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Research Paper

Approximate Analytical Solutions of Partial Differential Equations with Caputo Fractional Derivative

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ABSTRACT: *In this paper, For solving nonlinear fractional partial differential equations with Caputo fractional derivative, a fractional homotopy analysis approach is presented and applied. To exemplify the approach, examples were given, and the findings were compared to those produced using the fractional homotopy analysis approach.*

KEYWORDS: *Burger's equation; Homotopy analysis method; Fractional derivative -Caputo operator*

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I. INTRODUCTION

Fractional differential equations are widely used to describe lots of important phenomena and dynamic processes in physics, engineering, electromagnetics, acoustics, viscoelasticity electrochemistry, material science, stochastic dynamical system, plasma physics, controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics, etc. [1-4].

In recent years, a many of approximate analytical methods have been utilized to solve the ordinary and partial differential equations in the Caputo sense such as the fractional variational iteration method, fractional differential transform method, fractional series expansion method, fractional Sumudu variational iteration method, fractional Laplace transform method, fractional homotopy perturbation method, fractional Sumudu decomposition method, fractional Fourier series method, fractional reduced differential transform method, fractional Adomian decomposition method, and another methods [5-69]. Our aim is to present the HPM, and to used it to solve the nonlinear FRDE. The remaining sections of this work are organized as follows. In Section 2, some background notations of fractional calculus are presented. In Section 3, the analysis of fractional HAM is discussed. Applications of fractional HAM are shown in Section 4. The conclusion of this paper is given in Section 5.

II. PRELIMINARIES

Definition 1.[3] Areal function $\Psi(x,\tau), x \in R, \tau > 0$ is said to be in the space C_ε , $\varepsilon \in R$ if there exists a real number $q, (q > \varepsilon)$, such that $\Psi(x,\tau) = \tau^q \Psi_1(x,\tau)$, where $\Psi_1(x,\tau) \in c[0,\infty]$, and it is said to be in the space C_ε^m if $\Psi^{(m)}(x,\tau) \in C_\varepsilon, m \in N$.

Definition 2.[3] The Riemann Liouville fractional integral operator of order $\alpha \geq 0$, of a function $\Psi(\tau) \in C_\varepsilon, \varepsilon \geq -1$ is defined as

$$I_\tau^\alpha \Psi(\tau) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1} \Psi(s) ds & ,\alpha>0, \tau>0 \\ \Psi(\tau) & ,\alpha = 0 \end{cases} \qquad (1)$$

where $\Gamma(\cdot)$ is the well-Known Gamma function.

Definition 3.[3] The Liouville-caputo operator (c) with order $(\alpha > 0)$ of $\Psi(\tau)$ is defined as follows:

$$ {}^{c}D_{\tau}^{\alpha}\Psi(\tau) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_{0}^{\tau}(\tau-s)^{m-\alpha-1}\Psi^{(m)}(s)\,ds, m-1<\alpha\leq m \\ \frac{\partial^{n}}{\partial\tau^{n}}\Psi(\tau) \qquad , \quad \alpha = n \in \mathbb{N} \end{cases} \qquad (2) $$

for $m \in N$, $\tau > 0$, $\Psi \in C_{-1}^{m}$

The following are the basic properties of the operator D^{α}:

1. $D^{\alpha}I^{\alpha}\Psi(x,\tau) = \Psi(x,\tau)$
2. $I^{\alpha}D^{\alpha}\Psi(x,\tau) = \Psi(x,\tau) - \sum_{k=0}^{m-1}\frac{\tau^{k}}{k!}\Psi^{(k)}(x,0)$
3. $D^{\alpha}\tau^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\tau^{\beta-\alpha} \quad , \alpha > 0$

Definition 4.[3] The Mittag-Leffler function $E_{\alpha}(z)$ with $\alpha > 0$ is defined as.

$$ E_{\alpha}(z) = \sum_{m=0}^{\infty}\frac{z^{m}}{\Gamma(\alpha+1)}. \qquad (3) $$

III. ANALYSIS OF METHOD

Let us consider a general ifractional nonlinear PDE of the form:

$$ D_{\tau}^{\alpha}\Psi(x,\tau) + R\Psi(x,\tau) + N\Psi(x,\tau) = G(x,\tau), \qquad m-1 < \alpha \leq m, x \in R, \tau > 0 \qquad (4) $$

