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Research Paper

Adomian Decomposition Method for Solving Nonlinear Fractional PDEs

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ABSTRACT: *In this paper, we obtain the approximate analytical solution of fractional differential equations with Caputo-Fabrizio fractional derivative by using the fractional Adomian decomposition method. The approximate solutions of nonlinear differential equations with fractional order are successfully obtained using this method , and the result is compared with the result of the existing methods.*
KEYWORDS: *Korteweg-de Vries; Caputo-Fabrizio fractional derivative; Adomian decomposition method***.**

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I. INTRODUCTION

Fractional calculus and its various applications in mathematics, physics and engineering have received considerable attention. Fractional calculus applications are found in many areas, such as dynamic device control theory, chemical mechanics, probability and statistics, electrical networks, corrosion electrochemistry, and optics and signal processing. Linear and nonlinear fractional-order differential equations may be successfully modeled. A fractional PDE is obtained from the classical diffusion equation of mathematical physics by replacing the n^{th} order time derivative with a fractional-order derivative α, which is now the area of increasing interest apparent in the literature study. A significant phenomenon of these evolution equations is that it produces the Brownian fractional movement, a Brownian motion generalization. In several articles and books, different definitions of fractional calculus are available [1-4].

In recent years, many researchers have paid attention to study the behavior of physical problems by using various analytical and numerical techniques which are not described by the common observations, such as the fractional variational iteration method, fractional differential transform method, fractional series expansion method, fractional Sumudu variational iteration method, fractional Laplace transform method, fractional homotopy perturbation method, fractional Sumudu decomposition method, fractional Fourier series method, fractional reduced differential transform method, fractional Adomian decomposition method, and another methods [5-69]. Our aim is to present the ADM, and to used it to solve the nonlinear FPDE. The remaining sections of this work are organized as follows. In Section 2, some background notations of fractional calculus are presented. In Section 3, the analysis of fractional ADM is discussed. Applications of fractional ADM are shown in Section 4. The conclusion of this paper is given in Section 5.

II. Preliminaries

Definition 1. Let $\varphi \in H(a,b), a > b$, $a \in (-\infty,\tau), 0 \le \alpha \le 1$, then The definition of the new Caputo fraction derivative is [70,71]:

$$ {}_{a}^{CF}D_{\tau}^{\alpha}\varphi(\tau) = \frac{Z(\alpha)}{(1-\alpha)}\int_{a}^{\tau}\varphi'(s)\, exp\left(-\frac{\alpha}{1-\alpha}(\tau-a)\right)ds, \tag{1} $$

where $Z(\alpha)$ is a normalization function satisfying $Z(0) = Z(1) = 1$.

Some properties fractional derivative

(i). ${}_{a}^{CF}D_{\tau}^{\alpha}[\varphi_1(\tau)+\varphi_2(\tau)] = {}_{a}^{CF}D_{\tau}^{\alpha}\varphi_1(\tau) + {}_{a}^{CF}D_{\tau}^{\alpha}\varphi_2(\tau)$.

(ii). ${}_{a}^{CF}D_{\tau}^{\alpha}(c) = 0$, where c is constant.

(iii). ${}_{a}^{CF}D_{\tau}^{\alpha}\varphi(\tau) = \varphi(\tau)$, where $\alpha = 0$.

Definition 2. Let $\varphi \in H(a,b), a > b$, $a \in (-\infty,\tau), 0 < \alpha \le 1$, then The fractional integral of $\varphi(\tau)$ of order α is defined by [70,71]:

$$ {}_{a}^{CF}I_{\tau}^{\alpha}\varphi(\tau) = \frac{1-\alpha}{Z(\alpha)}\varphi(\tau) + \frac{\alpha}{Z(\alpha)}\int_{a}^{\tau}\varphi(s)ds. \tag{2} $$

Some properties fractional integral

(i). ${}_{a}^{CF}I_{\tau}^{\alpha}[\lambda\varphi_1(\tau)+\varphi_2(\tau)] = \lambda\, {}_{a}^{CF}I_{\tau}^{\alpha}\varphi_1(\tau) + {}_{a}^{CF}I_{\tau}^{\alpha}\varphi_2(\tau)$.

