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Research Paper

Positive Solution to Singular Sixth–Order Differential System with Variable Parameters

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Abstract. In this paper, we investigate the existence of positive solution for the singular sixth-order differential system with sixth variable parameters

$$-u^{(6)} + A_1(t)u^{(4)} + B_1(t)u'' + C_1(t)u = \varphi u + f(t, u, \varphi), \quad 0 < t < 1$$

$$-\varphi^{(6)} + A_2(t)\varphi^{(4)} + B_2(t)\varphi'' + C_2(t)\varphi = \mu g(t, u, u'', u^{(4)}), \quad 0 < t < 1$$

$$u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0,$$

$$\varphi(0) = \varphi(1) = \varphi''(0) = \varphi''(1) = \varphi^{(4)}(0) = \varphi^{(4)}(1) = 0,$$

where $\mu > 0$ is a constant, and the nonlinear terms f, g may be singular with respect to the time and space variables. Using a fixed point theorem in cones and an operator spectral theorem we give an new existence result for singular differential system. The existence of the positive solution depends on μ, i.e. there exists a positive number $\overline{\mu}$ such that if $0 < \mu < \overline{\mu}$, the boundary value problem has a positive solution.

Keywords: positive solution; fixed point theorem; singular solution; bending of an elastic beam; cone; boundary value problem; existence; operator spectral theorem.

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I. INTRODUCTION

Boundary value problems for ordinary differential equations can be used to describe a large number of chemical, biological and physical phenomena. The existence of positive solutions for such problems has become an important area of investigation in recent years. It is well known that the bending of an elastic beam can be described with fourth-order boundary value problems. An elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \tag{1}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{2}$$

Existence of solutions for problem (1) was established for example by Gupta [1,2], Liu [3], Ma [4], Ma et. al. [5], Ma and Wang [6], Aftabizadeh [7], Yang [8], Del Pino and Manasevich [9], RP Agarwal et.al. [10,11,12] (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions.

Recently, Wang and An [13] studied the existence of positive solutions for a second-order differential system by using the fixed point theorem of cone expansion and compression.
It is well known that the deformation of the equilibrium state, an elastic circular ring segment can be described by a boundary value problem for a sixth-order ordinary differential equation. However, there are only a handful of articles on this topic.
In this paper we shall discuss the existence of positive solutions for the sixth-order boundary value problem

$$-u^{(6)} + A_1(t)u^{(4)} + B_1(t)u'' + C_1(t)u = \varphi u + f(t,u,\varphi), \quad 0<t<1$$

$$-\varphi^{(6)} + A_2(t)\varphi^{(4)} + B_2(t)\varphi'' + C_2(t)\varphi = \mu g(t,u,u'',u^{(4)}), \quad 0<t<1$$

$$u(0)=u(1)=u''(0)=u''(1)=u^{(4)}(0)=u^{(4)}(1)=0,$$

$$\varphi(0)=\varphi(1)=\varphi''(0)=\varphi''(1)=\varphi^{(4)}(0)=\varphi^{(4)}(1)=0, \tag{3}$$

where μ is a positive parameter, $A_k(t), B_k(t), C_k(t) \in C[0,1]$, $(k=1,2)$, and $f(t,u,\varphi):(0,1)\times[0,+\infty)\times[0+\infty)\longrightarrow(0,+\infty)$ and $g(t,u,v,w):(0,1)\times(0,+\infty)\times(-\infty,0)\times(0,+\infty)\longrightarrow(0,+\infty)$ is continuous. In fact as we will see below one could consider in Section 2 and 3 $f(t,u,\varphi)=f_1(t)\,f_2(t,u,\varphi)$ with $f_2(t,u,\varphi):[0,1]\times[0,+\infty)\times[0,+\infty)\longrightarrow(0,+\infty)$ and $f_1:(0,1)\to(0,+\infty)$ is continuous, provided

$$\int_0^1\int_0^1\int_0^1 G_{1,1}(\tau,\tau)\,G_{2,1}(\tau,s)G_{3,1}(s,v)\,f_1(v)dv\,dsd\tau < +\infty;$$

here $G_{i,1},(i=1,2,3)$ is as defined in Section 2. Moreover, our hypotheses allow but do not require $g(t,u,v,w):[0,1]\times(0,+\infty)\times(-\infty,0)\times(0,+\infty)\to(0,+\infty)$ to be singular at $u=0,\ v=0$ and at $w=0$. The existence of the positive solution depends on μ, i.e. there exists a positive number $\overline{\mu}$ such that if $0<\mu<\overline{\mu}$, the boundary value problem (3) has a positive solution. For this, we shall assume the following conditions throughout:

$(H1)$ $a_k=\sup_{t\in[0,1]}A_k(t)>-\pi^2$, $b_k=\inf_{t\in[0,1]}B_k(t)>0$, $c_k=\sup_{t\in[0,1]}C_k(t)<0$, $\pi^6+a_k\pi^4-b_k\pi^2+c_k>0$, where $a_k,b_k,c_k\in R$, $a_k=\lambda_{1,k}+\lambda_{2,k}+\lambda_{3,k}>-\pi^2$, $b_k=-\lambda_{1,k}\lambda_{2,k}-\lambda_{2,k}\lambda_{3,k}-\lambda_{1,k}\lambda_{3,k}>0$, $c_k=\lambda_{1,k}\lambda_{2,k}\lambda_{3,k}<0$ and $\lambda_{1,k}\geq0\geq\lambda_{2,k}>-\pi^2, 0\leq\lambda_{3,k}<-\lambda_{2,k}$, $(k=1,2)$.

Assumption $(H1)$ involves a three-parameter nonresonance condition.

2. Preliminaries

Let $Y=C[0,1]$ and $Y_+=\{u\in Y: u(t)\geq0,\ t\in[0,1]\}$. It is well known that Y is a Banach space equipped with the norm $\|u\|_0=\sup_{t\in[0,1]}|u(t)|$.

We denote the norm $\|u\|_2$ by

$$\|u\|_2=\max\left\{\|u\|_0,\|u''\|_0\right\}.$$

It is easy to show that $Z=\left\{u\in C^2[0,1]: u(0)=u(1)=0\right\}$ is complete with the norm $\|u\|_2$ and $\|u\|_2\leq\|u\|_0+\|u''\|_0\leq2\|u\|_2$.

Set $X=\left\{u\in C^4[0,1]: u(0)=u(1)=u''(0)=u''(1)=0\right\}$. For given $\chi\geq0$ and $\nu\geq0$, we denote the norm $\|\cdot\|_{\chi,\nu}$ by

$$\|\cdot\|_{\chi,\nu}=\sup_{t\in[0,1]}\left\{\left|u^{(4)}(t)\right|+\chi\left|u''(t)\right|+\nu\left|u(t)\right|\right\},\quad u\in X.$$

We also need the space X equipped with the norm

$$\|u\|_4=\max\left\{\|u\|_0,\|u''\|_0,\left\|u^{(4)}\right\|_0\right\}.$$

In [11], it is shown that X is complete with the norms $\|\cdot\|_{\chi,\nu}$ and $\|u\|_4$, and moreover $\forall u\in X,\ \|u\|_0\leq\|u''\|_0\leq\left\|u^{(4)}\right\|_0$.

We will investigate the existence of positive solutions for problem (3) by the following fixed point theorem of cone expansion and compression of norm type:

Lemma 1 ([14]). Let E be a real Banach space and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subset of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; or
(ii) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.
Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Firstly, we will transform the problem (3) into a new form.

For $h \in Y$, consider the following linear boundary value problem:

$$-w^{(6)} + a_k w^{(4)} + b_k w'' + c_k w = h(t), \quad 0 < t < 1$$
$$w(0) = w(1) = w''(0) = w''(1) = w^{(4)}(0) = w^{(4)}(1) = 0, \tag{4}$$

where a_k, b_k, c_k satisfy the assumption

$$\pi^6 + a_k\pi^4 - b_k\pi^2 + c_k > 0 \tag{5}$$

and let $\Gamma_k = \pi^6 + a_k\pi^4 - b_k\pi^2 + c_k$. The inequality (5) follows immediately from the fact that $\Gamma_k = \pi^6 + a_k\pi^4 - b_k\pi^2 + c_k$ is the first eigenvalue of the problem $-w^{(6)} + a_k w^{(4)} + b_k w'' + c_k w = \lambda w$, $w(0) = w(1) = w''(0) = w''(1) = w^{(4)}(0) = w^{(4)}(1) = 0$ and $\phi_1(t) = \sin \pi t$ is the first eigenfunction, i.e. $\Gamma_k > 0$. Because the line $l_1 = \{(a, b, c) : \pi^6 + a_k\pi^4 - b_k\pi^2 + c_k = 0\}$ is the first eigenvalue line of the three-parameter boundary value problem $-w^{(6)} + a_k w^{(4)} + b_k w'' + c_k w = 0$, $w(0) = w(1) = w''(0) = w''(1) = w^{(4)}(0) = w^{(4)}(1) = 0$, if (a_k, b_k, c_k) lies in l_1, then by the Fredholm alternative the existence of a solution of the boundary value problem (4) cannot be guaranteed.

