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**Research Paper**

# Derivation and Applications of Grad-Shafranov Equation InMagnetohydrodynamics(MHD)

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***ABSTRACT***
*The Grad-shafranov equation is derived from the magnetohydrodynamic (MHD) equilibrium equations given by the momentum equation for a plasma in equilibrium, Amperes law relating the current density $J$ to the curl of the magnetic flux density B and The divergence free postulate, stating that there are no sources of magnetic flux, that is, no magnetic monopoles, the equilibrium equation in ideal magnetohydrodynamics (MHD) for a two dimensional plasma, for example the axisymmetric toroidal plasma in a tokamak. Tokamak equilibrium can be considered as an internal balance between the plasma pressure and the forces from the magnetic field. This gives rise to the shape and position of the plasma, controlled by the currents in the external coils. Cylindrical coordinate system is used to derive the grad-shafranov equation, we consider the right hand system (r, $\phi$, z) that is $e_r . e_\phi \times e_z > 0$. The largest component of the magnetic field is the toroidal field produced by the poloidal currents in the external coils. The two main applications of magnetohydrodynamics (MHD) are technological—to liquid metals and to plasmas. There is little doubt that the former has had the greater impact on society. It includes the casting and stirring of liquid metals, levitation melting, vacuum-arc re-melting, induction furnaces, electromagnetic valves, and aluminum reduction cells.*
***KEYWORDS:*** *Magnetohydrodynamics(MHD), Tokamak Equilibrium, Amperes Law, Divergence, Flux Function, Plasma, Magnetic Flux, Toroidal Field, Poloidal Currents, Current Density.*

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## I. INTRODUCTION-MHD EQUILIBRIUM

The Grad-shafranov equation (H. Grad and H. Rubin (1958); VitaliDmitrievich Shafranov (1966) is the equilibrium equation in ideal magnetohydrodynamics (MHD) for a two dimensional plasma, for example the axisymmetric toroidal plasma in a tokamak. This equation takes the same form as the Hick's equation from fluid dynamics. This equation is a two dimensional, nonlinear, elliptic partial differential equation obtained from the reduction of the ideal MHD equations to two dimensions, often for the case of toroidal axisymmetric (the case relevant in a tokamak). Taking $(r, \phi, z)$ as the cylindrical coordinates, the flux function $\Psi$ is governed by the equation

$$-\mu_\circ r^2 \frac{d}{d\Psi} - \frac{1}{2}\frac{dF^2}{d\Psi} = \frac{\partial^2\Psi}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi}{\partial z^2} + \frac{\partial^2\Psi}{\partial z^2} \qquad \text{(i)}$$

Where $\mu_\circ$ is the magnetic permeability, $P(\Psi)$ is the pressure, $F(\Psi) = rB\phi$ and the magnetic field and current are respectively given by

$$B = \frac{1}{r}\nabla\Psi \times e_\phi + \frac{F}{r}e_\phi \qquad \text{(ii)}$$

$$\mu_\circ J = \frac{1}{r}\frac{dF}{d\Psi}\nabla\Psi \times e_\phi - \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\Psi}{\partial z^2}\right] e_\phi \qquad \text{(iii)}$$

The nature of the equilibrium whether it be a tokamak, reversed field pinch, etc. is largely determined by the choices of the two functions $F(\Psi)$ and $P(\Psi)$ as well as the boundary conditions.

**FLUX FUNCTIONS**

The fundamental equilibrium condition is that the forces are zero at all points. Assuming the absence of plasma resistance, (*i.e.* ideal MHD conditions) we require that the magnetic and pressure forces balance at all points,

$$J \times B = \nabla P \qquad \text{(a)}$$

Where, $J$ is the current density, $B$ is the magnetic field and $P$ is the pressure.

