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Research Paper

Theory of Quotient Matrix

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ABSTRACT
Generally, when we discuss operations on matrices, we introduce addition, subtraction, scalar multiplication and even multiplication. We never discuss the concept of division of two square matrices. In this paper, I have introduced the concept of division of two square matrices under certain conditions. In fact, we have introduced the term 'Quotient Matrix' $\frac{A}{B}$ *for two square matrices A and B of the same order provided AB = BA and B is a non-singular matrix. We have also established all the parallel results for 'Quotient Matrix' related to algebra of Quotient Matrices, adjoint of a Quotient matrix, inverse of a Quotient matrix and determinant of a quotient matrix.*

KEYWORDS
Quotient Matrix

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I. INTRODUCTION

- **Why do we stop at matrix multiplication while doing algebra of matrices?**
- **Why did we not talk about matrix division?**

<u>NOTE:</u>Let us go to real number system where we learnt division. If *a* and b are two real numbers, b ≠ 0 then $\frac{a}{b}$ is defined as solution of the equations b*x* = *a* and *x*b = *a*. We know that b*x* = *x*b (by commutativity in real numbers), therefore uniqueness of *a* is preserved so the quotient $\frac{a}{b}$ is meaningful.

Now, if we consider A and B as two square matrices (of same order), B ≠ 0 and suppose $\frac{A}{B} = C$ (where C is a square matrix of same order as of A and B) then A = BC or A = CB are the consequent matrix equation BC ≠ CB (in general), therefore uniqueness of A gets violated. Hence $\frac{A}{B}$ does not make sense in case of matrices.

II. MATERIAL AND METHODS

Let A and B are two square matrices of order n ×n such that:

(i) AB = BA (ii) | B | ≠ 0

Then we define quotient matrix $\frac{A}{B}$ as a matrix C of order n ×n such that C = AB^{-1}

NOTE: If A and B are commuting matrices (of same order) then $AB^{-1} = B^{-1}A$.

$\therefore$ C is uniquely determined.

Example 1: $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, $B = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Then AB = BA and |B| = 1 ($\neq 0$)

$\therefore \frac{A}{B} = C$ where C = AB^{-1}

$$C = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Example 2: $A = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Then AB = BA and |B| = 2 ($\neq 0$)

$\therefore \frac{A}{B} = C$ where C = AB^{-1}

$$C = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 7/2 \end{bmatrix}$$

Algebra of Quotient Matrices

Let $\frac{A}{B}$ and $\frac{C}{D}$ are two quotient matrices (where A, B, C and D are square matrices of order n), then we can define.

$\frac{A}{B} + \frac{C}{D}$, $\frac{A}{B} - \frac{C}{D}$, $\alpha \frac{A}{B}$ (for α to be a scalar) and $\frac{A}{B} \cdot \frac{C}{D}$ in usual manner in which we have defined X + Y, X – Y, αX and XY. (for two suitable matrices x and y)

Some more properties which one can easily verify are:

(1) $adj\, \alpha \left(\frac{A}{B}\right) = \alpha^{n-1}\, adj. \left(\frac{A}{B}\right)$, where α is a scalar and n is the order of the square matrices A and B.

(2) For $n \in \square^{+}$, $adj\left(\frac{A}{B}\right)^{n} = \left(adj\left(\frac{A}{B}\right)\right)^{n}$

(3) $adj\left(\frac{A}{B} \cdot \frac{C}{D}\right) = adj\, \frac{C}{D} \cdot adj\, \frac{A}{B}$

(where *adj* denotes adjoint)

III. RESULTS AND DISCUSSION

Result 1: $\left|\frac{A}{B}\right| = \frac{|A|}{|B|}$. (where |A| = determinant of matrix A)

Proof: Consider, $\left|\frac{A}{B}\right| = |AB^{-1}| = |A||B^{-1}| = \frac{|A|}{|B|}$.

Result2: For $n \in \mathbb{Z}^+$, we have $\left|\left(\frac{A}{B}\right)^n\right| = \left|\frac{A}{B}\right|^n$.

Proof: Consider,

$$\left|\left(\frac{A}{B}\right)^n\right| = |(AB^{-1})^n| = |A^n(B^{-1})^n| \qquad (\because AB^{-1} = B^{-1}A)$$

$$= |A^n|\,|(B^{-1})^n| = \frac{|A|^n}{|B|^n} = \left|\frac{A}{B}\right|^n \qquad \text{(By Prop.1)}$$

Result3: $\left|adj\left(\frac{A}{B}\right)\right| = \frac{|adj\ A|}{|adj\ B|}$.