Let $P(\lambda_k) = \lambda_k^2 + \beta_k\lambda_k - \alpha_k$ where $\beta_k < 2\pi^2, \alpha_k \geq 0, (k = 1, 2)$. It is easy to see that equation $P(\lambda_k) = 0$ has two real roots $\lambda_{1,k}, \lambda_{2,k} = \frac{-\beta_k \pm \sqrt{\beta_k^2 + 4\alpha_k}}{2}$, with $\lambda_{1,k} \geq 0 \geq \lambda_{2,k} > -\pi^2$. Let $\lambda_{3,k}$ be a number such that $0 \leq \lambda_{3,k} < -\lambda_{2,k}$. In this case, (4) satisfies the following decomposition form:

$$-w^{(6)} + a_k w^{(4)} + b_k w'' + c_k w = (-\frac{d^2}{dt^2} + \lambda_{1,k})(-\frac{d^2}{dt^2} + \lambda_{2,k})(-\frac{d^2}{dt^2} + \lambda_{3,k})w, \quad 0 < t < 1. \tag{6}$$

It is obvious that $a_k = \lambda_{1,k} + \lambda_{2,k} + \lambda_{3,k} > -\pi^2, b_k = -\lambda_{1,k}\lambda_{2,k} - \lambda_{2,k}\lambda_{3,k} - \lambda_{1,k}\lambda_{3,k} > 0, c = \lambda_{1,k}\lambda_{2,k}\lambda_{3,k} < 0.$

Suppose that $G_{i,k}(t, s), (i = 1, 2, 3), (k = 1, 2)$ is the Green function associated with

$$-w'' + \lambda_{i,k} w = 0, \quad u(0) = u(1) = 0. \tag{7}$$

We need the following lemmas.

Lemma 2 ([14]). Let $\omega_{i,k} = \sqrt{|\lambda_{i,k}|}$, then $G_{i,k}(t, s)(i = 1, 2, 3)$ can be expressed as

(i) when $\lambda_{i,k} > 0, G_{i,k}(t, s) = \left\{ \begin{array}{ll} \frac{\sinh \omega_{i,k} t \sinh \omega_{i,k}(1-s)}{\omega_{i,k} \sinh \omega_{i,k}}, & 0 \leq t \leq s \leq 1 \\ \frac{\sinh \omega_{i,k} s \sinh \omega_{i,k}(1-t)}{\omega_{i,k} \sinh \omega_{i,k}}, & 0 \leq s \leq t \leq 1 \end{array} \right\}$

(ii) when $\lambda_{i,k} = 0, G_{i,k}(t, s) = \left\{ \begin{array}{ll} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{array} \right\}$

(iii)when $-\pi^2 < \lambda_{i,k} < 0, G_{i,k}(t, s) = \left\{ \begin{array}{ll} \frac{\sin \omega_{i,k} t \sin \omega_{i,k}(1-s)}{\omega_{i,k} \sin \omega_{i,k}}, & 0 \leq t \leq s \leq 1 \\ \frac{\sin \omega_{i,k} s \sin \omega_{i,k}(1-t)}{\omega_{i,k} \sin \omega_{i,k}}, & 0 \leq s \leq t \leq 1 \end{array} \right\}.$

Lemma 3 ([14]). $G_{i,k}(t, s)(i = 1, 2, 3)$ has the following properties:
(i) $G_{i,k}(t, s) > 0, \quad \forall t, s \in (0, 1)$;

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(ii) $G_{i,k}(t,s) \le C_{i,k} G_{i,k}(s,s), \quad \forall t,s \in [0,1]$;
(iii) $G_{i,k}(t,s) \ge \delta_{i,k} G_{i,k}(t,t) G_{i,k}(s,s), \quad \forall t,s \in [0,1]$;
(iv) $|G_{i,k}(t_1,s) - G_{i,k}(t_2,s)| \le \widehat{l}_{i,k} |t_1 - t_2|, \quad \forall t_1,t_2,s \in [0,1]$;

where $C_{i,k} = 1, \delta_{i,k} = \frac{\omega_{i,k}}{\sinh \omega_{i,k}}$, if $\lambda_{i,k} > 0$; $C_{i,k} = 1, \delta_{i,k} = 1$, if $\lambda_{i,k} = 0$; $C_{i,k} = \frac{1}{\sin \omega_{i,k}}, \delta_{i,k} = \omega_{i,k} \sin \omega_{i,k}$, if $-\pi^2 < \lambda_{i,k} < 0$, and $\widehat{l}_{i,k}$ is a positive constant.

In what follows, we shall let

$$D_{i,k} = \int_0^1 G_{i,k}(s,s)ds, \quad (i = 1,2,3), \quad (k = 1,2). \tag{8}$$

Now, since from the second equation of (3), we have

$$\begin{aligned} -\varphi^{(6)} + a_2\varphi^{(4)} + b_2\varphi'' + c_2\varphi &= (-\frac{d^2}{dt^2} + \lambda_{1,2})(-\frac{d^2}{dt^2} + \lambda_{2,2})(-\frac{d^2}{dt^2} + \lambda_{3,2})\varphi \\ &= (-\frac{d^2}{dt^2} + \lambda_{2,2})(-\frac{d^2}{dt^2} + \lambda_{1,2})(-\frac{d^2}{dt^2} + \lambda_{3,2})\varphi = h(t), \end{aligned} \tag{9}$$

where $h(t) = \mu g(t, u(t), u''(t), u^{(4)}(t))$.

The solution of boundary value problem (9) can be expressed by

$$\varphi(t) = \int_0^1 \int_0^1 \int_0^1 G_{1,2}(t,v) G_{2,2}(v,s) G_{3,2}(s,\tau) h(\tau) d\tau ds dv, \quad t \in [0,1]. \tag{10}$$

Thus, for every given $h \in Y$, the boundary value problem (9) has a unique solution $\varphi \in C^6[0,1]$ which is given by (10).

We now define a mapping $Q_2 : C[0,1] \to C[0,1]$ by

$$(Q_2 h)(t) = \int_0^1 \int_0^1 \int_0^1 G_{1,2}(t,v) G_{2,2}(v,s) G_{3,2}(s,\tau) h(\tau) d\tau ds dv, \quad t \in [0,1]. \tag{11}$$

Similarly, from the first equation of (3), we obtain

$$-u^{(6)} + a_1 u^{(4)} + b_1 u'' + c_1 u = (-\frac{d^2}{dt^2} + \lambda_{1,1})(-\frac{d^2}{dt^2} + \lambda_{2,1})(-\frac{d^2}{dt^2} + \lambda_{3,1})u = h_1(t) \tag{12}$$

where $h_1(t) = \varphi(t)u(t) + f(t, u(t), \varphi(t))$, and the solution of boundary value problem (12) can be expressed by

$$u(t) = \int_0^1 \int_0^1 \int_0^1 G_{1,1}(t,v) G_{2,1}(v,s) G_{3,1}(s,\tau) h_1(\tau) d\tau ds dv, \quad t \in [0,1]. \tag{13}$$

Similarly, we define a mapping $Q_1 : C[0,1] \to C[0,1]$ by

$$(Q_1 h)(t) = \int_0^1 \int_0^1 \int_0^1 G_{1,1}(t,v) G_{2,1}(v,s) G_{3,1}(s,\tau) h(\tau) d\tau ds dv, \quad t \in [0,1]. \tag{14}$$

It is useful to introduce the following notations:

$$(Q_{i,j} h)(t) = \int_0^1 G_{i,j}(t,s) h(s) ds, \quad (i = 1,2,3), \quad (j = 1,2), \tag{15}$$

and

$$\widehat{Q}_k h = \int_0^1 \int_0^1 \int_0^1 C_{1,k} G_{1,k}(v,v) G_{2,k}(v,s) G_{3,k}(s,\tau) h(\tau) d\tau ds dv, \quad (k = 1,2). \tag{16}$$

It is easy to see that

$$(Q_j h)(t) = Q_{1,j} Q_{2,j} Q_{3,j} h(t), \quad (j = 1,2),$$

and

$$(Q_j h)(t) \le \widehat{Q}_j h, \quad (j = 1, 2).$$

Lemma 4. $Q_k : Y \to (X, \|\cdot\|_{\chi_k,\nu_k})$ is linear and completely continuous where $\chi_k = \lambda_{1,k} + \lambda_{3,k}$, $\nu_k = \lambda_{1,k}\lambda_{3,k}$ and $\|Q_k\| \le D_{2,k}$, where $D_{2,k}$ is defined by (8). Moreover, $(1 + \chi_k + \nu_k)^{-1}\|.\|_{\chi_k,\nu_k} \le \|.\|_4 \le \|.\|_{\chi_k,\nu_k}$.