Subject to the initial conditions

$$ \Psi(x,0) = \Psi^{(k)}(x,0) \qquad , \qquad k = 1,2,\ldots,m-1 \qquad (5) $$

where $D_{\tau}^{\alpha}\Psi(x,\tau) = \mathcal{L}^{\alpha}$ is the CFD of the function $\Psi(x,\tau)$ defined as:

$$ D_{\tau}^{\alpha}\Psi(x,\tau) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_{0}^{\tau}(\tau-s)^{m-\alpha-1}\frac{\partial^{m}\Psi(x,s)}{\partial\tau^{m}}ds \quad , m-1 < \alpha < m \\ \frac{\partial^{m}\Psi(x,\tau)}{\partial\tau^{m}} \qquad , \alpha = m \in \mathbb{N} \end{cases} $$

and R is the linear differential operator, N represents the general nonlinear differential operator, and $G(x,\tau)$ is the source term.

We define the nonlinear operator

$$ N[\emptyset(x,\tau;q)] = \emptyset(x,\tau;q) - G(x,\tau) + R\Psi(x,\tau;q) + N\Psi(x,\tau;q) \qquad (6) $$

where $q \in [0,1]$ and $\emptyset(x,\tau;q)$ is a real function of x, τ *and* q

the so-called zero-order deformation equation of (6) has the form

$$ (1-q)\mathcal{L}^{\alpha}[\emptyset(x,\tau;q) - \Psi_{0}(x,\tau)] = qhH(x,\tau)N[\emptyset(x,\tau;q)] \qquad (7) $$

where $q \in [0,1]$ is the embedding parameter, $H(x,\tau)$ denotes a nonzero auxiliary function, $h \neq 0$ is an auxiliary parameter.

$\Psi_{0}(x,\tau)$ is an initial guess of $\Psi(x,\tau)$ and $\emptyset(x,\tau;q)$ is an unknown function.

Obviously, when the parameter $q = 0$ and $q = 1$, it holds

$$ \emptyset(x,\tau;0) = \Psi_{0}(x,\tau), \qquad \emptyset(x,\tau;1) = \Psi(x,\tau) \qquad (8) $$

respectively. Thus as q increases from 0 to 1

the solution $\emptyset(x,\tau;q)$ varies from the initial guess $\Psi_{0}(x,\tau)$ to the solution $\Psi(x,\tau)$.

Expanding $\emptyset(x,\tau;q)$ in Taylors serie's with respect to q,

we have

$$ \emptyset(x,\tau;q) = \Psi_{0}(x,\tau) + \sum_{m=1}^{\infty}\Psi_{m}(x,\tau)\,q^{m} \qquad (9) $$

Where

$$ \Psi_{m}(x,\tau) = \frac{1}{m!}\frac{\partial^{m}\emptyset(x,\tau;q)}{\partial q^{m}}\Big|_{q=0} \qquad (10) $$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are properly chosen.

The series (9) converges at $q = 1$, then we has

$$ \Psi(x,\tau) = \Psi_{0}(x,\tau) + \sum_{m=1}^{\infty}\Psi_{m}(x,\tau) \qquad (11) $$

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which must be one of the solution of the original nonlinear equations.
According to the definition (11), the governing equation can be deduced from the zero-order deformation (7)
Define the vectors

$$\vec{\Psi}_m(x,\tau) = \{\Psi_0(x,\tau), \Psi_1(x,\tau), \ldots, \Psi_m(x,\tau)\} \tag{12}$$

Differentiating the zero order deformation equation (7) m-times with respect to q and then dividing by m! and finally setting q=0 we get the following m^{th} – order deformation equation :

$$\mathcal{L}^{\alpha}[\Psi_m(x,\tau) - x_m\Psi_{m-1}(x,\tau)] = hH(x,\tau)R_m(\vec{\Psi}_{m-1}(x,\tau)) \tag{13}$$

Applying the Riemann Liouville fractional integral operator of order of order $\alpha \geq 0$, we have

$$\Psi_m(x,\tau) = x_m\Psi_{m-1}(x,\tau) + \mathcal{L}^{-\alpha}\left[hH(x,\tau)R_m\left(\vec{\Psi}_{m-1}(x,\tau)\right)\right], \tag{14}$$

where

$$R_m(\vec{\Psi}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\emptyset(x,\tau;q)]}{\partial q^{m-1}}\Big|_{q=0} \tag{15}$$

$$\text{and } X_m = \begin{cases} 0 & ,x \leq 1 \\ 1 & ,x > 1 \end{cases} \tag{16}$$