Proof: Consider,

$$\left|adj\left(\frac{A}{B}\right)\right| = |adj\ AB^{-1}| = |adj\,(B^{-1}).adj\ A|$$

$$= |adj\,(B^{-1})|\,|adj\ A|$$

$$= |adj\ B|^{-1}\,|adj\ A|$$

$$= \frac{|adj\ A|}{|adj\ B|}$$

(**NOTE:** $|B| \neq 0 \Rightarrow |adj\ B| \neq 0$)

Result4: For two quotient matrix $\frac{A}{B}$ and $\frac{C}{D}$, we have

$$\left|adj\left(\frac{A}{B}.\frac{C}{D}\right)\right| = \frac{|adj\,(CA)|}{|adj\,(DB)|} = \frac{|adj\,(AC)|}{|adj\,(BD)|}.$$

Proof: Consider,

$$\left|adj\left(\frac{A}{B}.\frac{C}{D}\right)\right| = \left|adj\ \frac{C}{D}.adj\ \frac{A}{B}\right|$$

$$= \left|adj\ \frac{C}{D}\right|\,\left|adj\ \frac{A}{B}\right|$$

$$= \frac{|adj\ C|}{|adj\ D|}\,\frac{|adj\ A|}{|adj\ B|} \qquad \text{.....(1)}$$

$$= \frac{|adj\ C\ .\ adj\ A|}{|adj\ D\ .\ adj\ B|}$$

$$= \frac{|adj\ (AC)|}{|adj\ (BD)|} = \left|\frac{adj\ (CA)}{adj\ (DB)}\right| \qquad \text{By (1)}$$

Inverse of quotient matrix:

Let $\frac{A}{B}$ be any quotient matrix with $|A| \neq 0$, then we say $\frac{A}{B}$ is invertible if $\exists$ a quotient matrix $\frac{C}{D}$ such that $\frac{A}{B} \cdot \frac{C}{D} = I = \frac{C}{D} \cdot \frac{A}{B}$.

In this case, $\frac{C}{D} = \left(\frac{A}{B}\right)^{-1}$

Now, $\boxed{\left(\frac{A}{B}\right)^{-1} = (AB^{-1})^{-1} = BA^{-1}}$

Prop.1 $\left(\left(\frac{A}{B}\right)^{-1}\right)^{-1} = \frac{A}{B}$

Proof: LHS = $\left(\left(\frac{A}{B}\right)^{-1}\right)^{-1} = \left(\left(AB^{-1}\right)^{-1}\right)^{-1} = \left(BA^{-1}\right)^{-1} = AB^{-1} = \frac{A}{B}$

Prop.2 $\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^{-1} = I = \left(\frac{A}{B}\right)^{-1}\left(\frac{A}{B}\right)$

Proof: Consider,

$$\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^{-1} = \left(AB^{-1}\right)\left(AB^{-1}\right)^{-1}$$

$$= \left(AB^{-1}\right)\left(BA^{-1}\right)$$

$$= A\left(B^{-1}B\right)A^{-1} = AA^{-1} = I$$

Similarly, $\left(\frac{A}{B}\right)^{-1}\left(\frac{A}{B}\right) = \left(\left(AB^{-1}\right)^{-1}\right)\left(AB^{-1}\right)$

$$= B\left(A^{-1}A\right)B^{-1} = I$$

Result5: In addition, if A is non-singular then $\left|\left(\frac{A}{B}\right)^{-1}\right| = \frac{|B|}{|A|}$.

Proof: Consider,

$$\left|\left(\frac{A}{B}\right)^{-1}\right| = \left|\left(AB^{-1}\right)^{-1}\right| = \left|\left(B^{-1}\right)^{-1}A^{-1}\right| = |B|\ \left|A^{-1}\right| = \frac{|B|}{|A|}$$

Result6: Further, if $|C| \neq 0$, we can see $\left|\frac{\left(\frac{A}{B}\right)}{\left(\frac{C}{D}\right)}\right| = \frac{|AD|}{|BC|} = \frac{|DA|}{|CB|}$

Proof: $\left|\frac{\left(\frac{A}{B}\right)}{\left(\frac{C}{D}\right)}\right| = \left|\left(\frac{A}{B}\right)\left(\frac{C}{D}\right)^{-1}\right| = \left|\left(AB^{-1}\right)\left(CD^{-1}\right)^{-1}\right|$

$$= \left|AB^{-1}\right|\left|DC^{-1}\right|$$

$$= \left|A\right| \left|B^{-1}\right| \left|D\right| \left|C^{-1}\right|$$

$$= \frac{\left|A\right|}{\left|B\right|} \frac{\left|D\right|}{\left|C\right|}$$

$$= \frac{|AD|}{|BC|} = \frac{|DA|}{|CB|}$$

Assumptions:

(i) $|C| \neq 0 \Rightarrow \frac{|C|}{|D|} \neq 0$ as $= \left|\frac{C}{D}\right| = \frac{|C|}{|D|} \neq 0$

(ii) $\frac{A}{B} \cdot \frac{C}{D} = \frac{C}{D} \cdot \frac{A}{B}$

Prob.1: If $\left(\frac{A}{B}\right)^n = I$ for some positive integer n, then show that $\left(\frac{A}{B}\right)^{-1}$ exists.