Proof. The proof of completely continuous is similar to the proof of Lemma 6 in [15], so we omit it. Next we will show that $\|Q_k\| \le D_{2,k}$. Assume that $h \in Y$ and $w = Q_k h$ is the solution the boundary value problem (4). It is clear that the operator Q_k maps Y into X. Now for all $\forall h \in Y, w = Q_k h \in X$, $w(0) = w(1) = w''(0) = w''(1) = w^{(4)}(0) = w^{(4)}(1) = 0$. Using (9) it is easy to see that

$$-w'' + \lambda_{i,k} w = \int_0^1 \int_0^1 G_{j,k}(t,v) G_{m,k}(v,\tau) h(\tau) d\tau dv, \quad t \in [0,1]. \tag{17}$$

and

$$w^{(4)} - (\lambda_{i,k} + \lambda_{j,k}) w'' + \lambda_{i,k}\lambda_{j,k} w = \int_0^1 G_{m,k}(t,v) h(v) dv, \quad t \in [0,1]. \tag{18}$$

where $i, j, m = 1, 2, 3$ and $i \ne j \ne m$.

We will now show $\|Q_k h\|_{\chi_k,\nu_k} \le D_{2,k} \|h\|_0, \forall h \in Y$, where $\chi_k = \lambda_{1,k} + \lambda_{3,k} \ge 0, \nu_k = \lambda_{1,k}\lambda_{3,k} \ge 0$. For this, $\forall h \in Y_+$, let $w = Q_k h$, and by Lemma 3, $w \in X \cap Y_+$. The equality (17) with the assumption $\lambda_{2,k} \le 0$ implies that $w'' \le 0$. Similarly, the equality (18) with the assumptions $\lambda_{2,k} + \lambda_{3,k} < 0$ and $\lambda_{2,k}\lambda_{3,k} \le 0$ implies that $w^{(4)} \ge 0$.

From (18) with $\chi_k = \lambda_{1,k} + \lambda_{3,k} \ge 0$, $\nu_k = \lambda_{1,k}\lambda_{3,k} \ge 0$ and $w \ge 0$, $w'' \le 0$, $w^{(4)} \ge 0$ we immediately have

$$\left|w^{(4)}(t)\right| + \chi_k \left|w''(t)\right| + \nu_k \left|w(t)\right| = w^{(4)} - (\lambda_{1,k} + \lambda_{3,k}) w'' + \lambda_{1,k}\lambda_{3,k} w = \int_0^1 G_{2,k}(t,v) h(v) dv, \quad t \in [0,1]. \tag{19}$$

For any $h \in Y$, let $h = \overline{h}_1 - \overline{h}_2, w_1 = Q_k \overline{h}_1, w_2 = Q_k \overline{h}_2$, where h_1, h_2 are the positive part and negative part of h, respectively. Let $w = Q_k h$, then $w = w_1 - w_2$. From the above, we have $w_i \ge 0, w_i'' \le 0, w_i^{(4)} \ge 0, i = 1, 2$, and the following equality holds:

$$\left|w_i^{(4)}(t)\right| + (\lambda_{1,k} + \lambda_{3,k}) \left|w_i''(t)\right| + \lambda_{1,k}\lambda_{3,k} \left|w_i(t)\right| = \int_0^1 G_{2,k}(t,v) \overline{h}_i(v) dv = Q_{2,k}\overline{h}_i, \quad t \in [0,1], \quad i = 1, 2. \tag{20}$$

So, from (20), we have

$$\begin{aligned}
\left|w^{(4)}(t)\right| + (\lambda_{1,k} + \lambda_{3,k}) \left|w''(t)\right| + \lambda_{1,k}\lambda_{3,k} \left|w(t)\right| &= \left|w_1^{(4)}(t) - w_2^{(4)}(t)\right| \\
&+ (\lambda_{1,k} + \lambda_{3,k}) \left|w_1''(t) - w_2''(t)\right| + \lambda_{1,k}\lambda_{3,k} \left|w_1(t) - w_2(t)\right| \\
&\le \left(\left|w_1^{(4)}(t)\right| + (\lambda_{1,k} + \lambda_{3,k}) \left|w_1''(t)\right| + \lambda_{1,k}\lambda_{3,k} \left|w_1(t)\right|\right) \\
&+ \left(\left|w_2^{(4)}(t)\right| + (\lambda_{1,k} + \lambda_{3,k}) \left|w_2''(t)\right| + \lambda_{1,k}\lambda_{3,k} \left|w_2(t)\right|\right) \\
&= Q_{2,k}\overline{h}_1 + Q_{2,k}\overline{h}_2 = Q_{2,k}|h| \le D_{2,k} \||h|\|_0 = D_{2,k} \|h\|_0 .
\end{aligned}$$

Thus $\|Q_k h\|_{\chi_k,\nu_k} \le D_{2,k} \|h\|_0$, and hence $\|Q_k\| \le D_{2,k}$. ■

We consider the existence of a positive solution of the second equation of (3) (the function $\varphi \in C^6(0,1) \cap C^4[0,1]$ is a positive solution of the second equation of (3), if $\varphi(t) \ge 0$, $t \in [0,1]$, and $\varphi \ne 0$). It is easy to see that the second equation of (3) is equivalent to the following boundary value problem:

$$\begin{aligned}
-\varphi^{(6)} + a_2\varphi^{(4)} + b_2\varphi'' + c_2\varphi &= -\left(A_2(t) - a_2\right)\varphi^{(4)} - \left(B_2(t) - b_2\right)\varphi'' - \left(C_2(t) - c_2\right)\varphi \\
&+ \mu g(t, u, u'', u^{(4)}).
\end{aligned} \tag{21}$$

For any $\varphi \in X$, let

$$(G_2\varphi)(t) = -\left(A_2(t) - a_2\right)\varphi^{(4)} - \left(B_2(t) - b_2\right)\varphi'' - \left(C_2(t) - c_2\right)\varphi.$$

The operator $G_2 : X \to Y$ is linear. By Lemma 4 and Corollary 9, $\forall \varphi \in X, t \in [0,1]$, we have

$$|(G_2\varphi)(t)| \le [-A_2(t) + B_2(t) - C_2(t) - (-a_2 - b_2 - c_2)] \, \|\varphi\|_4$$
$$\le K_2\|\varphi\|_4 \le K_2\|\varphi\|_{\chi_2,\nu_2}$$

where $K_2 = \max_{t\in[0,1]} [-A_2(t) + B_2(t) - C_2(t) - (-a_2 + b_2 - c_2)]$, $\chi_2 = \lambda_{2,2} + \lambda_{3,2} \ge 0$, $\nu_2 = \lambda_{2,2}\lambda_{3,2} \ge 0$. Hence $\|G_2\varphi\|_0 \le K_2 \|\varphi\|_{\chi_2,\nu_2}$, and so $\|G_2\| \le K_2$. Also $\varphi \in C^4[0,1] \cap C^6(0,1)$ is a solution of (21) iff $\varphi \in X$ satisfies $\varphi = Q_2(G_2\varphi + h)$, where $h(t) = \mu g(t, u, u'', u^{(4)})(t)$ i.e.

$$\varphi \in X, \quad (I - Q_2G_2)\varphi = Q_2h. \tag{22}$$

The operator $I - Q_2G_2$ maps X into X. From $\|Q_2\| \le D_{2,2}$ together with $\|G_2\| \le K_2$ and condition $D_{2,2}K_2 < 1$, and applying the operator spectra theorem, we find that $(I - Q_2G_2)^{-1}$ exists and bounded. Let $L_2 = D_{2,2}K_2$.

Let $H_2 = (I - Q_2G_2)^{-1}Q_2$. Then (22) is equivalent to $\varphi = H_2h$. By the Neumann expansion formula, H_2 can be expressed by

$$H_2 = (I + Q_2G_2 + ... + (Q_2G_2)^n + ...) \; Q_2 =$$
$$= Q_2 + (Q_2G_2)Q_2 + ... + (Q_2G_2)^n Q_2 + ... = Q_2(I + (G_2Q_2) + ... + (G_2Q_2)^n + ...) \tag{23}$$

The complete continuity of Q_2 with the continuity of $(I - Q_2G_2)^{-1}$ guarantees that the operator $H_2 : Y \to X$ is completely continuous.

Now $\forall h \in Y_+$, let $\varphi = H_2h$, then $\varphi \in X \cap Y_+$, and it is easy to see that $\varphi''(t) \le 0$, $\varphi^{(4)}(t) \ge 0$, $t \in [0,1]$ Indeed, using by Lemma 4, and from (23), we have

$$\varphi''(t) = Q_2''(I + (G_2Q_2) + ... + (G_2Q_2)^n + ...)h(t) =$$
$$= (\lambda_{2,2}Q_2 - Q_{1,1}Q_{3,1})(I + (G_2Q_2) + ... + (G_2Q_2)^n + ...)h(t) \le 0, \quad t \in [0,1],$$

and

$$\varphi^{(4)}(t) = (\lambda_{2,1} + \lambda_{3,1})Q_2'' - \lambda_{2,1}\lambda_{3,1}Q_1 + Q_{1,1})(I + (G_2Q_2) + ... + (G_2Q_2)^n + ...)h(t) \ge 0, \quad t \in [0,1].$$

Thus, we have

$$(G_2\varphi)(t) = -(A_2(t) - a_2)\varphi^{(4)} - (B_2(t) - b_2)\varphi'' - (C_2(t) - c_2)\varphi \ge 0, \quad t \in [0,1].$$

Hence

$$\forall h \in Y_+, \quad (G_2Q_2h)(t) \ge 0, \quad t \in [0,1] \tag{24}$$

and so $(Q_2G_2)(Q_2h)(t) = Q_2(G_2Q_2h)(t) \ge 0$, $t \in [0,1]$.