Hence

$$B.\nabla P = 0$$

(b)

$$B.(J \times B) = 0 \qquad (c)$$

Equation (ii) shows that the magnetic surfaces are surfaces of constant pressure and that

$$J.\nabla P = 0$$

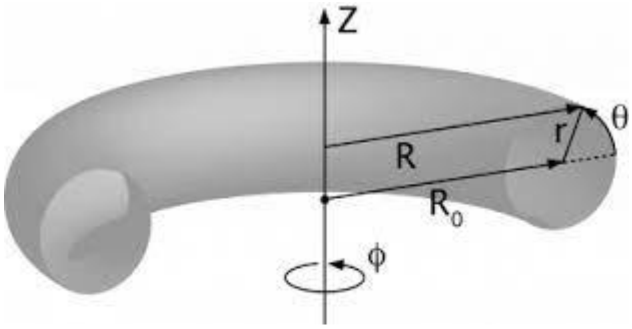


**Figure 1:** cylindrical coordinate system $R = 0$ is the major axis of the torus.

Introducing the poloidal magnetic function $\Psi$. This function is determined by the poloidal flux lying within each magnetic surfaces and is therefore constant on that surface. Hence

$$B.\nabla\Psi = 0$$

Defining the flux function$\Psi$ as the poloidal flux per unit radian in ϕ, the poloidal magnetic field is related to $\Psi$ by equation (d) below.

$$B_R = -\frac{1}{R}\frac{\partial\Psi}{\partial R}, B_Z = \frac{1}{R}\frac{\partial\Psi}{\partial R} \qquad (d)$$

Recall, from Maxwell equation of divergence$\nabla.\mathrm{B} = 0$, that is

$$\frac{1}{R}\frac{\partial R B_R}{\partial R} + \frac{\partial B_Z}{\partial Z} = 0$$

The flux function is arbitrary to an additive constant which is chosen for convenience. From the symmetry of $J$ and $B$ it is clean that a current flux function also exists. This function is related to the poloidal current density by

$$J_R = -\frac{1}{R}\frac{\partial F}{\partial Z}, J_Z = \frac{1}{R}\frac{\partial F}{\partial R}$$

(e)

Comparison of equation with ampere equation

$$J_R = -\frac{1}{\mu_0 R}\frac{\partial B_\phi}{\partial Z}, J_Z = \frac{1}{\mu_0 R}\frac{\partial B_\phi}{\partial R}$$

Gives the relation between $F$ and the toroidal magnetic field as

$$F = \frac{R B_\phi}{\mu_0}$$

(f)

It can be shown that $F$ function of $\Psi$ by taking the scalar product of equation (a) with $J$ to obtain $J.\nabla P = 0$, and then substituting equation (f) for $J$. thus

$$\frac{\partial F}{\partial R}\frac{\partial P}{\partial Z} - \frac{\partial F}{\partial Z}\frac{\partial P}{\partial R} = 0$$

And so

$$\nabla f \times \nabla P = 0,$$

Proving that $F$ is a function of $P$. then since $P = P(\Psi)$, it follows that $F = F(\Psi)$. The flux function as defined here give the poloidal flux per radian in $\phi$. It is possible to define the total flux function for the torus which is simply $2\pi\Psi$.

## THE GRAD-SHAFRANOV EQUATION

The grad-shafranov equation is derived from the magnetohydrodynamic (MHD) equilibrium equations given by:

1. The momentum equation for a plasma in equilibrium

$$\mathrm{J} \times B = \nabla P \qquad (1)$$

No time variation, therefore $\frac{dv}{dt} = 0$

2. Amperes law relating the current density $J$ to the curl of the magnetic flux density $B$.

$$\mu_0 \mathrm{J} = \nabla \times B \qquad (2)$$

3. The divergence free postulate, stating that there are no sources of magnetic flux, that is, no magnetic monopoles.

$$\nabla . B = 0 \quad (3)$$

Note, equation (1) establishes the force balance needed for equilibrium, the pressure gradient (expansion force) needs to be equal to the magnetic force (confinement force). In this way the plasma is in equilibrium. Importantly, the plane defined by $J$ and $B$ everywhere tangent to the isosurfaces of $P$.