(ii). ${}_{a}^{CF}I_{\tau}^{\alpha}\, {}_{a}^{CF}D_{\tau}^{\alpha}\varphi(\tau) = \varphi(\tau) - \varphi(a)$.

(iii). ${}_{a}^{CF}I_{\tau}^{\alpha}(c) = \frac{c}{Z(\alpha)}[1-\alpha+\alpha(\tau-a)]$.

III. Analysis of Method

We consider the fractional partial differential equation:

$$ {}^{CF}D_{\tau}^{\alpha}\varphi(\mu,\tau) + R\big(\varphi(\mu,\tau)\big) + N\big(\varphi(\mu,\tau)\big) = g(\mu,\tau), 0 \le \alpha \le 1, \mu \in \mathbb{R} > 0 \tag{3} $$

with the initial condition

$$ \varphi(\mu,0) = \varphi_0(\mu), \tag{4} $$

where ${}^{CF}D_{\tau}^{\alpha}$ is Caput- Fabrizio operator, R is a linear operator, N is a nonlinear operator and g is a source term. Taking integral of Caput- Fabrizio to both side, of (3), we get

$$ {}^{CF}I_{\tau}^{\alpha}\, {}^{CF}D_{\tau}^{\alpha}\varphi(\mu,\tau) + {}^{CF}I_{\tau}^{\alpha}\, R\big(\varphi(\mu,\tau)\big) + {}^{CF}I_{\tau}^{\alpha}\, N\big(\varphi(\mu,\tau)\big) = {}^{CF}I_{\tau}^{\alpha}\, g(\mu,\tau). \tag{5} $$

By properties fractional integral, we obtain

$$ \varphi(\mu,\tau) - \varphi(\mu,0) = {}^{CF}I_{\tau}^{\alpha}\, g(\mu,\tau) - {}^{CF}I_{\tau}^{\alpha}\, R\big(\varphi(\mu,\tau)\big) - {}^{CF}I_{\tau}^{\alpha}\, N\big(\varphi(\mu,\tau)\big). \tag{6} $$

From (4), we get

$$ \begin{aligned} \varphi(\mu,\tau) = \varphi_0(\mu) + {}^{CF}I_{\tau}^{\alpha}\, g(\mu,\tau) - {}^{CF}I_{\tau}^{\alpha}\, R\big(\varphi(\mu,\tau)\big) \\ - {}^{CF}I_{\tau}^{\alpha}\, N\big(\varphi(\mu,\tau)\big). \end{aligned} \tag{7} $$

let $f = \varphi_0(\mu) + {}^{CF}I_{\tau}^{\alpha}[g(\mu,\tau)]$

then

$$ \varphi(\mu,\tau) = f - {}^{CF}I_{\tau}^{\alpha}\, R\big(\varphi(\mu,\tau)\big) - {}^{CF}I_{\tau}^{\alpha}\, N\big(\varphi(\mu,\tau)\big). \tag{8} $$

Now, we represent solution as an infinite series given below

$$ \varphi(\mu,\tau) = \sum_{n=0}^{\infty}\varphi_n(\mu,\tau) \tag{9} $$

and the nonlinear term can be decomposed as

$$ N\big(\varphi(\mu,\tau)\big) = \sum_{n=0}^{\infty}A_n(\varphi_0,\varphi_1,\varphi_2) \tag{10} $$

where

$$ A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i\varphi_i\right)\right]_{\lambda=0}, n = 0,1,2,\dots \tag{11} $$

By substituting (9)and (10)in (8), we have

$$\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)=f-{}^{CF}I_\tau^\alpha\left[\left(R\sum_{n=0}^{\infty}\varphi_n+\sum_{n=0}^{\infty}A_n\right)\right] \tag{12}$$

On comparing both sides of the Eq.(12) we get

$$\begin{aligned}\varphi_0(\mu,\tau)&=f\\ \varphi_1(\mu,\tau)&=-{}^{CF}I_\tau^\alpha[(R(\varphi_0(\mu,\tau)+A_0))]\\ \varphi_2(\mu,\tau)&=-{}^{CF}I_\tau^\alpha[(R(\varphi_1(\mu,\tau)+A_1))]\\ &\vdots\end{aligned} \tag{13}$$