It is easy to see [15] that the following inequalities hold: $\forall h \in Y_+$,

$$\frac{1}{1 - L_2}(Q_2h)(t) \ge (H_2h)(t) \ge (Q_2h)(t), \quad t \in [0,1], \tag{25}$$

moreover,

$$\|(H_2h)\|_0 \le \frac{1}{1 - L_2}\|(Q_2h)\|_0. \tag{26}$$

For any $u \in X_+$, it is easy to see that $\varphi \in C^4[0,1] \cap C^6(0,1)$ being a positive solution of the second equation of (3) is equivalent to $\varphi \in Y_+$ being a nonzero solution of

$$\varphi(t) = \mu H_2 g(s, u(s), u''(s), u^{(4)}(s))(t). \tag{27}$$

Obviously, $H_2 : Y_+ \to Y_+$ is completely continuous.

Thus inserting (27) into the first equation of (3), we have

$$-u^{(6)} + A_1(t)u^{(4)} + B_1(t)u'' + C_1(t)u =$$
$$= \mu u(t)H_2g(s, u(s), u''(s), u^{(4)}(s))(t) + f(t, u(t), \mu H_2g(s, u(s), u''(s), u^{(4)}(s))(t)),$$
$$u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0. \tag{28}$$

Now we consider the existence of a positive solution of (28). The function $u \in C^6(0,1) \cap C^4[0,1]$ is a positive solution of (28), if $u(t) \ge 0$, $t \in [0,1]$, and $u \ne 0$.

Then, similarly as the solution of (21), the solution of (28) can be expressed as

$$u(t) = H_1(u(s)\mu H_2 g(v, u(v), u''(v), u^{(4)}(v))(s) + f(s, u(s), \mu H_2 g(v, u(v), u''(v), u^{(4)}(s))(s)). \tag{29}$$

Similarly, as (23) and (25), we obtain

$$(H_1 h)(t) = Q_1(I + (G_1Q_1) + ... + (G_1Q_1)^n + ...)h(t), \tag{30}$$

and

$$\frac{1}{1-L_1}(Q_1h)(t) \geq (H_1h)(t) \geq (Q_1h)(t), \quad t \in [0,1] \tag{31}$$

where $(G_1u)(t) = -(A_1(t) - a_1)u^{(4)}(t) - (B_1(t) - b_1)u''(t) - (C_1(t) - c_1)u(t) \geq 0$, $L_1 = D_{2,1}K_1$, and $K_1 = \max_{t\in[0,1]}[-A_1(t) + B_1(t) - C_1(t) - (-a_1 + b_1 - c_1)]$

We recall that $X = \left\{u \in C^4[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\right\}$ is complete with the norm $\|u\|_4 = \max\left\{\|u\|_0, \|u''\|_0, \left\|u^{(4)}\right\|_0\right\}$ and using Lemma 8 and Corollary 9, we have $\|u\|_4 = \left\|u^{(4)}\right\|_0$. Throughout this paper, we use the Banach space $(X, \left\|u^{(4)}\right\|_0)$ to solve the problem (28).

Set

$$P = \{u \in X,\ u(t) \geq \widehat{k}G_{1,1}(t,t)\|u\|_0,\ \ -u''(t) \geq \widehat{k}G_{1,1}(t,t)\|u''\|_0,$$

$$u^{(4)}(t) \geq \widehat{k}G_{1,1}(t,t)\left\|u^{(4)}\right\|_0,\ t \in [0,1]\}$$

where $\widehat{k} = \frac{(1-L_1)\delta_{1,1}}{C_{1,1}}$.

Note, P is a cone in X. For $R > 0$, write $B_R = \{u \in C^4[0,1] : \|u\|_4 < R\}$.

It is easy to see that if $u \in P$ than

$$u^{(4)}(t) \geq \sigma\left\|u^{(4)}\right\|_0 \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \tag{32}$$

where $\sigma = \widehat{k}\min_{t\in[\frac{1}{4},\frac{3}{4}]} G_{1,1}(t,t)$.

We now define a mapping $T : P \to C[0,1]$ by

$$Tu(t) = H_1(u(s)\mu H_2 g(v, u(v), u''(v), u^{(4)}(v))(s) + f(s, u(s), \mu H_2 g(v, u(v), u''(v), u^{(4)}(v))(s)). \tag{33}$$

Let us introduce the notation:

$$q(\tau) = \mu u(\tau)H_2 g(v, u(v), u''(v), u^{(4)}(v))(\tau) + f(\tau, u(\tau), \mu H_2 g(t, u, u'', u^{(4)})(\tau)),$$

and rewrite (33) in the following form:

$$\begin{aligned}(Tu)(t) &= H_1(q(\tau)) = \\ &= Q_1(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q(\tau)).\end{aligned} \tag{34}$$

Using by (34), it is easy to see that

$$\begin{aligned}(Tu)''(t) &= Q_1''(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q(\tau)) = \\ &(\lambda_{2,1}Q_1 - Q_{1,1}Q_{3,1})(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q(\tau)) \leq 0, \quad t \in [0,1],\end{aligned} \tag{35}$$

and similarly, we have

$$\begin{aligned}(Tu)^{(4)}(t) &= Q_1^{(4)}(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q(\tau)) = \\ = ((\lambda_{2,1}+\lambda_{3,1})&(\lambda_{2,1}Q_1 - Q_{1,1}Q_{3,1}) - \lambda_{2,1}\lambda_{3,1}Q_1 + Q_{1,1})(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q(\tau)) \geq 0, \quad t \in [0,1].\end{aligned} \tag{36}$$

Lemma 5. Let $u \in P$. Then the following relations hold:
(a) $(Tu)(t) \geq \widehat{k}G_{1,1}(t,t)\|Tu\|_0$, for $t \in [0,1]$,
(b) $-(Tu)''(t) \geq \widehat{k}G_{1,1}(t,t)\|(Tu)''\|_0$, for $t \in [0,1]$,

(c) $(Tu)^{(4)}(t) \geq \widehat{k}G_{1,1}(t,t)\left\|(Tu)^{(4)}\right\|_0$, for $t \in [0,1]$.

where $\widehat{k} = \frac{\delta_{1,1}(1-L_1)}{C_{1,1}}$.

Proof.
By using (14) an (31), we have

$$Tu(t) = H_1(q(\tau)) \leq \frac{1}{1-L_1}Q_1(q(\tau))$$

$$= \frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)q(\tau)d\tau dsdv$$

where we recall that

$$q(\tau) = \mu u(\tau)H_2 g(v,u(v),u''(v),u^{(4)}(v))(\tau) + f(\tau,u(\tau),\mu Hg(t,u,u'',u^{(4)})(\tau)).$$

For simplicity we denote

$$I_1 = \int_0^1\int_0^1\int_0^1 G_{1,1}(v,v)G_{2,1}(v,s)G_{3,1}(s,\tau,)q(\tau)d\tau dsdv$$

$$I_2 = |\lambda_{2,1}|I_1 + \int_0^1\int_0^1 G_{1,1}(s,s)G_{3,1}(s,\tau)q(\tau)d\tau ds,$$

$$I_3 = |\lambda_{2,1}+\lambda_{3,1}|I_2 + |\lambda_{2,1}\lambda_{3,1}|I_1 + \int_0^1 G_{1,1}(\tau,\tau)q(\tau)d\tau.$$

From Lemma 3 it is easy to see that

$$\delta_{1,1}G_{1,1}(t,t)I_1 \leq Tu(t) \leq \frac{C_{1,1}}{1-L_1}I_1 \qquad t \in [0,1]\,. \tag{37}$$

$$\delta_{1,1}G_{1,1}(t,t)I_2 \leq -(Tu)''(t) \leq \frac{C_{1,1}}{1-L_1}I_2, \qquad t \in [0,1] \tag{38}$$

$$\delta_{1,1}G_{1,1}(t,t)I_3 \leq (Tu)^{(4)}(t) \leq \frac{C_{1,1}}{1-L_1}I_3, \qquad t \in [0,1] \tag{39}$$

Using (37-39), we have

$$\|Tu\|_0 \leq \frac{C_{1,1}}{1-L_1}I_1, \quad \|-(Tu)''\|_0 \leq \frac{C_{1,1}}{1-L_1}I_2, \text{ and } \left\|(Tu)^{(4)}\right\|_0 \leq \frac{C_{1,1}}{1-L_1}I_3,$$

hence

$$(Tu)(t) \geq \frac{\delta_{1,1}(1-L_1)}{C_{1,1}}G_{1,1}(t,t)\,\|Tu\|_0 \text{ for } t \in [0,1],$$

$$-(Tu)''(t) \geq \frac{\delta_{1,1}(1-L_1)}{C_{1,1}}G_{1,1}(t,t)\,\|(Tu)''\|_0 \text{ for } t \in [0,1] \text{ and}$$

$$(Tu)^{(4)}(t) \geq \frac{\delta_{1,1}(1-L_1)}{C_{1,1}}G_{1,1}(t,t)\left\|(Tu)^{(4)}\right\|_0 \text{ for } t \in [0,1].$$

This finishes the proof. ■

Throughout this paper, we assume additionaly that the continuous function $f(t,u,\varphi) : (0,1)\times[0,+\infty)\times[0+\infty) \longrightarrow (0,+\infty)$ satisfies

$(H2)$

$$f(t,u,v) \leq f_1(t)f_2(uv), \qquad t \in (0,1), \qquad u,v \in R^+,$$

where $f_1 : (0,1) \to (0,+\infty)$ and $f_2 : [0,+\infty) \to (0,+\infty)$ is continuous, $R^+ = [0,+\infty)$, $R^- = (-\infty,0]$.