**DERIVATION VECTOR CALCULUS**

For the derivation of the grad-shafranov equation, an axisymmetric geometry and a standard cylindrical coordinate system (r, ϕ, z)

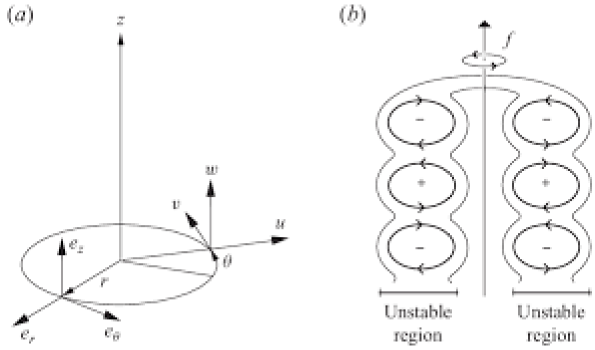


**Figure 2:**Cylindrical coordinate system is used to derive the grad-shafranov equation, note that we consider the right hand system (r, ϕ, z) that is $e_r . e_\phi \times e_z > 0$.

Since we assume axisymmetric, for all functions defined on the domain **Ω**, spatial derivatives with respect to ϕ are zero.

$$\frac{\partial F}{\partial \phi} = 0, \forall\, F \,\boldsymbol{\epsilon}\, w_\phi^1(\Omega) \quad (4)$$

Since the magnetic flux density B is divergence free, from equation (3)Poincare's theorem.
For example, states that there must exist a magnetic vector potential A.

$$\nabla . B = 0 \Rightarrow \exists\, A \in H\,(curl, \nabla) \quad / \quad \nabla \times \mathrm{A} = \mathrm{B} \quad (5)$$

In cylindrical coordinates this vector potential A and ∇can be written respectively as

$$A = A_r e_r + A_\phi e_\phi + A_z e_z \,, \nabla = r\frac{\partial}{\partial r} + \phi \frac{1}{r}\frac{\partial}{\partial \phi} + z\frac{\partial}{\partial z} \quad (6)$$

Therefore, the magnetic flux field becomes

$$B = \nabla \times A = \frac{1}{r}\begin{pmatrix} e_r & e_\phi r & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{pmatrix} \quad (7)$$

Due to the axisymmetric assumption equation from equation (4)

$$\frac{\partial A_r}{\partial \phi} = \frac{\partial A_Z}{\partial \phi} = 0$$

$$B = \nabla \times A = \frac{1}{r}\left\{ e_r \left( \frac{\partial A_Z}{\partial \phi} - \frac{r\partial A_\phi}{\partial z} \right) - e_\phi r \left( \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \right) + \ e_z \left( \frac{r\partial A_\phi}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) \right\}$$

$$B = \nabla \times A = \frac{1}{r}\left\{ e_r \left( -\frac{r\partial A_\phi}{\partial z} \right) - e_\phi r \left( \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \right) + \ e_z \left( \frac{r\partial A_\phi}{\partial r} \right) \right\} \quad (8)$$

Opening the outer bracket, we obtain equation (8)

$$= -\frac{\partial A_\phi}{\partial z}\frac{e_r}{1} - e_\phi \left( \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \right) + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\frac{e_z}{1}$$

Since, $\nabla \times (A_\phi e_\phi) = -\frac{\partial A_\phi}{\partial z}\frac{e_r}{1} + \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r}\frac{e_z}{1}$ (9a)

considering $A$ in the direction of $\phi$, hence we can write that

$$B = \nabla \times (A_\phi e_\phi) - e_\phi \left( \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \right) \quad (9b)$$

Defining the following two variables

$$\Psi = -rA_\phi \quad (10)$$

$$g = rB_\phi = r\left( \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \right) \quad (11)$$

Recall also that,

$$\nabla \phi = \frac{1}{r} e_\phi \quad (12)$$

Hence, equation (9b) can be rewritten as,

$$B = \nabla \phi \times \nabla \Psi + g \nabla \phi \quad (13)$$

By substituting equation (13) into equation (2), $\boldsymbol{\mu_\circ J} = \boldsymbol{\nabla} \times \boldsymbol{B}$.

$$\mu_\circ J = \nabla \times B.$$
$$= \nabla \times (\nabla\emptyset \times \nabla\Psi) + \nabla \times (g\nabla\emptyset)$$
$$= \nabla \times \left[\frac{1}{r} e_\phi \times \left(\frac{\partial\Psi}{\partial r} e_r + \frac{\partial\Psi}{\partial z} e_z\right)\right] + \nabla g \times \nabla\phi \qquad (14)$$

