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Research Paper

Monotone Method for Riemann – Liouville Fractional Differential Equations

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ABSTRACT: *The main purpose of this paper is to review the work on monotone method for Riemann-Liouville fractional differential equations. Monotone method developed for Riemann-Liouville fractional differential equation with integral boundary conditions and initial conditions is considered. The developed monotone method is applied to obtain existence and uniqueness of solutions of problem.*
KEYWORDS:*Riemann – Liouville fractional differential equation, monotone method, existence and uniqueness, lower and upper solution, integral boundary value problem, initial value problem*.

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I. INTRODUCTION

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelof system and processes in the field of physics, chemistry, aerodynamics, electrodynamics of complex medium, etc. (see [1 – 4] and references therein). The basic theory of fractional differential equation involving Riemann – Liouville fractional derivative is developed[4 – 9] by Lakshmikantham et. al.
Numerous methods have been proposed in literature to study the fractional differential equations such as power series method [2], compositional method [2], variationalLypanov method [10], quasilinearization method [11], homotopy analysis method [12], homotopy perturbation method [13], collocation method [14], finite difference method [15], Adomain decomposition method [16 – 18], iteration method [19, 20] and monotone method [5, 8, 9, 21] etc.
Among the available techniques,monotone method is powerful technique for nonlinear equations and dynamical systems because it reduce the problem to sequences of linear equations. Many researchers attracted towards the monotone method and developed it for initial value problems [9, 25, 27, 28, 36], boundary value problems [37, 38], integral boundary value problems [21 – 24, 30 – 35] and periodic boundary value problems [39, 40] and proved existence and uniqueness of solutions of these problems. In this paper monotone method developed for Riemann-Liouville fractional differential equations with initial conditions and integral boundary conditions is reviewed. The rest of the paper is presented as under: Section II is devoted for the basic definitions and results that are used to develop monotone method for the problem under investigation. In section III, monotone method for fractional integral boundary value problem (IBVP), fractional initial value problem (IVP) is considered. In section IV, monotone method for system of fractional differential equations is considered and its applications are considered. Last section highlights the results on monotone method for finite system of fractional differential equations.

II. PRELIMINARIES

In this section, we discuss some basic definitions and results which are required to develop monotone method for Riemann-Liouville fractional differential equations with initial conditions and integral boundary conditions etc.

Let $q \in R_+$ and $n = [q]$, where $[\cdot]$ is the greatest integer function.

Definition 2.1: The Riemann – Liouville fractional integral of order q is defined as

$$ {}_aI_t^q u(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} u(\tau)\, d\tau $$

Definition 2.2: The Riemann – Liouville fractional derivative of order q is defined as

$$ {}_aD_t^q u(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-q-1} u(\tau)\, d\tau, \text{ for } a \le t \le b. $$

For n = 1, we have

$$ {}_aD_t^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t (t-\tau)^{-q} u(\tau)\, d\tau,\ 0<q<1. $$

If $u(t)$ is continuous for $t \ge a$, then fractional derivative and fractional integral has the following properties:

1. ${}_aI_t^p \left[{}_aI_t^q u(t) \right] = {}_aI_t^{p+q} u(t)$
2. ${}_aD_t^q \left[{}_aI_t^q u(t) \right] = u(t),$ for $q > a$ and $t > a$
3. ${}_aD_t^q \left[\lambda u_1(t) + \mu\, u_2(t) \right] = \lambda \left[{}_aD_t^q u_1(t) \right] + \mu \left[{}_aD_t^q u_2(t) \right]$

Lemma 2.1:[6] Let $m \in C_p[J,R]$ be locally Holder continuous with exponent $\lambda > q$ and for $t_1 \in (t_0,T]$ we have $m(t_1) = 0$ and $m(t) \le 0$ for $t_0 \le t \le t_1$. Then $D^q m(t_1) \ge 0$.

Lemma 2.2:[6]Let $\{u_\in(t)\}$ be a family of continuous functions of J , for each $\in > 0$ where $D^q u_\in(t) = f(t,u_\in(t)),\ D^q u_\in(t) = f(t,u_\in(t)),\ for\ \ u_\in(t_0) = u_\in(t)(t-t_0)^{1-q}\Big\}_{t=t_0}$ and $\left| f(t,u_\in(t)) \right| \le M$ for $t_0 \le t \le T$. Then the family $\{u_\in(t)\}$ is equi-continuous on $[t_0,T]$.

The Lemma 2.1 is improved by J.V.Devi et.al [10] for the class of continuous functions which is stated below:

Lemma 2.3:[10] Let $m \in C_p([t_0,T],R)$ and for any $t_1 \in (t_0,T)$ we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \le t \le t_1$. Then it follows that $D^q m(t_1) \ge 0$.

III. MONOTONE METHOD FOR RIEMANN – LIOUVILLE FRACTIONAL DIFFERENTIAL EQUAITONS

In this section we consider the Monotone method developed by Wang et.al. [21]forfractional integral boundary value problem (IBVP), McRae [9] for fractional initial value problem (IVP) and monotone method developed by Nanwareet.al.

In 2008, Wang and Xie [21] developed monotone method for following IBVP with Holder continuity and obtained existence and uniqueness of solution of the problem

$$ \left. \begin{aligned} & D^q u(t) = f(t,u),\ t \in J = [0,T],\ \ T \ge 0 \\ & u(0) = \lambda \int_0^T u(s)\, d(s), \quad d \in R \end{aligned} \right\} \qquad (3.1) $$

where $0 < q < 1,\ \lambda = \pm 1$ and $f \in C[J \times R,\ R]$, D^q is the Riemann – Liouville fractional derivative of order q.

Nanware and Dhaigude [23] have improved results obtained by Wang et.al. [21] for the IBVP (3.1) without locally Holder continuous functions. The improvedexistence and uniqueness results are

Theorem 3.1:[23] Assume that:

(i) $f(t,\ u(t))$ isnon-decreasing in *u* for each *t*.