Moreover the function $g(t,u,v,w) : [0,1]\times(0,+\infty)\times(-\infty,0)\times(0,+\infty) \to [0,+\infty)$ satisfies

$(H3)$ There exists an $a > 0$ such that $g(t,u,v,w)$ is nonincreasing in $u,w \leq a$ and $|v| \leq a$ for each fixed $t \in [0,1]$ i.e. if $-a \leq v_2 \leq v_1 < 0$, $0 < u_1 \leq u_2$ and $0 < w_1 \leq w_2$ then $g(t,u_1,v_1,w_1) \geq g(t,u_2,v_2,w_2)$.

$(H4)$ There exists an function $g_1(t,w):[0,1]\times(0,+\infty)\to[0,+\infty)$ such that $g_1(t,w)$ is nonincreasing in $u\le a$ for each fixed $t\in[0,1]$, i.e. if $0<w_1\le w_2$ then $g(t,w_1)\ge g(t,w_2)$
and each fixed $0<r\le a$

$$0<\int_0^1 g_1(s,r\widehat{k}G_{1,1}(s,s)ds<\infty.$$

So, we assume additionaly that the function $g(t,u,v),w$ satisfies

$$g(t,u,v,w)\le g_1(t,u+|v|+w),\qquad t\in[0,1],\qquad u,w\in(0,+\infty),\qquad v\in(-\infty,0).$$

Let us introduce the following notations:

$$D_1=\int_0^1\int_0^1\int_0^1 G_{1,1}(x,x)\,G_{2,1}(x,\tau)\,G_{3,1}(\tau,s)ds\,d\tau dx,$$

$$D_2=\int_0^1\int_0^1\int_0^1 G_{1,1}(v,v)\,G_{2,1}(v,s)\,G_{3,1}(s,\tau)f_1(\tau)\,d\tau dsdv,$$

$$D_3=\int_0^1\int_0^1 G_{1,1}(x,x)\,G_{3,1}(x,\tau)\,d\tau dx,\quad D_6=\int_0^1 G_{1,1}(s,s)f_1(s)\,ds,$$

$$D_4=\int_0^1\int_0^1 G_{1,1}(s,s)\,G_{3,1}(s,\tau)f_1(\tau)\,d\tau ds,\quad D_5=\int_0^1 G_{1,1}(x,x)dx,$$

$$D_7=\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34} G_{1,2}(v,v)G_{2,2}(v,z)G_{3,2}(z,x)dxdzdv,$$

$$D_8=\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34}\int_{\frac14}^{\frac34} G_{1,1}(\frac12,y)G_{2,1}(y,\tau)G_{3,1}(\tau,s)\cdot$$
$$\cdot G_{1,2}(s,v)G_{2,2}(v,z)G_{3,2}(z,x))dxdzdv\,dsd\tau dy.$$

Lemma 6. Let $(H1),(H2),(H3)$ and $(H4)$ hold. Then for all $u\in P\cap\overline{B}_R/B_r$ where $r<a<R$ the following hold

$$(Tu)(t)\le\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_5M_r\|u\|_0+\frac{C_{1,1}}{1-L_1}D_6\sup_{\tau\in(0,1)}f_2(\mu u(\tau)H_2g(v,u(v),u'',u^{(4)})(\tau)),$$

$$-(Tu)''(t)\le(1+|\lambda_{2,1}|)[\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_5M_r\|u\|_0+$$
$$+\frac{C_{1,1}}{1-L_1}D_6\sup_{\tau\in(0,1)}f_2(\mu u(\tau)H_2g(v,u(v),u'',u^{(4)})(\tau))],$$

and

$$(Tu)^{(4)}(t)\le(1+|\lambda_{2,1}+\lambda_{3,1}|(1+|\lambda_{2,1}|)+|\lambda_{2,1}\lambda_{3,1}|)[\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_5M_r\|u\|_0+$$
$$+\frac{C_{1,1}}{1-L_1}D_6\sup_{\tau\in(0,1)}f_2(\mu u(\tau)H_2g(v,u(v),u'',u^{(4)})(\tau))],$$

where

$$M_r=\int_0^1\int_0^1\int_0^1 G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g_1(v,r\widehat{k}G_{1,1}(v,v))dvdzdw+$$
$$+\int_0^1\int_0^1\sup_{y\in(0,R]}\sup_{e\in(0,R]}\sup_{p\in[r,R]}\int_0^1 G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,y,e,p)\,dvdzdw.$$

Proof. It is easy to see that $D_1\le D_3\le D_5$, and $D_2\le D_4\le D_6$. Let $u\in P\cap\overline{B}_R/B_r$, then by Lemma 8, $\|u\|_0\le\|u''\|_0\le\left\|u^{(4)}\right\|_0$ and by Corollary 9, $\|u\|_4=\left\|u^{(4)}\right\|_0$. Thus $r\le\left\|u^{(4)}\right\|_0\le R$. Also, since $u\in P$, we have

$u(t) \geq \widehat{k}G_{1,1}(t,t)\left\|u\right\|_0, \ -u''(t) \geq \widehat{k}G_{1,1}(t,t)\left\|u''\right\|_0, u^{(4)}(t) \geq \widehat{k}G_{1,1}(t,t)\left\|u^{(4)}\right\|_0,$ and $u(t)+|u''(t)|+u^{(4)}(t) \geq r\widehat{k}G_{1,1}(t,t),$ for all $t \in [0,1]$.

Let us introduce the following notation:

$$\widehat{a}(v) = u(v) + |u''(v)| + u^{(4)}(v).$$

By Lemma 1. and $(H3)-(H5)$ we have

$$Tu(t) = H_1(u(s)\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s) + f(s,u(s),\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s))$$

$$\leq \frac{1}{1-L_1}Q_1(u(s)\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s) + f(s,u(s),\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s))$$

$$= \frac{\mu}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)u(\tau)H_2 g(v,u,u'',u^{(4)})(\tau)d\tau ds dv+$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f(\tau,u(\tau),\mu H_2 g(v,u,u'',u^{(4)})(\tau))d\tau ds dv$$

$$\leq \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)u(\tau)Q_2 g(v,u,u'',u^{(4)})(\tau)d\tau ds dv+$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f(s,u(s),\mu H_2 g(v,u,u'',u^{(4)})(\tau))d\tau ds dv$$

$$\leq \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)u(s)\int_0^1\int_0^1\int_{\widehat{a}(v)\leq a}\cdot$$

$$\cdot G_{1,2}(s,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,u(v),u''(v),u^{(4)}(v))\, dv dz dw\, ds\, d\tau dx+$$

$$+\frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)u(s)\int_0^1\int_0^1\int_{\widehat{a}(v)\geq a}\cdot$$

$$\cdot G_{1,2}(s,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,u(v),u''(v),u^{(4)}(v))\, dv dz dw\, ds\, d\tau dx+$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f_1(\tau)f_2(\mu u(\tau)H_2 g(\tau,u,u'',u^{(4)})(\tau))d\tau ds dv$$

$$\leq \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\int_{\widehat{a}(v)\leq a}\cdot$$

$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g_1(v,r\widehat{k}G_{1,1}(v,v))\, dv dz dw\, ds\, d\tau dx+$$

$$+\frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\sup_{y\in(0,R]}\sup_{e\in(0,R]}\sup_{p\in[r,R]}\int_{\widehat{a}(v)\geq a}\cdot$$

$$\cdot G_{1,2}(s,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,y,e,p)\, dv dz dw\, ds\, d\tau dx+$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f_1(\tau)\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2 g(\tau,u,u'',u^{(4)})(\tau))d\tau ds dv$$

$$\leq \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 C_{1,1}G_{1,1}(x,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\int_0^1\cdot$$

$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g_1(v,r\widehat{k}G_{1,1}(v,v))\, dv dz dw\, ds\, d\tau dx+$$

$$+\frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 C_{1,1}G_{1,1}(x,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1 \sup_{y\in(0,R]}\sup_{e\in(0,R]}\sup_{p\in[r,R]}\int_0^1\cdot$$
$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,y,e,p)\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 C_{1,1}G_{1,1}(v,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f_1(\tau)\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2g(\tau,u,u'',u^{(4)})(\tau))d\tau dsdv$$
$$\le\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_1M_r\|u\|_0+\frac{C_{1,1}}{1-L_1}D_2\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2g(v,u(v),u'',u^{(4)})(\tau))$$
$$\le\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_3M_r\|u\|_0+\frac{C_{1,1}}{1-L_1}D_4\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2g(v,u(v),u'',u^{(4)})(\tau)),$$
$$\le\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_5M_r\|u\|_0+\frac{C_{1,1}}{1-L_1}D_6\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2g(v,u(v),u'',u^{(4)})(\tau)),$$

and similarly we also have

$$-(Tu)''(t)\le(1+|\lambda_{2,1}|)[\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_5M_r\|u\|_0+$$
$$+\frac{C_{1,1}}{1-L_1}D_6\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2g(v,u(v),u'',u^{(4)})(\tau))],$$

and

$$(Tu)^{(4)}(t)\le(1+|\lambda_{2,1}+\lambda_{3,1}|(1+|\lambda_{2,1}|)+|\lambda_{2,1}\lambda_{3,1}|)[\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_5M_r\|u\|_0+$$
$$+\frac{C_{1,1}}{1-L_1}D_6\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2g(v,u(v),u'',u^{(4)})(\tau))],$$

This finishes the proof. ■

Lemma 7. $T(P)\subset P$ and $T:P\cap(\overline{B}_R/B_r)\to P$ is completely continuous.