In this way, it is easily to obtain $\Psi_m(x,\tau)$ for $m \geq 1$, at m^{th} - order, $h = -1$, we have

$$\Psi(x,\tau) = \sum_{m=0}^{\infty}\Psi_m(x,\tau) \tag{17}$$

IV. APPLICATIONS

Example 1: Consider the following nonlinear fractional Burger's equation.

$$D_\tau^\alpha\Psi - \Psi_{xx} - \Psi_{yy} = 0, \qquad 0 < \alpha \leq 1 \tag{18}$$

with the initial condition

$$\Psi(x,y,0) = sin(x)\,sin(y)\,. \tag{19}$$

We now define a nonlinear operator is

$$N[\emptyset(x,y,\tau;q)] = \emptyset(x,y,\tau;q) - \frac{\partial^2\emptyset(x,y,\tau;q)}{\partial x^2} - \frac{\partial^2\emptyset(x,y,\tau;q)}{\partial y^2}, \tag{20}$$

and thus

$$R_m(\vec{\Psi}_{m-1}) = D_\tau^\alpha\Psi_{m-1} - \frac{\partial^2\Psi_{m-1}(x,y,\tau;q)}{\partial x^2} - \frac{\partial^2\Psi_{m-1}(x,y,\tau;q)}{\partial y^2} \tag{21}$$

The m^{th} – order deformation Eq. (21) is

$$\mathcal{L}^{\alpha}[\Psi_m - \mathrm{x}_m\Psi_{m-1}] = hH(x,y,\tau)R_m(\vec{\Psi}_{m-1}) \tag{22}$$

Applying The Riemann Liouville fractional integral operator $\mathcal{L}^{-\alpha}$ of order $\alpha \geq 0,$, we have

$$\Psi_m = \mathrm{x}_m\Psi_{m-1} + h\mathcal{L}^{-\alpha}[H(x,y,\tau)R_m(\vec{\Psi}_{m-1})] \tag{23}$$

Solving above the Eq.(23) for m=1,2,... and choosing H(x,y,τ)=1
Let us take the initial condition

$$\Psi_0(x,y,\tau) = sin(x)\,sin(y)$$

$$\begin{aligned}
\Psi_1(x,y,\tau) &= \mathrm{x}_1\Psi_0 + h\mathcal{L}^{-\alpha}[R_1(\vec{\Psi}_0)] \\
&= (0)\,(sin(x)\,sin(y)) + h\mathcal{L}^{-\alpha}[2\,sin(x)\,sin(y)] \\
&= \frac{2h\tau^\alpha\,sin(x)\,sin(y)}{\Gamma(\alpha+1)}
\end{aligned}$$

$$\begin{aligned}
\Psi_2(x,y,\tau) &= \mathrm{x}_2\Psi_1 + h\mathcal{L}^{-\alpha}[R_2(\vec{\Psi}_1)] \\
&= (1)\left(\frac{2h\tau^\alpha\,sin(x)\,sin(y)}{\Gamma(\alpha+1)}\right) + h\mathcal{L}^{-\alpha}\left[\frac{\partial^\alpha\Psi_1}{\partial\tau^\alpha} - \frac{\partial^2\Psi_1}{\partial x^2} - \frac{\partial^2\Psi_1}{\partial y^2}\right] \\
&= \frac{2h\tau^\alpha\,sin(x)\,sin(y)}{\Gamma(\alpha+1)} + h\mathcal{L}^{-\alpha}\left[2h\,sin(x)\,sin(y) + \frac{4h\tau^\alpha\,sin(x)\,sin(y)}{\Gamma(\alpha+1)}\right] \\
&= \frac{2h\tau^\alpha\,sin(x)\,sin(y)}{\Gamma(\alpha+1)} + \frac{2h^2\tau^\alpha\,sin(x)\,sin(y)}{\Gamma(\alpha+1)} + \frac{4h^2\tau^{2\alpha}\,sin(x)\,sin(y)}{\Gamma(2\alpha+1)}
\end{aligned}$$

⋮

and so on. Then we have

$$\Psi(x,y,\tau) = sin(x)\, si\, n(y) + \frac{2h\tau^{\alpha}\, sin(x)\, sin(y)}{\Gamma(\alpha+1)} + \frac{2h\tau^{\alpha}\, sin(x)\, sin(y)}{\Gamma(\alpha+1)} + \frac{2h^2\tau^{\alpha}\, sin(x)\, sin(y)}{\Gamma(\alpha+1)} + \frac{4h^2\tau^{2\alpha}\, sin(x)\, sin(y)}{\Gamma(2\alpha+1)} \cdots \quad (24)$$