(ii) $v_0(t)$ and $w_0(t)$ in $C_p(J,R)$ are lower and upper solutions of (3.1) such that $v_0(t) \le w_0(t)$ on $J=[0,T]$

(iii) $f(t,u)$ satisfies one-sided Lipschitz condition,

$f(t,u) - f(t,v) \le -L(u-v), \; L \ge 0$

Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_p(J,R)$ such that $\{v_n(t)\} \to v(t)$ and $\{w_n(t)\} \to w(t)$ as $n \to \infty$, where $v(t)$ and $w(t)$ are minimal and maximal solutions of (3.1) respectively.

Theorem 3.2:[23] Assume that:

(i) $f(t,u(t))$ in $C[J \times R, R]$ is non-decreasing in *u* for each t.

(ii) $v_0(t)$ and $w_0(t)$ in $C(J,R)$ are lower and upper solutions of (3.1) such that $v_0(t) \le w_0(t)$ on J.

(iii) Functions $f(t,u)$ satisfy Lipschitz conditions, $|f(t,u) - f(t,v)| \le L|u-v|, \; L \ge 0$

(iv) $\lim_{n\to\infty} \|w_n(t) - v_n(t)\| = 0$, where the norm is defined by $\|f\| = \int_0^T |f(s)|\,ds,$

then the solution of (3.1) is unique.

In 2008, J. Vasundhara Devi developed [39] general monotone method of periodic boundary value problems (PBVP) of Caputo fractional differential equations when the function is sum of non-decreasing and non-increasingfunctions. Nanware and Dhaigude developed monotone method [24] for Riemann – Liouville IBVP (3.2).

$$\left.\begin{aligned} D^q u(t) &= f(t,u) + g(t,u), \; t \in J = [0,T], \; T \ge 0 \\ u(0) &= \lambda \int_0^T u(s)\,ds + d, \;\; d \in R \end{aligned}\right\} \qquad (3.2)$$

where $0 < q < 1, \; \lambda = \pm 1, \; f, g \in C[J \times R, R], \; f$ is non-decreasing and *g* is non-increasing, D^q is the Riemann-Liouville fractional derivative of order *q*. The developed monotone method is used to obtain existence and uniqueness of the problem (3.2). The developed monotone method existence and uniqueness results are stated here.

Theorem 3.3:[24] Assume that:

(i) $f(t,u(t))$ and $g(t,u(t))$ in $C[J \times R, R]$, $f(t,u)$ is non-decreasing in u for each t and $g(t,u)$ is non-increasing in u for each t.

(ii) $v_0(t)$ and $w_0(t)$ in $C(J,R)$ are coupled lower and upper solutions of the problem (3.2) such that $v_0(t) \le w_0(t)$ on J.

(iii) $f(t,u), g(t,u)$ satisfy one-sided Lipschitz conditions,

$f(t,u) - f(t,v) \le -L_1(u-v), \; L_1 \ge 0,$

$g(t,u) - g(t,v) \le -L_2(u-v), \; L_2 \ge 0$

Then there exists monotones sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C(J,R)$ such that $\{v_n(t)\} \to \{v(t)\}$ and $\{w_n(t)\} \to \{w(t)\}$ as $n \to \infty$, Where $v(t)$ and $w(t)$ are minimal and maximal solutions of the problem (3.2) respectively which satisfy

$D^q v(t) = f(t, v(t)) + g(t, w(t))$

$D^q w(t) = f(t, w(t)) + g(t, v(t))$ on J.

Theorem 3.4: [24] Assume that:

(i) $f(t,u(t))$ and $g(t,u(t))$ in $C[J\times R,R]$, $f(t,u)$ is non-decreasing in u for each t and $g(t,u)$ is non-increasing in u for each t.

(ii) $v_0(t)$ and $w_0(t)$ in $C(J,R)$ are coupled lower and upper solutions of the problem (3.2) such that $v_0(t)\geq w_0(t)$ on J.

(iii) $f(t,u)$, $g(t,u)$ satisfy Lipschitz conditions

$\left|f(t,u)-f(t,v)\right|\leq L_1|u-v|,\ L_1\geq 0,$

$\left|g(t,u)-g(t,v)\right|\leq L_2|u-v|,\ L_2\geq 0.$

(iv) $\lim_{n\to\infty}\left\|w_n(t)-v_n(t)\right\|=0$, where the norm is given by $\|f\|=\int_0^T|f(s)|ds$, then the solution of the problem (3.2) is unique.

X. Wang, L. Wang and Q. Zeng [22] have improved results obtained by Wang and Xie in [21] for the IBVP (3.1) taking $\lambda\geq 0$. The improved results are

Theorem 3.5:[22] Assume that:

(H_1): $f\in C(J\times\Omega,R)$,

(H_2): There exists $M>0$ such that $f(t,v)-f(t,u)\leq M[u-v]$ if $v\leq u,\ u,v\in\Omega,\ t\in J$, and $u,v\in D$ are upper and lower solutions of problem (3.1) with $\lambda\geq 0$, respectively and $v(t)\leq u(t)$ on J. If

$$D^q y(t)=f(t,u(t))-M\left[y(t)-u(t)\right],\ t\in J,\ y(0)=\lambda\int_0^T u(s)ds+d,$$

$$D^q z(t)=z(t,v(t))-M\left[z(t)-v(t)\right],\ t\in J,\ z(0)=\lambda\int_0^T v(s)ds+d,$$

then $v(t)\leq z(t)\leq y(t)\leq u(t), t\in J$ and y,z are upper and lower solutions of problem (3.1) respectively.

Note that in above theorem, $\Omega=\{u: y_0(t)\leq u\leq z_0(t)\}$ and

$D=\{w\in C^1(J,R): y_0(t)\leq w(t)\leq z_0(t), t\in J\}$ and $M=\sup_{t\in J}M(t)$.