Proof. First, we prove that $T(P)\subset P$. To do this, let $u\in P$, then we define mapping $T:P\to C^2[0,1]$ by (33). Then for any $u\in P$, it is clear that

$$-(Tu)''(t)\ge 0,\quad (Tu)^{(4)}(t)\ge 0,\ \text{for } t\in[0,1].$$

By Lemma 5,

$$(Tu)(t)\ge\frac{\delta_{1,1}(1-L_1)}{C_{1,1}}G_{1,1}(t,t)\|Tu\|_0\ \text{ for } t\in[0,1],$$
$$-(Tu)''(t)\ge\frac{\delta_{1,1}(1-L_1)}{C_{1,1}}G_{1,1}(t,t)\left\|(Tu)''\right\|_0\ \text{ for } t\in[0,1] \text{ and}$$
$$(Tu)^{(4)}(t)\ge\frac{\delta_{1,1}(1-L_1)}{C_{1,1}}G_{1,1}(t,t)\left\|(Tu)^{(4)}\right\|_0\ \text{ for } t\in[0,1].$$

Hence $T(P)\subset P$.
We recall that

$$\widehat{Q}_2h=\int_0^1\int_0^1\int_0^1 C_{1,2}G_{1,2}(v,v)G_{2,2}(v,s)G_{3,2}(s,\tau)h(\tau)d\tau dsdv,\tag{40}$$

and it is easy to see that

$$(H_2h)(t)\le\frac{1}{1-L_2}(Q_2h)(t)\le\frac{1}{1-L_2}\widehat{Q}_2h,\quad\forall t\in[0,1].$$

Let us introduce the following notation:

$$N_r = \widehat{Q}_2 g_1(\tau, r\widehat{k}G_{1,1}(\tau,\tau)) = \int_0^1\int_0^1\int_0^1 C_{1,2}G_{1,2}(v,v)G_{2,2}(v,s)G_{3,2}(s,\tau)g_1(\tau, r\widehat{k}G_{1,1}(\tau,\tau))d\tau ds dv. \quad (41)$$

Let $V \subset P \cap (\overline{B}_R/B_r)$ be a bounded set. Then there exists a $d > 0$, such that $\sup\{\|u\|_4 : u \in V\} = d$.

First we prove $T(V)$ is bounded. Since $\|u\|_4 = \max\left\{\|u\|_0, \|u''\|_0, \left\|u^{(4)}\right\|_0\right\} = \left\|u^{(4)}\right\|_0$, we have $u(t) + |u''(t)| + u^{(4)}(t) \le \|u\|_0 + \|u''\|_0 + \left\|u^{(4)}\right\|_0 \le 3d$, and $|\mu u(t)H_2 g(v,u(v),u''(v),u^{(4)}(v))| \le \frac{\mu}{1-L_2}\|u\|_0\widehat{Q}_2 g_1(v, r\widehat{k}G_{1,1}(v,v)) = \frac{\mu}{1-L_2}dN_r$ for all $t \in [0,1]$. Let $M_d = \sup\{f_2(w) : w \in [0, \frac{\mu}{1-L_2}dN_r]\}$. Now, from Lemma 3 and Lemma 6, we have for any $u \in V$ and $t \in [0,1]$ that

$$|Tu(t)| = |H_1(u(s)\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s) + f(s,u(s),\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s))|$$

$$\le \frac{1}{1-L_1}Q_1(u(s)\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s) + f(s,u(s),\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s))$$

$$\le \frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_3M_r\|u\|_0 + D_4\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2 g(v,u(v),u'',u^{(4)})(\tau)),$$

$$\le \frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_3M_r d + D_4\sup_{\tau\in(0,1)} f_2(\mu u(\tau)H_2 g(v,u(v),u'',u^{(4)})(\tau)),$$

$$\le \frac{\mu C_{1,1}C_{1,2}D_3 dM_r}{(1-L_1)(1-L_2)} + M_d D_4. \quad (42)$$

We have a similar type inequality for $|(Tu)''(t)|$ and $|(Tu)^{(4)}(t)|$. Therefore $T(V)$ is bounded.

Next, we prove that $T(V)$ is equicontinuous. Now, from Lemma 3 and Lemma 6, we have for any $u \in V$ and any $t_1, t_2 \in [0,1]$ that

$$|(Tu)(t_1) - (Tu)(t_2)| =$$

$$= |Q_1(I + (G_1Q_1) + \ldots + (G_1Q_1)^n\ldots)(q(\tau))(t_1) - Q_1(I + (G_1Q_1) + \ldots + (G_1Q_1)^n\ldots)(q(\tau))(t_2)|$$

$$= |\int_0^1\int_0^1\int_0^1 G_{1,1}(t_1,s)G_{2,1}(s,v)G_{3,1}(v,\tau)(I + (G_1Q_1) + \ldots + (G_1Q_1)^n\ldots)(q(\tau))d\tau dv ds -$$

$$-\int_0^1\int_0^1\int_0^1 G_{1,1}(t_2,s)G_{2,1}(s,v)G_{3,1}(v,\tau)(I + (G_1Q_1) + \ldots + (G_1Q_1)^n\ldots)(q(\tau))d\tau dv ds\,|$$

$$\le \int_0^1\int_0^1\int_0^1 |\,G_{1,1}(t_1,s) - G_{1,1}(t_2,s)\,|\,G_{2,1}(s,v)G_{3,1}(v,\tau)(I + (G_1Q_1) + \ldots + (G_1Q_1)^n\ldots)(q(\tau))d\tau dv ds$$

$$\le \frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 |\,G_{1,1}(t_1,s) - G_{1,1}(t_2,s)\,|\,G_{2,1}(s,v)G_{3,1}(v,\tau)q(\tau)d\tau dv ds$$

$$\le \widehat{l}_{1,1}\left(\frac{\mu C_{1,1}C_{1,2}}{(1-L_1)(1-L_2)}D_5M_r d + \frac{C_{1,1}}{1-L_1}D_6M_d\right)|t_1 - t_2|,$$

where $\widehat{l}_{1,1}$ is a constant.

We have a similar type inequality for $|(Tu)''(t_1) - (Tu)''(t_2)|$ and $|(Tu)^{(4)}(t_1) - (Tu)^{(4)}(t_2)|$.

Therefore $T(V)$ is equicontinuous.

Next, we prove that T is continuous. Suppose $u_n, u \in P \cap (\overline{B}_R/B_r)$ and $\|u_n - u\|_4 \to 0$ which implies that $u_n(t) \to u(t), u_n''(t) \to u''(t)$ and $u_n^{(4)}(t) \to u^{(4)}(t)$ uniformly on $[0,1]$. Similarly for $f(t,u,v) \le f_1(t)\,f_2(|u| + |v| + |w|)$, $f_2(|u_n(t)| + |u_n''(t)| + |u_n^{(4)}(t)|) \to f_2(|u(t)| + |u''(t)| + |u^{(4)}(t)|)$ uniformly on $[0,1]$ and $g_1(t,u_n(t)) \to g_1(t,u(t))$ uniformly on $[0,1]$. We recall that

$$q(\tau) = \mu u(\tau)H_2 g(v,u(v),u''(v),u^{(4)}(v))(\tau) + f(\tau,u(\tau),\mu H_2 g(t,u,u'',u^{(4)})(\tau))$$

so, we have

$$q_n(\tau) = \mu u_n(\tau) H_2 g(v, u_n(v), u_n''(v), u_n^{(4)}(v))(\tau) + f(\tau, u(\tau), \mu H_2 g(t, u_n, u_n'', u_n^{(4)})(\tau)).$$

The assertion follows from the estimate

$$|Tu_n(t) - Tu(t)| =$$

$$= |Q_1(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q_n(\tau))(t) - Q_1(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q(\tau))(t)|$$

$$= \Big| \int_0^1 \int_0^1 \int_0^1 G_{1,1}(t,s) G_{2,1}(s,v) G_{3,1}(v,\tau)(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q_n(\tau)) d\tau dv ds -$$

$$- \int_0^1 \int_0^1 \int_0^1 G_{1,1}(t,s) G_{2,1}(s,v) G_{3,1}(v,\tau)(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q(\tau)) d\tau dv ds \Big|$$

$$\leq \int_0^1 \int_0^1 \int_0^1 G_{1,1}(t,s) G_{2,1}(s,v) G_{3,1}(v,\tau) \mid (I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q_n(\tau))$$

$$-(I + (G_1Q_1) + \ldots + (G_1Q_1)^n \ldots)(q(\tau)) \mid d\tau dv ds,$$

and the similar estimate for $|(Tu_n)''(t) - (Tu)''(t)|$ and $\left|(Tu_n)^{(4)}(t) - (Tu)^{(4)}(t)\right|$ by an application of the standard theorem on the convergence of integrals.

