 6\cdot 8}-\frac{x^6}{2\cdot 4\cdot 6^2\cdot 8\cdot 10}+\cdots\right)+c_2\left[b_0x^{-2}\log x\left(-\frac{1}{2^2\cdot 4}+\frac{x^2}{2^3\cdot 4\cdot 6}-\frac{x^4}{2^3\cdot 4^2\cdot 6\cdot 8}+\cdots\right)+\right.$$

b0x−21+x222+x422·42+… (16)

Problem 2
Solve in series the differential equation

$$x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+x^2y=0 \quad (17)$$

Solution

Comparing the given equation with the form

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, we get

$$P(x) = \frac{5}{x} Q(x) = 1$$

at $x = 0$, since $P(x)$ is not analytic therefore $x = 0$ is a singular point. $x\,P\,(x) = 5$

$$x^2\,Q\,(x) = x^2$$

Since both $xP\,(x)$ and $x^2\,Q\,(x)$are analytic at $x = 0$ therefore, $x = 0$ is a regular singular point.

Let us assume

$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \cdots$ (18)

$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \cdots$ (19)$\frac{d^2y}{dx^2} =$

$m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \cdots$ (20)

Substituting the above values in given equation, we get

$x^2[m(m-1)a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \cdots]$

$+5x[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \cdots]$

$+x^2[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \cdots] = 0$ (21)

Equating the coefficient of lowest power of x to zero, we get

$m(m-1)a_0 + 5ma_0 = 0\,[coeff.\ x^m = 0]$

$$\Rightarrow (m^2 + 4m)a_0 = 0$$

$\Rightarrow m\,(m+4) = 0$ (indicial equation) $(\therefore a_0 \neq 0)$

$$\Rightarrow\ m = 0, \quad 4$$

Hence, the roots are distinct and differing by an integer. Equating to zero, the coefficients of successive powers of x, we get

Coefficient of $x^{m+1} = 0$

$$(m+1)ma_1 + 5(m+1)a_1 = 0$$

$$\Rightarrow\ (m+5)(m+1)a_1 = 0 \quad \Rightarrow \quad a_1 = 0[\therefore m \neq -5, -1]$$

Coefficient of $x^{m+2} = 0$

$$(m+2)(m+1)a_2 + 5(m+2)a_2 + a_0 = 0$$

$(m+2)(m+6)a_2 + a_0 = 0$

$a_2 = \frac{-a_0}{(m+2)(m+6)}$

Coefficient of $x^{m+3} = 0$

$$(m+3)(m+2)a_3 + 5(m+3)a_3 + a_1 = 0$$

$(m+3)(m+7)a_3 + a_1 = 0$

$\Rightarrow\ a_3 = \frac{-a_1}{(m+3)(m+7)}$

$\Rightarrow\ a_3 = 0$

Similarly,$a_5 = a_7 = a_9 = \cdots \quad = 0$

Now, Coefficient of $x^{m+4} = 0$

$$(m+4)(m+3)a_4 + 5(m+4)a_4 + a_2 = 0$$

$(m+4)(m+8)a_4 = -a_2$

$$\Rightarrow \quad a_4 = \frac{-a_2}{(m+4)(m+8)} = \frac{a_0}{(m+2)(m+4)(m+6)(m+8)} \text{etc.}$$

These give $y = a_0 x^m \left[1 - \frac{x^2}{(m+2)(m+6)} + \frac{x^4}{(m+2)(m+4)(m+6)(m+8)} \dots\right]$ (22)

Putting $m = 0$ in (22), we get

$y_1 = (y)_{m=0} = a_0 \left[1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} \dots\right]$ (23)

If we put $m = -4$ in the series given by equation (22), the coefficients become infinite. To avoid this difficulty, we put $a_0 = b_0(m+4)$, so that

$y = b_0 x^m \left[(m+4) - \frac{(m+4)x^2}{(m+2)(m+6)} + \frac{x^4}{(m+2)(m+6)(m+8)} - \cdots\right]$ (24)

Now, $$\frac{\partial y}{\partial m} = \log x \, b_0 x^m \left[1 - \frac{m^2+8m+20}{(m^2+8m+12)^2} x^2 - \frac{(3m^2+32m+76)}{(m^3+16m^2+76m+96)^2} x^4 + \cdots\right]$$

Second solution is given by

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-4} = b_0 x^{-4} \log x \left[0 - 0 + \frac{x^4}{(-2)(2)(4)} - \frac{x^6}{16} + \cdots\right] + b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \cdots\right)$$

$$= b_0 x^{-4} \log x \left(-\frac{x^4}{16} - \frac{x^6}{16} + \cdots\right) + b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \cdots\right)$$

Hence, the complete solution is given by

$y = c_1 y_1 + c_2 y_2$

$$= c_1 a_0 \left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \cdots\right) + c_2 b_0 x^{-4} \log x \left(-\frac{x^4}{16} - \frac{x^6}{16} - \cdots\right) + c_2 b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \cdots\right)$$

$$y = A\left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \cdots\right) + Bx^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \cdots\right) - B \log x \left(\frac{1}{16} - \frac{x^2}{16} + \cdots\right)$$

where $A = c_1 a_0$ and $B = c_2 b_0$.