Theorem 3.6:[22] Assume that the conditions $(H_1),(H_2)$ and (H_3) holds.: $y_0,z_0\in C^1(J,R)$ are upper and lower solutions of (3.1), $\lambda\geq 0$, respectively and such that $y_0(t)\geq z_0(t),\ t\in J$ are satisfied. Then there exists monotone sequence $\{z_n,y_n\}$ such that $z_n(t)\to z(t),\ y_n(t)\to y(t),\ t\in J$ as $n\to\infty$ and this convergence is uniformly and monotonically on J. Moreover z,y are extremal solution of (3.1) in D.

In 2009, McRay [9] developed monotone method for following IVP (3.3) and obtained existence and uniqueness of solution of

$$D^q u(t)=f(t,u(t)),u(t_0)=u_0=u(t)(t-t_0)^{1-q}\Big\}_{t=t_0} \tag{3.3}$$

where $f\in C(J\times R,R),\ J=[t_0,T]$

In 2015, Nanware [25] developed monotone iterative technique for following IVP involving the difference of two monotone functions and successfully applied this technique to obtain existence of solution of the problem (3.4).

$$\left.\begin{array}{l} {}_0D_t^q u(t) = f(t,u(t)) - g(t,u(t)),\ t \in J[0,\ T] \\ \text{With initial condition } u(0) = u_0 \end{array}\right\} \quad (3.4)$$

where $f, g \in C(J \times R, R)$ are both non-decreasing in $u(t)$, uniformly in *t*. The results obtained are stated as under:

Theorem 3.7:[25] Assume that:

(i) $f(t,u(t))$ and $g(t,u(t))$ in $C[J \times R, R]$ are non-decreasing in $u(t)$.

(ii) $v_0(t)$ and $w_0(t)$ in $C(J,R)$ are coupled lower and upper solutions of IVP (3.4) such that $v_0(t) \le w_0(t),\ t \in J = [0,\ T]$.

Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C(J,R)$ such that $\{v_n(t)\} \to v(t)$ and $\{w_n(t)\} \to w(t)$ as $n \to \infty$, uniformly and monotonically on J and the functions $v(t)$ and $w(t)$ are the coupled minimal and maximal solutions of nonlinear IVP (3.4) respectively.

In 2012,Yaker and Koskal [27] have studied initial value problem (3.5) for Riemann – Liouville fractional differential equations and proved existence results by using concept of lower and upper solutions and local existence results under the strong hypothesis that the functions are locally holder continuous.

$$D^q u(t) = f(t,u(t)) + g(t,u(t)), \qquad u^0 = (ut)(t-t_0)^{1-q}\Big\}_{t=t_0},\ \ t \in [t_0, T] \qquad (3.5)$$

where $f, g \in C[J \times R, R],\ J = [t_0, T],\ f(t,u)$ is non-decreasing in *u*, $g(t,u)$ isnoni-increasing in u for each t.

In 2017,Nanware et.al. [28] developed monotone method without locally Holder continuity for the IVP (3.5). Developed monotone method is applied to obtain existence and uniqueness of solution of IVP (3.5). The Improved results are as follows:

Theorem 3.8:[28] Suppose that:

(i) $v(t)$ and $w(t)$ is $C_p(J,R)$ are coupled lower and upper solutions of IVP (3.5) with $v(t) \le w(t)$ on J.

(ii) $f(t,u), g(t,u) \in C[\Omega, R]$ and $g(t,u(t))$ is non-increasing in *u* for each *t* on J .Then there exists a solution $u(t)$ of IVP (3.5) satisfying $v(t) \le u \le w(t)$ on J .

Theorem 3.9:[23] Assume that:

(i) $f(t,u(t))$ and $g(t,u(t))$ in $C[\Omega, R^2]$ and $f(t,u(t))$ non-increasing in u for each $t \in [t_0, T]$.

(ii) $v_0(t)$ and $w_0(t)$ in $C[J,R]$ are coupled lower and upper solutions of IVP (3.5) such that $v_0(t_0) \le w_0(t_0)$ on J .

(iii) $f(t,u(t)),\ g(t,u(t))$ satisfies one-sided Lipschitz condition,

$$f(t,u(t)) - f(t,\bar{u}(t)) \ge -M(u-\bar{u}),\ M > 0,\ \bar{u} \ge u\ ,$$

$$g(t,u(t)) - g(t,\bar{u}(t)) \ge -N(u-\bar{u}),\ N > 0,\ \bar{u} \ge u.$$

Then there exists monotone sequence $\{v_n(t)\}$ and $\{w_n(t)\}$ such that $\{v_n(t)\} \to v(t)$ and $\{w_n(t)\} \to w(t)$ as $n \to \infty$, where $v(t)$ and $w(t)$ are minimal and maximal solutions of the IVP (3.5).

Theorem 3.10:[28] Assume that (i) – (ii) of Theorem 3.9 holds and if

$$\left| f(t,u(t)) - f(t,\bar{u}(t)) \right| \le M\left|u-\bar{u}\right|, \qquad v_0 \le \bar{u} \le u \le w_0,\ M > 0$$

Then $v(t) = w(t) = u(t)$ is the unique solution of IVP (3.5)

In 2011, C. Yaker and A. Yaker [26] developed monotoneiterative method for the following nonlinear fractional differential equations with initial time difference with locally holder continuous function and obtained existence and uniqueness of solution of the problem.

$$D^q u(t) = f(t,u), \quad u^0 = u(t)(t-t_0)^{1-q}\Big\}_{t=t_0} \qquad (3.6)$$

where $0<q<1$ and $f \in C\left[R^+ \times R, R\right]$.

In 2020, authors [29] improved the results obtained by Yakeret. al. without locally Holder continuity for the class of all continuous functions.

$u(t)\in C_p(J,R) = \{u(t)\in C(J,R)\}$ and $u(t)(t-t_0)^p \in C(J,R)\}$, $J=[t_0,T]$.