The Ascoli-Arzela theorem guarantees that $T : P \to P$ is completely continuous.

This finishes the proof. ■

Lemma 8. If $u(0) = u(1) = 0$ and $u \in C^2[0,1]$, then $\|u\|_0 \leq \|u''\|_0$, and so, $\|u\|_2 = \|u''\|_0$.

Proof. Since $u(0) = u(1)$, there is a $\alpha \in (0,1)$ such that $u'(\alpha) = 0$, and so $u'(t) = \int_\alpha^t u''(s) ds$, $t \in [0,1]$. Hence $|u'(t)| \leq \int_\alpha^t |u''(s)| ds \leq \int_0^1 |u''(s)| ds \leq \|u''\|_0$, $t \in [0,1]$. Thus $\|u'\|_0 \leq \|u''\|_0$. Since $u(0) = 0$, we have $u(t) = \int_0^t u'(s) ds$, $t \in [0,1]$, and so $|u(t)| \leq \int_0^1 |u'(s)| ds \leq \|u'\|_0$. Thus $\|u\|_0 \leq \|u'\|_0 \leq \|u''\|_0$. Since $\|u\|_2 = \max\left\{\|u\|_0, \|u''\|_0\right\}$ and $\|u\|_0 \leq \|u''\|_0$, we obtain that $\|u\|_2 = \|u''\|_0$.

This finishes the proof. ■

Corollary 9. $\forall u \in X$, $\|u\|_0 \leq \|u''\|_0 \leq \|u^{(4)}\|_0$, so we have $\|u\|_4 = \left\|u^{(4)}\right\|_0$.

Corollary 10. Let $r > 0$ and let $u \in \partial B_r \cap P$. Then $\|u\|_4 = \left\|u^{(4)}\right\|_0 = r$.

3. Main results

Theorem 1. Let $(H1)$,$(H2)$,$(H3)$ and $(H4)$ hold. Assume that the following condition holds $(H5)$

$$\limsup_{w \to 0^+} \frac{f_2(w)}{w} \leq c_1,$$

$$\liminf_{\varphi \to \infty} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \inf_{u \in [0, +\infty)} \frac{f(t,u,\varphi)}{\varphi} \geq c_2,$$

and

$$\liminf_{w \to \infty} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \inf_{u \in [0, +\infty)} \inf_{|v| \in [0, +\infty)} \frac{g(t,u,v,w)}{w} = \infty,$$

where c_1 and c_2 is positive real number.

Then there exists $\mu^* > 0$, such that if $\mu \in (0, \mu^*]$, then problem (3) has at least one positive solution.

Proof.

We divide the rather long proof into three steps.

(I) Firstly, we will prove that the first part of assumptions (i) of Lemma 1 is satisfied.

To do this, by $(H5)$, there exist $0 < r < a$ such that

$$f_2(w) \leq c_1 w, \quad \forall w \in [0, r]. \tag{43}$$

Let $u \in \partial B_r \cap P$, by Lemma 8, $\|u\|_0 \le \|u''\|_0 \le \left\|u^{(4)}\right\|_0$ and by Corollary 9, $\|u\|_4 = \left\|u^{(4)}\right\|_0$, then we have $u^{(4)}(t) \le \left\|u^{(4)}\right\|_0 = r, \quad -u''(t) \le \|u''\|_0 \le r$ and $u(t) \le \|u\|_0 \le r, \quad \forall t \in [0,1]$. Also, since $u \in P$, we have $u^{(4)}(t) \ge \widehat{k}G_{1,1}(t,t)\left\|u^{(4)}\right\|_0, \quad -u''(t) \ge \widehat{k}G_{1,1}(t,t)\|u''\|_0, \quad u(t) \ge \widehat{k}G_{1,1}(t,t)\|u\|_0$, and $u(t)+|u''(t)|+u^{(4)}(t) \ge r\widehat{k}G_{1,1}(t,t), \ t \in [0,1]$. We recall that $N_r = \widehat{Q}_2 g_1(v, r\widehat{k}G_{1,1}(t,t))$ and $0 < \mu \le \min\{\frac{1-L_2}{N_r}, \frac{(1-L_1)(1-L_2)}{M_r C_{1,1}C_{1,2}D_5+c_2 D_6 C_{1,1}N_r}\}$.

We now show that

$$0 \le \mu u(t) H_2 g(v, u(v), u''(v), u^{(4)}(v))(t) \le r, \quad \forall t \in [0,1].$$

To see this, since $\mu \le \frac{1-L_2}{N_r}$ and by (25), we have

$$\mu u(s)\, H_2 g(v, u(v), u''(v), u^{(4)}(v))(s) \ \le \mu\|u\|_0 \frac{1}{1-L_2} Q_2 g(v, u(v), u''(v), u^{(4)}(v)(t))(s)$$

$$\le \mu r \frac{1}{1-L_2} Q_2 g_1(v, r\widehat{k}G_{1,1}(t,t)) \le \frac{\mu r}{1-L_2}\widehat{Q}_2 g_1(v, r\widehat{k}G_{1,1}(t,t)) = \frac{\mu r N_r}{1-L_2} \le r.$$

So, using by (43), we have

$$f_2(\mu\ u(s) H_2 g(v, u(v), u''(v), u^{(4)}(v)(t))) \le c_1 \left(\mu u(s) H_2 g(v, u(v), u''(v), u^{(4)}(v)(t))\right), \quad \text{if} \quad \mu \le \frac{1-L_2}{N_r}.$$

We recall that $\widehat{a}(v) = u(v) + |u''(v)| + u^{(4)}(v)$.
Thus, by Lemma 3, $(H1)$,$(H2)$,$(H3)$ and $(H4)$, we have

$$Tu(t) = H_1(u(s)\mu H_2 g(v, u(v), u''(v), u^{(4)}(v))(s) + f(s, u(s), \mu H_2 g(v, u(v), u''(v), u^{(4)}(v))(s))$$

$$\le \frac{1}{1-L_1} Q_1(u(s)\mu H_2 g(v, u(v), u''(v), u^{(4)}(v))(s) + f(s, u(s), \mu H_2 g(v, u(v), u''(v), u^{(4)}(v))(s))$$

$$= \frac{\mu}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)u(\tau)H_2 g(v,u,u'',u^{(4)})(\tau)d\tau ds dv +$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f(\tau,u(\tau),\mu H_2 g(v,u,u'',u^{(4)})(\tau))d\tau ds dv$$

$$\le \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)u(\tau)Q_2 g(v,u,u'',u^{(4)})(\tau)d\tau ds dv +$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f(s,u(s),\mu H_2 g(v,u,u'',u^{(4)})(\tau))d\tau ds dv$$

$$= \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)u(s)\int_0^1\int_0^1\int_{\widehat{a}(v)\le a} \cdot$$

$$\cdot G_{1,2}(s,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,u(v),u''(v),u^{(4)}(v))\, dvdzdw\, ds\, d\tau\, dx +$$

$$+\frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)u(s)\int_0^1\int_0^1\int_{\widehat{a}(v)\le a} \cdot$$

$$\cdot G_{1,2}(s,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,u(v),u''(v),u^{(4)}(v))\, dvdzdw\, ds\, d\tau\, dx +$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f_1(\tau)f_2(\mu u(\tau)H_2 g(\tau,u,u'',u^{(4)})(\tau))d\tau ds dv$$