Problem 3

Find the general series solution about $x = 0$ of the differential equation

$x^2 y'' + 4xy' + (x^2 + 2)y = 0$ (25)

Solution

let $y = \sum a_k x^{m+k}$ (26)

$y' = \sum (m+k) a_k x^{m+k-1}$ (27)

$y'' = \sum (m+k)(m+k-1) a_k x^{m+k-2}$ (28)

Substituting equation (26), (27) and (28) in equation (25) above.

$$\sum (m+k)(m+k-1) a_k x^{m+k} + 4\sum (m+k) a_k x^{m+k} + \sum a_k x^{m+k+2} + 2\sum a_k x^{m+k} = 0$$

Collecting like terms

$[\sum (m+k)(m+k-1) + 4(m+k) + 2] a_k x^{m+k} + \sum a_k x^{m+k+2} = 0$ (29)

The coefficient of lowest power of x^m in (29) obtained by putting $k = 0$ in the first summation only and equates it to zero

$[m(m-1) + 4(m+0) + 2] a_0 = 0$

$[m^2 - m + 4m + 2] a_0 = 0$

$[m^2 + 3m + 2] a_0 = 0$

$(m+2)(m+1) = 0$ (indicial)

$m = -1, \quad m = -2.$

$$[(m+2)(m+k-1)+4(m+k)+2]a_k x^{m+k} + a_k x^{m+k+2} = 0$$

$$a_k = \frac{-a_{k-2}}{(m+2)(m+1)+4(m+k)+2} \quad (30)$$

Equation (30) is called recurrence relation when $k = 2, m = -1$

$$a_2 = \frac{-a_0}{6}$$

putting, $k = 3$; $a_3 = \frac{-a_1}{12}$

putting, $k = 4$; $a_4 = \frac{-a_2}{20}$ $but\ a_2 = \frac{-a_0}{6}$

$$a_4 = \frac{-a_0}{120}$$

Putting,$k = 5$; $a_5 = \frac{-a_3}{30}$ $but\ a_3 = \frac{-a_1}{12}$

$$a_5 = \frac{-a_1}{360}$$

$$y_1 = \left\{1 - \frac{x^2}{6} + \frac{x^4}{20} + \cdots\right\}a_0 + \left\{1 - \frac{x^2}{12} + \frac{x^4}{360} + \cdots\right\}a_1$$

putting, $m - 2,\ k = 1$

$$a_2 = \frac{-a_0}{2}$$

putting, $k = 3$; $a_3 = \frac{-a_1}{6}$

putting, $k = 4$; $a_4 = \frac{-a_2}{12}$ $but\ a_2 = \frac{-a_0}{2}$

$$a_4 = \frac{a_0}{24}$$

putting, $k = 5$; $a_5 = \frac{-a_3}{20}$ $but\ a_3 = \frac{-a_1}{6}$

$$a_5 = \frac{a_1}{120}$$

$$y_2 = x^{-2}\left\{1 - \frac{x^2}{2} + \frac{x^2}{24} + \cdots\right\}a_0 + \left\{\frac{1}{x} - \frac{x}{6} + \frac{x^3}{120} \ldots\right\}a_1.$$

IV. CONCLUSION

In this paper, we can clearly see that Frobenius method is an easier approach when dealing with power series solution of second order ordinary differential equation.

REFERENCES

[1]. Torabi, A. and Rohani, M. (2020) Frobenius Method for Solving Second-Order Ordinary Differential Equations. *Journal of Applied Mathematics and Physics*, **8**, 1269-1277. doi: 10.4236/jamp.2020.87097.

[2]. Haarsa, P. and Pothat, S. (2014). The Frobenius Method on a Second Order Homogeneous LinearODEs. *Advance Studies in Theoretical Physics*, 8, 1145 – 1148.

[3]. Syofra, A. H., Permatasari, R. and Nazara, A. (2016) The Frobenius Method for solving Ordinary Differential Equation with Coefficient Variable. *IJSR*, 5, 2233-2235.

[4]. Dass, H. K. (2013). *Advanced engineering mathematics.* India: 3. Ghana & Company PVT, LTD.

[5]. Kreyzig, E. (2011). *Advanced engineering mathematics.* (8th ed.). New York: John Wiley and SonsIncorporated.

[6]. Ross, J. M. (1989). *Introduction to ordinary differential equations.* New York: Harcourt Brace Jovanovich Incorporated.

[7]. Stroud, K. A. (2007). *Advanced engineering mathematics.* (6th ed.), New York: Palgrave PublishersLimited.

[8]. Adkins, W. and Davidson, M.G. (2010) Ordinary Differential Equation. Springer, New York.

[9]. Birkhoff, G. and Rota, G. C. (1989). Ordinary Differential Equation. 4th Edition, Wiley, New York.

[10]. Hartman, P. (2002). Ordinary Deferential Equation. 2nd Edition, SIAM, Philadelphia.

[11]. Robin, W. (2014). Frobenius series solution of Fuchs second-Order ordinary differential equations via complex integration, Inter. Math. Forum, 9(20), 953 - 965.

[12]. Lomen, D. and Mark, J. (1988). Differential equations, Prentice-Hall International, Inc., New Jersey.