Theorem 3.11:[29] Assume that:

(E_1) $v \in C_p\left[[t_0,t_0+T],R\right]$, $t_0, T>0$, $w \in C_p^*\left[[\tau_0,\tau_0+T],R\right]$ is continuous and $p=1-q$ where

$C_p(J,R) = \{u(t)\in C(J,R)$ and $u(t)(t-t_0)^p \in C(J,R)\}$, $J=[t_0,t_0+T]$,

$C_p^*(J^*,R) = \{u(t)\in C(J^*,R)$ and $u(t)(t-\tau_0)^p \in C(J^*,R)\}$, $J^*=[\tau_0,\tau_0+T]$,

$f \in C\left[[t_0,\tau_0+T]\times R, R\right]$ and

$D^q v(t) \le f(t,v(t))$, $t_0 \le t \le t_0+T$, $\quad D^q w(t) \ge f(t,w(t))$, $\tau_0 \le t \le \tau_0+T$

$v^0 = u^0 \le w^0$, where $v^0 = v(t)(t-t_0)^{1-q}\}t=t_0$, $w^0 = w(t)(t-\tau_0)^{1-q}\}t=\tau_0$

(E_2) $f(t,u)$ satisfies one –sided Lipschitz condition,

$f(t,u) - f(t,v) \le M[u-v]$ for $u \ge v$, $M \ge 0$.

(E_3) $f(t,u)$ isnon-decreasing in t for each u and, $v(t) \le (t+\eta), t_0 \le t \le t_0+T$ where $\eta = \tau_0 - t_0$ then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_p(J,R)$ which converges uniformly and monotonically on $[t_0,t_0+T]$ such that $\{v_n(t)\} \to v(t)$ and $\{w_n(t)\} \to w(t)$ as $n \to \infty$ where $v(t)$ and $w(t)$ are minimal and maximal solutions of IVP (3.6) respectively.

Theorem 3.12:[29] Assume that:

(U_1) Assumptions E_1 and E_3 of Theorem 3.11 holds.

(U_2) $f(t,u)$ satisfiesLipschitz condition,

$\left|f(t,u) - f(t,v)\right| \le L|u-v|$, for $u \ge v$, $M \ge 0$.

Then there exists unique solutions of IVP (3.6).

IV. MONOTONE METHOD FOR SYSTEM OF RIEMANN – LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

In this section monotone methods developed by Nanwareet. al. [30] for system of Riemann – Liouville Fractional Integral boundary value problem (IBVP) are considered.

Existence and uniqueness results obtained by Wang et.al. [21] are extended by Nanware and Dhaigude [30] for the system (4.1) of IBVP.

$$\left.\begin{aligned} D^q u_1(t) &= f_1(t,u_1(t),u_2(t)), \quad u_1(0) = \int_0^T u_1(s)\,ds + d \\ D^q u_2(t) &= f_2(t,u_1(t),u_2(t)), \quad u_2(0) = \int_0^T u_2(s)\,ds + d \end{aligned}\right\} \qquad (4.1)$$

where $d \in R, t \in J$, f_1, f_2 in $C\left(J \times R^2, R\right)$, $0 < q < 1$, u_1, u_2 are locally Holder continuous. The developed Monotone method and existence and uniqueness results for system (4.1) are given in the following Theorems:

Theorem 4.1:[30] Assume that:

(i) $f_i = f_i\left(t, u_1, u_2\right)$, $i = 1, 2$ in $C\left[J \in R^2, R\right]$ is quasi-monotonenon-decreasing.

(ii) $v^0 = \left(v_1^0, v_2^0\right)$ and $w^0 = \left(w_1^0, w_2^0\right)$ in $C_p\left(J, R\right)$ ordered lower and upper solutions of (4.1) such that $v_1^0\left(0\right) \le w_1^0\left(0\right)$, $v_2^0\left(0\right) \le w_2^0\left(0\right)$ on J.

(iii) $f_i \equiv f_i\left(t, u_1, u_2\right)$ satisfies one-sided Lipschitz condition,

$f_i\left(t, u_1, u_2\right) - f_i\left(t, \overline{u_1}, \overline{u_2}\right) \ge -M_i\left[u_i - \overline{u}_i\right]$, for $\overline{u}_1 \le M_i \ge 0$.

Then there exists monotone sequences $\left\{v^n\left(t\right)\right\}$ and $\left\{w^n\left(t\right)\right\}$ such that $\left\{v^n\left(t\right)\right\} \to v\left(t\right) = \left(v_1, v_2\right)$ and $\left\{w^n\left(t\right)\right\} \to w\left(t\right) = \left(w_1, w_2\right)$ as $n \to \infty$. Where $v\left(t\right)$ and $w\left(t\right)$ are minimal and maximal solutions of the problem (4.1).

Theorem 4.2:[30] Assume that:

(i) $f_i = f_i\left(t, u_1, u_2\right)$ in $C\left[J \times R^2, R\right]$ is quasi-monotonenon-decreasing

(ii) $v = \left(v_1, v_2\right)$ and $w = \left(w_1, w_2\right)$ in $C\left(J, R\right)$ are ordered lower and upper solutions of the problem (4.1) on J

(iii) $f_i = f_i\left(t, u_1, u_2\right)$ satisfies Lipschitz condition,

$\left|f_i\left(t, u_1, u_2\right) - f_i\left(t, \overline{u}_1, \overline{u}_2\right)\right| \ge -M_i\left|u_i - \overline{u}_i\right|$, $M_i \ge 0$

(iv) $\lim_{n\to\infty}\left\|w^n - v^n\right\| = 0$, where $\left\|f\right\| = \int_0^T\left|f\left(s\right)\right|ds$, then the solution of the problem (4.1) is unique.