$$\leq \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\int_{\widehat{a}(v)\leq a}\cdot$$
$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g_1(v,r\widehat{k}G_{1,1}(v,v))\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\sup_{y\in(0,R]}\sup_{e\in(0,R]}\sup_{p\in[r,R]}\int_{\widehat{a}(v)\geq a}\cdot$$
$$\cdot G_{1,2}(s,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,y,e,p)\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f_1(\tau)c_1\mu u(\tau)H_2g(\tau,u,u'',u^{(4)})(\tau)d\tau dsdv$$
$$\leq \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\int_0^1\cdot$$
$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g_1(v,r\widehat{k}G_{1,1}(v,v))\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\sup_{y\in(0,R]}\sup_{e\in(0,R]}\sup_{p\in[r,R]}\int_0^1\cdot$$
$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,y,e,p)\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f_1(\tau)c_1\mu\|u\|_0\frac{1}{1-L}Q_2g(v,u(v),u''(v),u^{(4)}(v)(t))(s)d\tau dsdv$$
$$\leq \frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\int_0^1\cdot$$
$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g_1(v,r\widehat{k}G_{1,1}(v,v))\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{\mu}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\|u\|_0\int_0^1\int_0^1\sup_{y\in(0,R]}\sup_{e\in(0,R]}\sup_{p\in[r,R]}\int_0^1\cdot$$
$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,y,e,p)\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{1}{1-L_1}\int_0^1\int_0^1\int_0^1 G_{1,1}(t,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f_1(\tau)c_1\mu\,\|u\|_0\,\frac{1}{1-L_2}Q_2g_1(v,r\widehat{k}G_{1,1}(t,t))d\tau dsdv$$
$$\leq \frac{\mu\,\|u\|_0}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 C_{1,1}G_{1,1}(x,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\int_0^1\int_0^1\int_0^1\cdot$$
$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g_1(v,r\widehat{k}G_{1,1}(v,v))\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{\mu\,\|u\|_0}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 C_{1,1}G_{1,1}(x,x)G_{2,1}(x,\tau)G_{3,1}(\tau,s)\int_0^1\int_0^1\sup_{y\in(0,R]}\sup_{e\in(0,R]}\sup_{p\in[r,R]}\int_0^1\cdot$$
$$\cdot C_{1,2}G_{1,2}(w,w)G_{2,2}(w,z)G_{3,2}(z,v)g(v,y,e,p)\,dvdzdw\,ds\,d\tau dx+$$

$$+\frac{c_1\mu\,\|u\|_0}{(1-L_1)(1-L_2)}\int_0^1\int_0^1\int_0^1 C_{1,1}G_{1,1}(v,v)G_{2,1}(v,s)G_{3,1}(s,\tau)f_1(\tau)\widehat{Q}_2g_1(v,r\widehat{k}G_{1,1}(t,t))d\tau dsdv$$

$$\leq \frac{\mu(D_1C_{1,1}C_{1,2}M_r + c_1D_2C_{1,1}N_r)}{(1-L_1)(1-L_2)}\|u\|_0$$

$$\leq \frac{\mu(D_5C_{1,1}C_{1,2}M_r + c_1D_6C_{1,1}N_r)}{(1-L_1)(1-L_2)}\|u\|_0 \leq \|u\|_0, \quad \forall u \in \partial B_r \cap P, \quad t \in [0,1].$$

Consequently,

$$\|Tu\|_0 \leq \|u\|_0 \leq \|u''\|_0, \qquad \forall u \in \partial B_r \cap P. \tag{44}$$

We have a similar type inequality for $\|(Tu)''\|_0$ and $\|(Tu)^{(4)}\|_0$:

$$\|(Tu)''\|_0 \leq \|u''\|_0, \qquad \|(Tu)^{(4)}\|_0 \leq \left\|u^{(4)}\right\|_0, \qquad \forall u \in \partial B_r \cap P. \tag{45}$$

This proves one of assumptions appearing Lemma 1.

(II) Secondly, we will prove that the second part of assumptions (i) of Lemma 1 is satisfied. To do this, by condition $(H4)$ there exists $R_2 > 0$ such that

$$f(t,u,w) \geq c_2 w, \quad \forall u \in R^+, \quad w \geq R_2, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Let us choose $c_3 \geq \frac{1}{c_2\mu\sigma D_8}$. Then by condition $(H5)$, there exists $R_1 > \frac{R_2}{\mu c_3 \delta_1 K_2 D_7} > 0$ such that,

$$g(t,u,v,w) \geq c_3 w, \quad \forall u, \in R^+, \quad \forall v \in R^-, \quad w \geq R_1, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Let $R > \max\{\frac{R_1}{\sigma}, a\}$. Let $u \in \partial B_R \cap P$, i.e. $\left\|u^{(4)}\right\|_0 = R$. Thus, using by (32) we have

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} u^{(4)}(t) \geq \sigma \left\|u^{(4)}\right\|_0 = \sigma R > R_1, \ \forall u \in \partial B_R \cap P.$$

It is easy to verify that

$$\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s) \geq \mu Q_2 g(v,u(v),u''(v),u^{(4)}(v))(s)$$

$$\geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,2}(s,v)G_{2,2}(v,z)G_{3,2}(z,x)g(x,u(x),u''(x),u^{(4)}(x))dxdzdv$$

$$\geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,2}(s,v)G_{2,2}(v,z)G_{3,2}(z,x)c_3R_1 dxdzdv$$

$$\geq \mu c_3 R_1 \delta_1 G_{1,2}(s,s) \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,2}(v,v)G_{2,2}(v,z)G_{3,2}(z,x)dxdzdv$$

$$\geq \mu c_3 R_1 \delta_1 \min_{s\in[\frac{1}{4},\frac{3}{4}]} G_1(s,s)D_7 = \mu c_3 R_1 \delta_1 \widehat{K}_2 D_7 > R_2, \quad s \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

where $\widehat{K}_2 = \min_{s\in[\frac{1}{4},\frac{3}{4}]} G_{1,2}(s,s)$.

Then, by Lemma 3, $(H1)$ and $(H5)$, we have

$$Tu(\frac{1}{2}) = (H_1(u(s)\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s) + f(s,u(s),\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s)) \,|_{t=\frac{1}{2}}$$

$$\geq (Q_1(u(s)\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s) + f(s,u(s),\mu H_2 g(v,u(v),u''(v),u^{(4)}(v))(s))) \,|_{t=\frac{1}{2}}$$

$$= \mu \int_0^1 \int_0^1 \int_0^1 G_{1,1}(\frac{1}{2},v)G_{2,1}(v,s)G_{3,1}(s,\tau)u(\tau)H_2 g(v,u,u'',u^{(4)})(\tau)d\tau dsdv +$$

$$+\int_0^1\int_0^1\int_0^1 G_{1,1}(\frac{1}{2},v)G_{2,1}(v,s)G_{3,1}(s,\tau)f(\tau,u(\tau),\mu H_2 g(v,u,u'',u^{(4)})(\tau))d\tau dsdv$$

$$=\mu\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,1}(\frac{1}{2},v)G_{2,1}(v,s)G_{3,1}(s,\tau)u(\tau)H_2 g(v,u,u'',u^{(4)})(\tau)d\tau dsdv+$$

$$+\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,1}(\frac{1}{2},v)G_{2,1}(v,s)G_{3,1}(s,\tau)f(\tau,u(\tau),\mu H_2 g(v,u,u'',u^{(4)})(\tau))d\tau dsdv$$

$$\geq c_2\mu\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,1}(\frac{1}{2},v)G_{2,1}(v,s)G_{3,1}(s,\tau)H_2 g(v,u,u'',u^{(4)})(\tau))d\tau dsdv$$

$$\geq c_2\mu\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,1}(\frac{1}{2},v)G_{2,1}(v,s)G_{3,1}(s,\tau)Q_2 g(v,u,u'',u^{(4)})(\tau))d\tau dsdv$$

$$\geq c_2\mu\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,1}(\frac{1}{2},y)G_{2,1}(y,\tau)G_{3,1}(\tau,s)\cdot$$

$$\cdot\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,2}(s,v)G_{2,2}(v,z)G_{3,2}(z,x)g(x,u(x),u''(x),u^{(4)})(x)dxdzdv\,dsd\tau dy$$

$$\geq c_2c_3\mu\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,1}(\frac{1}{2},y)G_{2,1}(y,\tau)G_{3,1}(\tau,s)\cdot$$

$$\cdot\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,2}(s,v)G_{2,2}(v,z)G_{3,2}(z,x)u^{(4)}(x)dxdzdv\,dsd\tau dy$$

$$\geq c_2c_3\mu\sigma\|u^{(4)}\|_0\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,1}(\frac{1}{2},y)G_{2,1}(y,\tau)G_{3,1}(\tau,s)\cdot$$

$$\cdot\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_{1,2}(s,v)G_{2,2}(v,z)G_{3,2}(z,x))dxdzdv\,dsd\tau dy$$

$$\geq c_2c_3\mu\sigma D_8\|u^{(4)}\|_0\geq\|u^{(4)}\|_0,$$

so

$$(Tu)(\frac{1}{2})\geq\|u^{(4)}\|_0,\qquad\forall u\in\partial B_R\cap P.$$

Consequently,

$$\|u^{(4)}\|_0\leq\|Tu\|_0\leq\left\|(Tu)^{(4)}\right\|_0,\qquad\forall u\in\partial B_R\cap P.\tag{46}$$

(III) Finally, we will prove that $T:P\cap(\overline{B}_R\backslash B_r)\to P$ is a completely continuous operator. By Lemma 7, the Ascoli-Arzela theorem guarantees that $T:P\cap(\overline{B}_R\backslash B_r)\to P$ is a completely continuous.

Then due to Lemma 1, by (45) and (46) inequality we see that the problem (3) has at least one positive solution.

This finishes the proof. ∎

4. Conclusions

This paper investigates the existence of positive solutions for a nonlinear sixth-order differential system using a fixed point theorem of cone expansion and compression type of norm type. The nonlinear terms may be singular with respect to the time and space variables. The problem comes from the deformation analysis of an elastic circular ring segment in the equilibrium state. The results obtained herein generalize and improve some known results including singular and non-singular cases.

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