Where $\nabla\phi$ and $\nabla\Psi$ have been expanded and the first term of the right hand side on the second term of the right hand side the vector calculus identity. $\nabla \times (gV) = g\nabla \times V + \nabla g + V$ has been used. Recall that in this case $V = \nabla\phi$ which implies that $\nabla \times V = \nabla \times \nabla\phi = 0$. Further expanding the curl term in equation (14) yields.

$$\mu_\circ J = \nabla \times \left[\frac{1}{r}\frac{\partial\Psi}{\partial z} e_r - \frac{1}{r}\frac{\partial\Psi}{\partial r} e_r\right] \times \nabla g \times \nabla\phi$$
$$= \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\Psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right)\right] e_\phi \times \nabla g \times \nabla\phi$$
$$= \left(\frac{1}{r}\frac{\partial^2\Psi}{\partial z^2} - \frac{1}{r^2}\frac{\partial\Psi}{\partial r} + \frac{1}{r}\frac{\partial^2\Psi}{\partial r^2}\right) e_\phi \times \nabla g \times \nabla\phi$$

Recall that, $\nabla\phi = \frac{1}{r} e_\phi$

$$= \left(\frac{\partial^2\Psi}{\partial z^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial r^2}\right)\nabla\phi \times \nabla g \times \phi \qquad (15)$$

Introducing the following elliptical operator, let

$$\Delta^*\Psi = \frac{\partial^2\Psi}{\partial z^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial r^2} \qquad (16)$$

Hence equation (15) can be rewritten as,

$$\mu_\circ J = \Delta^*\Psi\nabla\Psi \times \nabla g \times \nabla\phi \qquad (17)$$

Equation (17) decomposes the current density vector field into a toroidal component, $\Delta^*\Psi\nabla\phi$, and a poloidal component $\nabla g \times \nabla\phi$, the toroidal component is parallel to $\nabla\phi$ and the poloidal component us perpendicular to $\nabla\phi$ since the triple product $\nabla\phi . \nabla g \times \nabla\phi = 0$.

Substituting (13) and (17) into the equilibrium equation, equation (1) and assuming for compactness $\mu_\circ = 1$ we obtain

$$\nabla P = J \times B = (\Delta^*\Psi\nabla\phi + \nabla g \times \nabla\phi) \times (\nabla\phi \times \nabla\Psi + g\nabla\phi) \qquad (18)$$

To further simplify this expression it is fundamental to show that $P = P(\Psi)$ and $g = g(\Psi)$ we start by P. computing the inner product of B with both sides of the equilibrium equation (1) yields.

$$B.\nabla P = B.(J \times B) = 0 \qquad (19)$$

Combining (1) and (19) and noting that due to axisymmetric, $\nabla P = \frac{\partial P}{\partial r} e_r + \frac{\partial P}{\partial z} e_z$, since by equation (4) $\frac{\partial P}{\partial \phi} = 0$, we get.

$$B.\nabla P = (\nabla\phi \times \nabla\Psi + g\nabla\phi).\nabla P$$
$$= (\nabla\phi \times \nabla\Psi).\nabla P$$
$$= \left(\frac{e_\phi}{r} \times \nabla\Psi\right)\nabla P$$
$$= \frac{e_\phi}{r}.(\nabla\Psi \times \nabla P) \qquad (20)$$

Since both $\nabla\Psi$ and $\nabla P$ are vectors perpendicular to $e_\phi$ ( due to axisymmetric, equation (4) this implies that for equation (20) to satisfy equation (19), we have that.

$$\nabla\Psi \times \nabla P = 0 \ \Rightarrow \ \nabla P \| \nabla\Psi \qquad (21)$$

In turn, this means that pressure is a function only of the poloidal flux, $P = P(\Psi)$, and its gradient can be written as

$$\nabla P = \frac{dP}{d\Psi}\nabla\Psi \qquad (22)$$

To show that $g$ is a function only of $\Psi$, we need to follow a similar procedure to the one followed for P but using the current density J, the inner product of J with both sides of the equilibrium equation (1) yields.

$$J.\nabla P = J.(J \times B) = 0 \qquad (23)$$

Combining equation (1) with (23) and noting that due to axisymmetric

$$\nabla P = \frac{\partial P}{\partial r} e_r + \frac{\partial P}{\partial z} e_z$$