Further above results are obtained byJadhav et.al. [31] for the system (4.1) when the functions u_1, u_2 are continuous. The obtained results are stated in Theorem (4.3) and (4.4).

Assume that:

$\left(H_1\right)$ $v^0 = \left(v_1^0, v_2^0\right)$ and $w^0 = \left(w_1^0, w_2^0\right)$ in $C_p\left(J, R\right)$ are ordered lower and upper solutions of (4.1) such that $v_1^0\left(0\right) \le w_1^0\left(0\right), v_2^0\left(0\right) \le w_2^0\left(0\right)$ on J.

$\left(H_2\right)$ $f_i = f_i\left(t, u_1, u_2\right)$ $i = 1, 2$ in $C\left[J \times R^2, R\right]$ is quasi-monotonenon-decreasing,

$\left(H_3\right)$ $f_i = f_i\left(t, u_1, u_2\right)$ satisfiesLipschitz condition,

$\left|f_i\left(t, u_1, u_2\right) - f_i\left(t, u_1^*, u_2^*\right)\right| \ge -M_i\left|u_i - u_i^*\right|$, $M_i \ge 0$,

$\left(H_4\right)$ $\lim_{n\to\infty}\left\|w^n - v^n\right\| = 0$, where the norm is defined by $\left\|f\right\| = \int_0^T\left|f\left(s\right)\right|ds$.

Theorem 4.3:[31] Assume that $\left(H_1\right), \left(H_2\right)$ and $\left(H_3\right)$ holds. Then there exists monotone sequence $\left\{v^n\left(t\right)\right\}$ and $\left\{w^n\left(t\right)\right\}$ such that $\left\{v^n\left(t\right)\right\} \to v\left(t\right) = \left(v_1, v_2\right)$ and $\left\{w^n\left(t\right)\right\} \to w\left(t\right) = \left(w_1, w_2\right)$ as $n \to \infty$. Where s $v\left(t\right)$ and $w\left(t\right)$ are minimal and maximal solutions of the integral boundary value problem (2.1).

Theorem 4.4: [31] Assume that $\left(H_1\right)$, $\left(H_2\right), \left(H_3\right)$ and $\left(H_4\right)$ holds. Then integral boundary valueproblem (4.1) has a unique solution.

In 2017, Nanware [32] considered the system (4.2) of IBVP and developed monotone method for

$$\left.\begin{aligned} &D^q u_i(t) = f_i(t, u_1(t), u_2(t)),\ t \in J = [0,T]\ \ T \geq 0 \\ &u_i(t) = \int_0^T u_i(s)\,ds + d_i,\ d_i \in R,\ \ i = 1,2 \end{aligned}\right\} \qquad (4.2)$$

where f_1, f_2 in $C[J \times R^2, R]$, $\lambda = 1,\ 0 < q < 1$.

The developed monotone method and existence and uniqueness results of (4.2) are as under:

Theorem 4.5:[32] Assume that:

(i) $f_i(t, v_1(t), v_2(t))$ is quasi-monotone non-decreasing,

(ii) $v_0(t)$ and $w_0(t)$ in $C_p(J, R)$ are weakly coupled lower and upper solutions of (2.1) such that $v_0(t) \leq w_0(t)$ on J,

(iii) $f(t, u(t))$ satisfy one-sided Lipschitz condition ,

$f(t,u) - f(t,v) \leq -L(u - v),\ L \geq 0$.

Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_p(J, R)$ such that $\{v_n(t)\} \to v(t)$ and $\{w_n(t)\} \to w(t)$ as $n \to \infty$. Where $v(t)$ and $w(t)$ are minimal and maximal solutions of (4.2) respectively.

Theorem 4.6: [32] suppose that:

(i) $f_i(t, u_1(t), u_2(t))$ is quasi-monotone non-decreasing,

(ii) $v_0(t)$ and $w_0(t)$ in $C_p(J, R)$ are weakly coupled lower and upper solutions for (4.2) such that $v_0(t) \leq w_0(t)$ on J,

(iii) $f_i(t, u_1(t), u_2(t))$ satisfy lipschitz condition

$\left| f_i(t, u_1(t), u_2(t)) - f_i(t, V_1(t), V_2(t)) \right| \geq M_i |u_i - V_i|,\ M_i \geq 0$,

(iv) $\lim_{n\to\infty} \left\| w^n(t) - v^n(t) \right\| = 0$, where the norm is defined by $\|f\| = \int_0^T |f(s)|\,ds$.

Then the solution of problem (4.2) is unique.

Recently, Nanware and Dawkar [33] generalized the results of Nanware [32] for the system (4.3) of IBVP in which λ is any non-negative number.

$$D^q u_i(t) = f_i(t, u_1(t), u_2(t)),\quad u_i(0) = \lambda \int_0^T u_i(s)\,ds + d_i \quad i = 1,2 \qquad (4.3)$$

where $d_i \in R,\ t \in [0,T],\ f_i, f_2$ in $C(J \times R^2, R)$ $J = [0,T]$ $0 < q < 1$ and $\lambda \geq 0$.

The developed monotone method and obtained existence and uniqueness results are as follows:

Theorem 4.7:[33] Assume that :

(i) $f_i = f_i(t, u_1, u_2) \in C[J \times R^2, R]$ isquasi-monotonenon-decreasing.

(ii) $x^0 = (x_1^0, x_2^0)$ and $y^0 = (y_1^0, y_2^0) \in C_p(J, R^2)$ are ordered lower and upper solutions of problem (4.3) such that $x_1^0(0) \leq y_1^0(0), x_2^0(0) \leq y_2^0(0)$ on $[0,T]$

(iii) $f_i \equiv f_i(t, u_1, u_2)$ satisfies one sided Lipschitz condition

$f_i(t, u_1, u_2) - f_i(t, \bar{u}_1, \bar{u}_2) \geq -M_i[u_i - \bar{u}_i]$

whenever $x_i^0(0) \leq u_i \leq y_i^0(0),\ M_i \geq 0$ and $x_i^0(0) \leq \bar{u}_i \leq u_i \leq y_i^0(0)$

Then there exists monotone sequences $\{x^n(t)\} = (x_1^0, x_2^0)$ and $\{y^n(t)\} = (y_1^0, y_2^0)$ such that $\{x^n(t)\} \to x(t) = (x_1, x_2)$ and $\{y^n(t)\} \to y(t) = (y_1, y_2)$ as $n \to \infty$. Where $x(t)$ and $y(t)$ are minimal and maximal solutions of problem (4.3) respectively.