Put $h = -1$ to obtain

$$\Psi(x,y,\tau) = sin(x)\, sin(y) \left[1 - \frac{2\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{4\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \right] \quad (25)$$

The exact result of Example 1 when $\alpha = 1$ is

$$\Psi(x,y,\tau) = sin(x)\, sin(y) e^{-2\tau} \quad (26)$$

Example 2: Consider the following nonlinear fractional DEs.

$$D_{\tau}^{\alpha}\Psi + \Psi\Psi_x - \Psi_{xx} = 0, \qquad 0 < \alpha \le 1 \quad (27)$$

with the initial condition

$$\Psi(x,0) = x\,. \quad (28)$$

We now define a nonlinear operator is

$$N[\emptyset(x,y,\tau;q)] = \emptyset(x,\tau;q) + \emptyset(x,\tau;q)\frac{\partial\emptyset(x,\tau;q)}{\partial x} - \frac{\partial^2\emptyset(x,y,\tau;q)}{\partial x^2} \quad (29)$$

and thus

$$R_m(\vec{\Psi}_{m-1}) = D_{\tau}^{\alpha}\Psi_{m-1} + \left(\sum_{i=0}^{m-1}\Psi_i(\Psi_{m-1-i})_x\right) - \frac{\partial^2\Psi_{m-1}(x,\tau;q)}{\partial x^2} \quad (30)$$

The m^{th} – order deformation Eq. (21) is

$$\mathcal{L}^{\alpha}[\Psi_m - x_m\Psi_{m-1}] = hH(x,\tau)R_m(\vec{\Psi}_{m-1}) \quad (31)$$

Applying The Riemann Liouville fractional integral operator $\mathcal{L}^{-\alpha}$ of order $\alpha \ge 0,$, we have

$$\Psi_m = x_m\Psi_{m-1} + h\mathcal{L}^{-\alpha}[H(x,\tau)R_m(\vec{\Psi}_{m-1})] \quad (32)$$

Solving above the Eq.(23) for m=1,2,… and choosing H(x,y,τ)=1

Let us take the initial condition

$$\Psi_0(x,\tau) = x$$

$$\Psi_1(x,\tau) = x_1\Psi_0 + h\mathcal{L}^{-\alpha}[R_1(\vec{\Psi}_0)]$$

$$= (0)(x) + h\mathcal{L}^{-\alpha}\left[\frac{\partial^{\alpha}\Psi_0}{\partial\tau^{\alpha}} + x - \frac{\partial^2\Psi_0}{\partial x^2}\right]$$

$$= h\mathcal{L}^{-\alpha}[x]$$

$$= \frac{hx\tau^{\alpha}}{\Gamma(\alpha+1)}$$

$$\Psi_2(x,\tau) = x_2\Psi_1 + h\mathcal{L}^{-\alpha}[R_2(\vec{\Psi}_1)]$$

$$= (1)\left(\frac{hx\tau^{\alpha}}{\Gamma(\alpha+1)}\right) + h\mathcal{L}^{-\alpha}\left[\frac{\partial^{\alpha}\Psi_1}{\partial\tau^{\alpha}} + \frac{2hx\tau^{\alpha}}{\Gamma(\alpha+1)} - \frac{\partial^2\Psi_1}{\partial x^2}\right]$$

$$= \frac{hx\tau^{\alpha}}{\Gamma(\alpha+1)} + h\mathcal{L}^{-\alpha}\left[hx + \frac{2hx\tau^{\alpha}}{\Gamma(\alpha+1)}\right]$$

$$= \frac{hx\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2x\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{2h^2x\tau^{2\alpha}}{\Gamma(2\alpha+1)}$$

⋮

and so on. Then we have

$$\Psi(x,\tau) = x + \frac{hx\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{hx\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2x\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{2h^2x\tau^{2\alpha}}{\Gamma(2\alpha+1)} \cdots \quad (33)$$

Put $h = -1$ to obtain

$$\Psi(x,\tau) = x\left[1 - \frac{\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{2\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right] \quad (34)$$

The exact result of Example 2 when $\alpha = 1$ is

$$\Psi(x,\tau) = \frac{x}{1+\tau} \tag{35}$$

V. CONCLUSIONS

The successful implementation of method HAM yielded approximate solutions to nonlinear fractional order differential equations with temporal fractional derivatives. The answers discovered were in the form of infinite power series that could be stated in closed form.

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