Since by equation (4) $\frac{\partial P}{\partial \phi}$= 0, hence we obtain

$$J.\nabla P = (\Delta^*\nabla\Psi + \nabla g \times \nabla\phi).\nabla P$$
$$= (\nabla g \times \nabla\phi).\nabla P$$
$$= \left(\nabla g \times \frac{e_\phi}{r}\right).\nabla P$$
$$= \frac{e_\phi}{r}.(\nabla g \times \nabla P) \qquad (24)$$

Since both $\nabla g$ and $\nabla P$ are vectors perpendicular to $e_\phi$ ( due to axisymmetric equation (4)), this implies that for equation (24) to satisfy equation (23) we have that

$\nabla g \times \nabla P = 0 \Rightarrow \nabla P \| \nabla g \Rightarrow \nabla g \| \nabla P$ where we used for the last implication $\nabla P \| \nabla \Psi$, equation (21) in turn, this means that g is a function only of the poloidal flux $g = g(\Psi)$, and its gradient can be written as

$$\nabla g = \frac{dg}{d\Psi} \nabla \Psi \quad (25)$$

Combining now equation (22) and (25) into equation (18) we get

$$\frac{dP}{d\Psi} \nabla \Psi = \left( \Delta^* \Psi \nabla \phi + \frac{dg}{d\Psi} \nabla \Psi \times \nabla \phi \right) \times (\nabla \phi \times \nabla \Psi + g \nabla \phi)$$

$$= \Delta^* \Psi \nabla \phi \times (\nabla \phi \times \nabla \Psi) + g \left( \frac{dg}{d\Psi} \nabla \Psi \times \nabla \phi \right) \times \nabla \phi$$

$$= \frac{\Delta^* \Psi}{r^2} e_\phi \times (e_\phi \times \nabla \Psi) + \frac{g}{r^2} \frac{dg}{d\Psi} (\nabla \Psi \times e_\phi) \times e_\phi \quad (26)$$

Note that equation (12) was used on the last equality, recalling that due to axisymmetric equation (4) the gradient of the poloidal flux $\nabla \Psi$ is perpendicular to $e_\phi$and that $\|e_\phi\| = 1$, we have that $e_\phi \times (e_\phi \times \nabla \Psi) = (\nabla \Psi \times e_\phi) \times -\nabla \Psi$. Therefore, equation (26) can be simplified into.

$$\frac{dP}{d\Psi} \nabla \Psi = -\frac{\Delta^*}{r^2} \nabla \Psi - \frac{g}{r^2} \frac{dg}{d\Psi} \nabla \Psi \quad (27)$$

Which leads directly to the Grad-shafranov equation

$$\Delta^* \boldsymbol{\Psi} = -\boldsymbol{r}^2 \frac{\boldsymbol{dP}}{\boldsymbol{d\Psi}} - \boldsymbol{g} \frac{\boldsymbol{dg}}{\boldsymbol{d\Psi}} \quad \textbf{(28)}$$

## II. APPLICATIONS

The two main applications of magnetohydrodynamics (MHD) are technological—to liquid metals and to plasmas. There is little doubt that the former has had the greater impact on society. It includes the casting and stirring of liquid metals, levitation melting, vacuum-arc re-melting, induction furnaces, electromagnetic valves, and aluminum reduction cells. Another application, the flow of a liquid metal in the blanket surrounding a thermonuclear reaction chamber, touches on the other main area: plasma magnetohydrodynamics (MHD). The reactor contains a rarefied plasma of deuterium/tritium (DT) that is raised to a high enough temperature for these nuclei to fuse and release energy. The economic promise of such a device in generating magnetic fusion energy (MFE) has provided a powerful incentive for studying plasma magnetohydrodynamics (MHD) and has led to significant new insights, particularly into the structure and stability of MSE. In addition to these practical applications, the elucidation of a wide variety of magnetic phenomena in nature depends on an understanding of magnetohydrodynamics (MHD). Astrophysics and geophysics provide abundant examples, including the magnetism of the earth, planets, and satellites, that of the sun and other stars, and that of galaxies. Magnetohydrodynamics (MHD) is important in astrophysical processes such as magneto-convection, magnetic flux emergence, flux ropes, spots, atmospheric heating, wind acceleration, flares, and eruptions.

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