Theorem 4.8:[33] Assume that:

(i) $f_i = f_i(t, u_1, u_2)$ in $C[J \times R, R^2]$ is quasi-monotonenon-decreasing,

(ii) $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $C_p(J, R^2)$ are ordered lower and upper solutions of (4.3) on [0, T],

(iii) $f_i = f_i(t, u_1, u_2)$ satisfies Lipschitz condition,

$$\left| f_i(t, u_1, u_2) - f_i(t, \bar{u}_1, \bar{u}_2) \right| \le -M_i \left| u_i - \bar{u}_i \right|$$

(iv) $\lim_{n\to\infty} \left\| y^n - x^n \right\| = 0$, where the norm is defined by $\|f\| = \int_0^T |f(s)| ds$.

Then the solution of system (4.3) is unique.

Recently Nanwareet.al. [41], extended the results proved in [29], forsystem (4.4) of Riemann – Liouville fractional differential equations with initial time difference

$$\left.\begin{aligned} D^q u_1(t) &= f_1(t, u_1(t),\ u_2(t)),\ \ u_1(t)(t-t_0)^{1-q}\Big\}_{t=t_0} = u_1^0 \\ D^q u_2(t) &= f_2(t, u_1(t),\ u_2(t)),\ \ u_2(t)(t-t_0)^{1-q}\Big\}_{t=t_0} = u_2^0 \end{aligned}\right\} \quad (4.4)$$

where $t \in J = [0, T]$, f_1, f_2 in $C(J \times R^2, R)$, $0 < q < 1$.

The developed monotone method and obtained existence and uniqueness of solutions for problem (4.4) are as follows:

Theorem 4.9: [41] Assumethat:

(i) $v = (v_1, v_2) \in C_p[[t_0, t_0 + T], R^2]$ $t_0, T > 0$, $w = (w_1, w_2) \in C_p[[\tau_0, \tau_0 + T], R^2]$ iscontinuous and $p = 1 - q$ where

$$C_p(J, R^2) = \{u(t) \in (J, R^2) \text{ and } u(t)(t-t_0)^p \in C(J, R^2)\},\quad J = [t_0, t_0 + T]$$

$$C_p^*(J^*, R^2) = \{u(t) \in (J^*, R^2) \text{ and } u(t)(t - \tau_0{}^p) \in C(J^*, R^2)\},\ J^* = [\tau_0, \tau_0 + T]$$

$f \in C[[t_0, \tau_0 + T] \times R^2, R]$ and

$$D^q v(t) \le f(t, v_1(t), v_2(t)),\ t_0 \le t \le t_0 + T,$$

$$D^q w(t) \ge f(t, v_1(t), w_2(t)),\ \tau_0 \le t \le \tau_0 + T,$$

$v^0 \le u^0 \le w^0$, where $v^0 = v(t)(t-t_0)^{1-q}\Big\}_{t=t_0}$, $w^0 = w(t)(t-\tau_0)^{1-q}\Big\}_{t=t_0}$

(ii) $f_i(t, u_1, u_2)$ is quasi-monotonenon-decreasing in t for each u_i and

$v(t) \le w(t + \eta)$, $t_0 \le t \le t_0 + T$, where $\eta = \tau_0 - t_0$

(iii) f_i satisfies one- sided Lipschitz condition,

$f_i(t, u_1, u_2) - f_i(t, \bar{u}_1, \bar{u}_2) \ge -M_i[u_i - \bar{u}_i]$ for $\bar{u}_i \le u_i, M_i \ge 0$.

Then there exists monotone sequences $\{v^n(t)\}$ and $\{w^n(t)\}$ such that $\{v^n(t)\} \to v(t) = (v_1, v_2)$ and $\{w^n(t)\} \to w(t) = (w_1, w_2)$ as $n \to \infty$. where $v(t)$ and $w(t)$ are minimal and maximal solution of the problem (4.4) respectively.

Theorem 4.10:[41] Assume that:
(i) Assumption (i) and (iii) of Theorem 4.9 holds,
(ii) $f_i = f_i(t, u_1, u_2)$ satisfies Lipschitz conditions,
$\left| f_i(t, u_1, u_2) - f_i(t, \bar{u}_1, \bar{u}_2) \right| \ge -M_i \left| u_i - \bar{u}_i \right|, M_i \ge 0,$
Then the solution of the problem (4.4) is unique.

V. MONOTONE METHOD FOR FINITE SYSTEM OF RIEMANN – LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, some basic definitions are considered. Monotone method developed by Denton and Vastsala for finite system of IVP and Monotone method developed by Nanware et.al for finite system of IBVP are considered.

Definition 5.1: A function $f_i \in C\left(\left[t_0, T\right] \times R^N, R\right)$ is said to satisfy mixed quasimonotone property if for each i, $f_i\left(t, u_i, [u]_{r_i}, [u]_{s_i}\right)$ is monotone non-decreasing in $[u]_{r_i}$ and monotone non-increasing in $[u]_{s_i}$.