In general, the recursive relation is given as

$$\varphi_{n+1}(\mu,\tau)=-{}^{CF}I_\tau^\alpha[(R(\varphi_n(\mu,\tau)+A_n))] \tag{14}$$

Finally, we approximate the analytical solution (3) by truncated seriea:

$$\varphi(\mu,\tau)=\sum_{n=0}^{\infty}\varphi_n(\mu,\tau) \tag{15}$$

$$\varphi(\mu,\tau)=\varphi_0(\mu,\tau)+\varphi_1(\mu,\tau)+\varphi_2(\mu,\tau)+\cdots \tag{16}$$

IV. Applications

Example 1: We consider the following fractional KdV equation in the Caputo-Fabrizio sense

$${}^{CF}D_\tau^\alpha\,\varphi(\mu,\tau)+6\varphi\varphi_\mu+\varphi_{\mu\mu\mu}=0 \tag{17}$$

where $0\le\alpha\le 1,\mu\in\mathbb{R},\tau>0$ *and subject to the initial condition*

$$\varphi(\mu,0)=6\mu \tag{18}$$

Taking ${}^{CF}I_\tau^\alpha$ *to both sides we get*

$$\varphi(\mu,\tau)=6\mu+{}^{CF}I_\tau^\alpha[6\varphi\varphi_\mu+\varphi_{\mu\mu\mu}] \tag{19}$$

$let\ \varphi=\sum_{n=0}^{\infty}\varphi_n\ and\quad \varphi\varphi_\mu=\sum_{n=0}^{\infty}A_n$

then

$$\begin{aligned}\varphi_0(\mu,\tau)&=6\mu\\ \varphi_{n+1}(\mu,\tau)&={}^{CF}I_\tau^\alpha\left[6\sum_{n=0}^{\infty}A_n-(\varphi_n)_{\mu\mu\mu}\right]\end{aligned} \tag{20}$$

Now

$$\begin{aligned}A_0&=\varphi_0\varphi_{0\mu}\\ A_1&=\varphi_0\varphi_{1\mu}+\varphi_1\varphi_{0\mu}\\ A_2&=\varphi_0\varphi_{2\mu}+\varphi_2\varphi_{0\mu}+\varphi_1\varphi_{1\mu}\\ &\vdots\end{aligned}$$

$$\begin{aligned}\varphi_1(\mu,\tau)&={}^{CF}I_\tau^\alpha[6A_0-(\varphi_0)_{\mu\mu\mu}]\\ &={}^{CF}I_\tau^\alpha[6^3\mu-0]\\ &=6^3\mu\left[(1-\alpha)1+\alpha\int_0^\tau ds\right]\\ &=6^3\mu[(1-\alpha)+\alpha[s]_0^\tau]\\ &=6^3\mu(1-\alpha)+\alpha\tau\\ &=6^3\mu(1-\alpha+\alpha\tau)\end{aligned}$$

$$\begin{aligned}\varphi_2(\mu,\tau)&={}^{CF}I_\tau^\alpha[6A_1-(\varphi_1)_{\mu\mu\mu}]\\ &={}^{CF}I_\tau^\alpha[6(2(6^4\mu))(1-\alpha+\alpha\tau)-0]\\ &=2(6^5\mu)\,{}^{CF}I_\tau^\alpha[(1-\alpha+\alpha\tau)]\end{aligned}$$

$$= 2(6^5\mu)\left[(1-2\alpha+\alpha^2)+2(\alpha-\alpha^2)\tau+\tfrac{1}{2}\alpha^2\tau^2\right]$$

$$\vdots$$

Then the approximate solution of $\varphi(\mu,\tau)$ *is given by*

$$\varphi(\mu,\tau)=\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)=\varphi_0+\varphi_1+\varphi_2+\cdots$$

$$=6\mu+6^3\mu(1-\alpha+\alpha\tau)+2(6^5\mu)\left[(1-2\alpha+\alpha^2)+2(\alpha-\alpha^2)\tau+\tfrac{1}{2}\alpha^2\tau^2\right] + \cdots \tag{21}$$