Sol. Given: $\left(\frac{A}{B}\right)^n = I$

$\Rightarrow \quad \left|\left(\frac{A}{B}\right)^n\right| = |I|$

$\Rightarrow \quad \left|\frac{A}{B}\right|^n = 1$

$\Rightarrow \quad \left|\frac{A}{B}\right| = \pm 1$

$\Rightarrow \quad |A| \neq 0 \qquad (\because \ |B| \neq 0)$

$\therefore \quad \left(\frac{A}{B}\right)^{-1}$ exists.

Prob.2: If $\left(\frac{A}{B}\right)$ is a 3×3 quotient matrix, suchthat $\left|\frac{A}{B}\right| = 4$. Find $\left|2 \ adj \ \frac{A}{B}\right|$.

Sol. Consider,

$$\left|2adj\,\frac{A}{B}\right| = 8\left|adj\,\frac{A}{B}\right| = 8\left(\left|\frac{A}{B}\right|\right)^{3-1}$$

$$= 8 \times \left(\left|\frac{A}{B}\right|\right)^2$$

$$= 8 \times 16$$

$$= 128$$

Prob.3: $\frac{A}{B} \ adj\left(\frac{A}{B}\right) = \left|\frac{A}{B}\right| I = adj\left(\frac{A}{B}\right) \frac{A}{B}$.

Sol. Consider,

$$\frac{A}{B} \ adj\left(\frac{A}{B}\right)$$

$$= \left(AB^{-1}\right) adj\left(AB^{-1}\right)$$

$$= \left(AB^{-1}\right) adj\left(B^{-1}\right) adj \ A$$

$$= A\left(B^{-1} adj\left(B^{-1}\right)\right) adj \ A$$

$$= A \left|B^{-1}\right| I \ adj \ A \quad (\because A\cdot \text{adj } A = |A|I)$$

$$= \left|B^{-1}\right| (A \ adj \ A)$$

$$= \frac{|A|}{|B|} I$$

Similarly, $$adj\left(\frac{A}{B}\right) . \frac{A}{B} = \frac{|A|}{|B|} I$$

Prob.4: If $\frac{A}{B}$ is a 3×3 matrix satisfying $\left|\frac{A}{B}\right| = 1$ and $\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^T = I$. Prove that $\left|\frac{A}{B} - I\right| = 0$.

Sol. Consider,

$$\left|\frac{A}{B} - I\right| = \left|\frac{A}{B} - \left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^T\right|$$

$$= \left|\frac{A}{B}\left(I - \left(\frac{A}{B}\right)^T\right)\right|$$

$$= \left|\frac{A}{B}\right| \left|I^T - \left(\frac{A}{B}\right)^T\right|$$

$$= \left|\left(I - \left(\frac{A}{B}\right)\right)^T\right|$$

$$= \left|I - \frac{A}{B}\right| \quad (\square\ |A| = |A^T|)$$

$$= (-1)^3 \left|\frac{A}{B} - I\right|$$

$$\Rightarrow \quad 2\left|\frac{A}{B} - I\right| = 0$$

$$\Rightarrow \quad \left|\frac{A}{B} - I\right| = 0$$

Prob.5: If $\frac{A}{B}$ is a matrix order 2, such that $\left|2\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^T\right| = 16$ and $|B| = 1$. Find | A |.

Sol. Given,

$$\left|2\left(\frac{A}{B}\right)\left(\frac{A}{B}\right)^T\right| = 16$$

$$\Rightarrow \quad 4\left|\frac{A}{B}\right| \left|\left(\frac{A}{B}\right)^T\right| = 16$$

$$\Rightarrow \quad \left|\frac{A}{B}\right|^2 = 4 \quad \left(\because \left|\frac{A}{B}\right| = \left|\left(\frac{A}{B}\right)^T\right|\right)$$

$$\Rightarrow \quad \left|\frac{A}{B}\right| = \pm 2$$

$$\Rightarrow \quad \frac{|A|}{|B|} = \pm 2$$

$$\Rightarrow \quad |A| = \pm 2$$

Prob.6: If $\frac{A}{B}$ is a 2×2non-singular matrix, show that $adj\left(\frac{A}{B}\right) = adj\left(\frac{A}{B}\right)^{-1}$. Find $\left|\frac{A}{B}\right|$.