When either r_i or s_i is equal to zero a special case of the mixed quasimonotone property is defined as follows:

Definition 5.2: A function $f_i \in C\left(\left[t_o, T\right] \times R^N, R\right)$ is said to be quasimonotonenondecreasing (nonincreasing) if for each $i,\ u_i \le v_i$ and $u_j = v_j,\ i \ne j$, then

$f_i(t, u_1, u_2, ..., u_N) \le f_i(t, v_1, v_2, ..., v_N)\left(f_i(t, u_1, u_2, ..., u_N) \ge f_1(t, v_1, ...v_N)\right)$

Define the following function space:

$C_p\left(\left[t_0, T\right], R^N\right) = \left\{ u(t) \in C\left(\left[t_0, T\right], R^N\right) \text{ and } (t - t_0)^p \in C\left(\left[t_0, T\right] R^N\right)\right\}$

where $u(t) = \left(u_1(t), u_2(t), ..., u_N(t)\right)$ and sector

$\Omega = \left\{ (t, u_1, u_2, ..., u_N) \in [0, T] \times R^N : v_i(t) \le u_i(t) \le w_i(t), 0 \le t \le T \right\},\ \ i = 1, 2, ...N$.

Consider the following finite system of IVP

$$D^q u(t) = f\left(t, u(t)\right),\ u(t) = (u_1, u_2, ..., u_N),\ u(t_0) = u^0 = u(t)(t - t_0)^{1-q}\Big\}_{t=t_0} \quad (5.1)$$

where $f \in C\left(J \times R^N, R\right)$ is mixed quasi-monotone, $J = (t_0, T)$, $0 < q < 1$ and D^q denotes Riemann – Liouvillefractional derivative.

In 2011, following results are obtained by Denton and Vatsala [36] for IVP (5.1) when the function is mixed quasimonotone. These results are the generalization of the results due to McRae in [9], without Holder continuity assumption.

Theorem 5.1:[36] Assume that:
(i) $f(t, u)\ C\left[\Omega, R^N\right]$ is mixed quasi-monotone on $\left[t_0, T\right]$,
(ii) $v^0(t)$ and $w^0(t)$ in $C(J, R)$ are ordered lower and upper solutions of IVP (5.1) such that $v^0(t_0) \le w^0(t_0)$ on $\left[t_0, T\right]$
(iii) $f(t, u)$ satisfies one-sided Lipschitz condition
$\left| f(t, u) - f(t, \bar{u}) \right| \ge -M\left[u - \bar{u}\right],$
Whenever $v^0 \le u \le w,\ M \ge 0,\ v^0 \le \bar{u} \le u \le w^0,$ then the exists monotone sequences $\left\{ v^n(t) \right\}$ and $\left\{ w^n(t) \right\}$ such that $\left\{ v^n(t) \right\} \to v(t)$ and $\left\{ w^n(t) \right\} \to w(t)$ as $n \to \infty$, and $v(t)$ and $w(t)$ are minimal and maximal solutions of FIVP (5.1).

Theorem 5.2:[36] Assume that (i) – (iii) of Theorem 5.1 hold and if
$\left| f(t, u) - f(t, \bar{u}) \le M\left[u - \bar{u}\right], v^0 \le \bar{u} \le u \le w^0,\ M > 0\right.$,

then $v = w = u$ is the unique solution of IVP (5.1).

In 2014, Nanware et.al [34] developedmonotone iterative technique and obtained existence and uniqueness of the solution of the following finite system (5.2) of nonlinear Riemann- Liouville fractional differential equations with integral boundary conditions.

$$D^q u_i = f_i\left(t, u_1, u_2, ..., u_N\right),\ u_i\left(0\right) = \int f_i\left(s\right)ds + d \qquad (5.2)$$

where $f_i \in C\left(\left[0, T\right] \times R^N, R\right)$ is mixed quasi-monotone.

The problem (5.2) can be written as

$$D^q u_i\left(t\right) = f_i\left(t, u_i, [u]_{r_i}, [u]_{s_i}\right),\ u_i\left(0\right) = \int u_i\left(s\right)ds + d \qquad (5.3)$$

where r_i, s_i be two non-negative integers such that $r_i + s_i = N - 1$. The improved results are as follows:

Theorem 5.3:[34] Suppose that:

(i) $f_i \in C\left(J \times R^N, R\right)$ possess mixed quasi-monotone property,

(ii) $v_0\left(t\right) = \left(v_{01}, v_{02}, ..., v_{0N}\right)$ and $w_0\left(t\right) = \left(w_{01}, w_{02}, ..., w_{0N}\right)$ in $C_p\left(J, R^N\right)$ are coupled lower and upper quasisolutions of the problem (5.3) such that $v_{0i}\left(0\right) \le w_{0i}\left(0\right)$ on $J\left[0, T\right]$

(iii) f_i satisfies one – sided Lipschitz condition :

$$f_i\left(t, u_i, [u]_{r_i} - [u]_{s_i}\right) - f_i\left(t, v_i, [u]_{r_i}, [u]_{s_i}\right) \ge -M_i\left[u_i - v_i\right],\ M_i \ge 0,\ u_i \le v_i\ .$$

Then there exists monotone sequences $\left\{v_{n_i}\left(t\right)\right\}$ and $\left\{w_{n_i}\left(t\right)\right\}$ such that $\left\{v_{n_i}\left(t\right)\right\} \to v_i\left(t\right)$ and $\left\{w_{n_i}\left(t\right)\right\} \to w_i\left(t\right)$ as $n \to \infty$, uniformly and monotonically on J and $v_i\left(t\right)$ and $w_i\left(t\right)$ are coupled minimal and maximal periodic quasisolutions of problem (5.3).

Furthermore, if $u = \left(u_1, u_2, ..., u_n\right)$ is any solution of the problem (5.3) such that $v_{0_i}\left(0\right) \le u_{0_i}\left(0\right) \le w_{0_i}\left(0\right)$ on J, then $v_i\left(t\right) \le u_i\left(t\right) \le w_i\left(t\right)$ on J .