Sol. Given,

$$\left|adj\left(\frac{A}{B}\right)\right| = \left|adj\left(\frac{A}{B}\right)^{-1}\right|$$

$$\Rightarrow \left|adj\left(\frac{A}{B}\right)\right| = \left|\left(adj\left(\frac{A}{B}\right)\right)\right|^{-1} = \frac{1}{\left|adj\left(\frac{A}{B}\right)\right|}$$

$$\Rightarrow \left|adj\left(\frac{A}{B}\right)\right|^{2} = 1$$

$$\Rightarrow \left|adj\left(\frac{A}{B}\right)\right| = \pm 1$$

$$\Rightarrow \left|\frac{A}{B}\right| = \pm 1 \left(\because \left|adj\frac{A}{B}\right| = \left|\frac{A}{B}\right|^{n-1}\right)$$

$$\Rightarrow \left|\frac{A}{B}\right| = \pm 1$$

Prob.7: Let $\frac{A}{B}$ and $\frac{C}{D}$ are two non-singular matrixof order 2×2, such that $|B| = 2 = |D|$, $\left(\frac{A}{B}\right)^2 = \frac{A}{B}.\frac{C}{D}$ and $\left(\frac{C}{D}\right)^2 = \left(\frac{A}{B}.\frac{C}{D}\right)^{-1}$. Find | A | and | C |.

Sol. Given,

$$\left(\frac{A}{B}\right)^2 = \frac{A}{B}.\frac{C}{D}$$

$$\Rightarrow \left|\frac{A}{B}\right|^2 = \left|\frac{A}{B}\right| \left|\frac{C}{D}\right|$$

$$\Rightarrow \left|\frac{A}{B}\right| = \left|\frac{C}{D}\right|$$

Also, $$\left|\left(\frac{C}{D}\right)\right|^2 = \left|\frac{A}{B}.\frac{C}{D}\right|^{-1}$$

$$\Rightarrow \left|\frac{C}{D}\right|^2 = \frac{1}{\left|\frac{A}{B}\right|\left|\frac{C}{D}\right|}$$

$$\Rightarrow \left|\frac{C}{D}\right|^4 = 1 \qquad \left(\because \left|\frac{A}{B}\right| = \left|\frac{C}{D}\right|\right)$$

$$\Rightarrow \left|\frac{C}{D}\right| = \pm 1$$

$$\therefore \left|\frac{A}{B}\right| = \left|\frac{C}{D}\right| = \pm 1$$

Also, $$\left|\frac{A}{B}\right| = \pm 1 \quad \Rightarrow \quad \frac{|A|}{|B|} = \pm 1$$

$$\Rightarrow |A| = \pm |B| \qquad \therefore |A| = \pm 2$$

Similarly, $|C| = \pm 2$

Prob.8: $\left(\left(\dfrac{A}{B}\right)^{T}\right)^{-1} = \left(\left(\dfrac{A}{B}\right)^{-1}\right)^{T}$.

Sol. Consider,

$$\left(\left(\frac{A}{B}\right)^{-1}\right)^{T} = \left(\left(AB^{-1}\right)^{-1}\right)^{T}$$

$$= \left(\left(AB^{-1}\right)^{T}\right)^{-1}$$

$$= \left(\left(\frac{A}{B}\right)^{T}\right)^{-1}$$

Prob.9: For any quotient matrix $\dfrac{A}{B}$ (with $|A| \neq 0$) and a non-zero scalar α, we have:

$$\left(\alpha\frac{A}{B}\right)^{-1} = \frac{1}{\alpha}\left(\frac{A}{B}\right)^{-1}. \text{ (Verify yourself)}$$

Prob.10: For two quotient matrix $\dfrac{A}{B}$ and $\dfrac{C}{D}$ (with $|A| \neq 0$, $|C| \neq 0$) (for which $\dfrac{A}{B}.\dfrac{C}{D}$ is defined), we have:

$$\left(\frac{A}{B}.\frac{C}{D}\right)^{-1} = \left(\frac{C}{D}\right)^{-1}\left(\frac{A}{B}\right)^{-1}. \text{ (Verify yourself)}$$

IV. CONCLUSION

The concept of division of two square matrices can be defined under certain assumed conditions. In fact, we can talk about the quotient matrix '$\dfrac{A}{B}$' and verify that all the parallel results related to algebra of matrices, adjoint of a matrix, inverse of a matrix and determinant of a matrix hold true in case of quotient matrix.

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