Theorem 5.4:[34] Suppose that:

(i) $f_i \in C\left(J \times R^N, R\right)$ possess mixed quasi-monotone property,

(ii) $v_0\left(t\right) = \left(v_{0_1}, v_{0_2}, ..., v_{0_N}\right)$ and $w_0\left(t\right) = \left(w_{0_1}, w_{0_2}, ..., w_{0_N}\right)$ in $C_p\left(J, R^N\right)$ are coupled lower and upper quasisoluitons of the problem (5.3) such that $v_0\left(0\right) \le w_0\left(0\right)$ on J.

(iii) f_i satisfy one- sided Lipschitz condition ,

$$f_i\left(t, u_i, [u]_{r_i}, [u]_{s_i}\right) - f_i\left(t, v_i, [u]_{r_i}, [u]_{s_i}\right) \ge -M\left[u_i - v_i\right], M_i \ge 0, u_i \le v_i$$

Furthermore f satisfy the condition $\left(u - v, f\left(t, u\left(t\right)\right)\right) - f\left(t, v\left(t\right)\right) \le -L\left\|u - v\right\|_2 L > 0$ whenever $v_0\left(t\right) \le u,\ v \le w_0\left(t\right)$, where $u\left(t\right) = \left(u_1, u_2, ..., u_N\right)$ & $v\left(t\right) = \left(v_1, v_2, ..., v_N\right)$. Then there exists a unique periodic quasisolution of the problem (5.2).

RecentlyNanwareet. al. [35] considered the finite system (5.4) of fractional differential equations with integral boundary conditions when the right hand side function is mixed quasi-monotone and λ is any non-negative number.

$$\left.\begin{aligned} &D^q u_i\left(t\right) = g_i\left(t, u_1\left(t\right), u_2\left(t\right), ..., u_N\left(t\right)\right), \\ &u_i\left(0\right) = \lambda\int_0^T u_i\left(s\right)ds + d_i,\ i = 1, 2, ...N,\ \lambda \ge 0, \end{aligned}\right\} \qquad (5.4)$$

where $g_i \in C\left(\left[0, T\right] \times R^N, R\right)$ is nonlinear mixed quasimonotonefunction.

The problem (5.4) is rewritten as follows:

$$D^q u_i(t) = g_i\left(t, u_i, [u]_{r_i}, [u]_{s_i}\right), \quad u_i(0) = \lambda\int_0^T u_i(s)\,ds + d_i \qquad (5.5)$$ where

r_i, s_i are two nonnegative integers such that $r_i + s_i = N-1$.

The developed monotone method and existence uniqueness results are as follows :

Theorem 5.5:[35] suppose that:

(i) $g_i \in C\left(J\times R^N, R\right)$ possess mixed quasi-monotone property

(ii) $v_0(t) = \left(v_{0_1}, v_{0_2}, ..., v_{0_N}\right)$ and $w_0(t) = \left(w_{0_1}, w_{0_2}, ..., w_{0_N}\right)$ in $C_p\left(J, R^N\right)$ are coupled lower and upper quasisolutions of the problem (2.2) such that $v_{0_i}(0) \le w_{0_i}(0)$ on J = [0, T]

(iii) g_i satisfies one –sided Lipschitz condition

$$\left|g_i\left(t, u_i, [u]_{r_i}, [u]_{s_i}\right) - g_i\left(t, \bar{u}_i, [\bar{u}]r_i, [\bar{u}]_{s_i}\right)\right| \ge -M_i\left(u_i - \bar{u}_i\right),\ M_i \ge 0,\ \bar{u}_i \le u_i$$

Then there exists monotones sequences $\{v_{n_i}(t)\}$ and $\{w_{n_i}(t)\}$ such that $\{v_{n_i}(t)\} \to v_i(t)$ and $\{w_{n_i}(t)\} \to w_i(t)$ as $n \to \infty$ uniformly and monotonically on J and $v_i(t)$ and $w_i(t)$ are coupled minimal and maximal periodic quasisolutions of the problem (5.5)…Furthermore, if $u = (u_1, u_2, ..., u_N)$ is a solution of the problem (5.5) such that $v_{0i}(0) \le u_{0_i}(0) \le w_{0_i}(0)$ on J, then $v_i(t) \le u_i(t) \le w_i(t)$ on J .

Theorem 5.6:[35] Suppose that:

(i) $g_i \in C\left(J\times R^N, R\right)$ possess mixed quasi-monotone property.

(ii) $v_0(t) = \left(v_{0_1}, v_{0_2}, ..., v_{0_N}\right)$ and $w_0(t) = \left(w_{0_1}, w_{0_2}, ..., w_{0_N}\right)$ in $C_p\left(J, R^N\right)$ are coupled lower and upper quasisolutions of the problem (5.5) such that $v_0(0) \le w_0(0)$ on J.

(iii) g_i satisfies Lipschitz condition

$$\left|g_i\left(t, u_i, [u]_{r_i}, [u]_{s_i}\right) - g_i\left(t, \bar{u}_i, [\bar{u}]_{r_i}, [\bar{u}]_{s_i}\right)\right| \le M_i\left|u_i - \bar{u}_i\right|,\ M_i \ge 0,\ \bar{u}_i < u_i$$

Furthermore f satisfies the condition $\left(u - v, g(t, u(t)) - g(t, v(t)\right) \le -L\|u - v\|^2,\ L > 0$ whenever $v_0(t) \le u\ ,\ v \le w_0(t)$, where $u(t) = (u_1, u_2, ..., u_N)$, $v(t) = (v_1, v_2, ..., v_N)$ and $g = (g_1, g_2, ..., g_N)$ .Then there exists a unique quasi-solution of the problem (5.5).

VI. CONCLUSION

Monotone method for Riemann-Liouville fractional differential equations with integral boundary conditions and initial conditions is reviewed. As an application of monotone method, the qualitative properties of solutions such as existence and uniqueness are proved for the problem under investigation. It is found that two monotone sequences constructed using lower and upper solutions of the problem converges uniformly to minimal and maximal solutions of the problem.

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