Therefore, the analytical solution when $\alpha\to 1$ *is given by*

$$\varphi(\mu,\tau)=\frac{6\mu}{1-6^2\tau}. \tag{22}$$

Example 2: We consider the following nonlinear equation in the Caputo-Fabrizio sense

$${}^{CF}D_\tau^\alpha\,\varphi(\mu,\tau)+\varphi\varphi_\mu+\varphi\varphi_{\mu\mu\mu}=0. \tag{23}$$

where $0\le\alpha\le 1, \mu\in\mathbb{R}, \tau>0$ *and subject to the initial condition*

$$\varphi(\mu,0)=\mu. \tag{24}$$

Taking ${}^{CF}I_\tau^\alpha$ *to both sides we get*

$$\varphi(\mu,\tau)=\mu-{}^{CF}I_\tau^\alpha\left[\varphi\varphi_\mu+\varphi\varphi_{\mu\mu\mu}\right]. \tag{25}$$

$$let\ \varphi=\sum_{n=0}^{\infty}\varphi_n,\qquad \varphi\varphi_\mu=\sum_{n=0}^{\infty}A_n\ \ and\ \ \varphi\varphi_{\mu\mu\mu}=\sum_{n=0}^{\infty}B_n$$

then

$$\varphi_0(\mu,\tau)=\mu$$

$$\varphi_{n+1}(\mu,\tau)=-{}^{CF}I_\tau^\alpha\left[\sum_{n=0}^{\infty}A_n-\sum_{n=0}^{\infty}B_n\right] \tag{26}$$

Now

$$A_0=\varphi_0\varphi_{0\mu}$$
$$B_0=\varphi_0\varphi_{0\mu\mu\mu}$$
$$A_1=\varphi_0\varphi_{1\mu}+\varphi_1\varphi_{0\mu}$$
$$B_1=\varphi_0\varphi_{1\mu\mu\mu}+\varphi_1\varphi_{0\mu\mu\mu}$$
$$A_2=\varphi_0\varphi_{2\mu}+\varphi_2\varphi_{0\mu}+\varphi_1\varphi_{1\mu}$$
$$B_2=\varphi_0\varphi_{2\mu\mu\mu}+\varphi_2\varphi_{0\mu\mu\mu}+\varphi_1\varphi_{1\mu\mu\mu}$$
$$\vdots$$

$$\begin{aligned}\varphi_1(\mu,\tau)&=-{}^{CF}I_\tau^\alpha[A_0-B_0]\\&=-{}^{CF}I_\tau^\alpha[\mu+0]\\&=-\mu\left[(1-\alpha)1+\alpha\int_0^\tau ds\right]\\&=-\mu[(1-\alpha)+\alpha[s]_0^\tau]\\&=-\mu(1-\alpha+\alpha\tau)\end{aligned}$$

$$\begin{aligned}\varphi_2(\mu,\tau)&=-{}^{CF}I_\tau^\alpha[A_1-B_1]\\&=-{}^{CF}I_\tau^\alpha[-2\mu(1-\alpha+\alpha\tau)+0]\\&=2\mu\,{}^{CF}I_\tau^\alpha[(1-\alpha+\alpha\tau)]\\&=2\mu\left[(1-2\alpha+\alpha^2)+2(\alpha-\alpha^2)\tau+\tfrac{1}{2}\alpha^2\tau^2\right]\end{aligned}$$

$$\vdots$$

Then the approximate solution of $\varphi(\mu,\tau)$ *is given by*

$$\varphi(\mu,\tau)=\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)$$
$$=\varphi_0+\varphi_1+\varphi_2+\cdots$$
$$=\mu-(1-\alpha+\alpha\tau)\mu+2\mu\left[(1-2\alpha+\alpha^2)+2(\alpha-\alpha^2)\tau+\tfrac{1}{2}\alpha^2\tau^2\right]+\cdots \qquad (27)$$

Therefore, the analytical solution when $\alpha\to 1$ *is given by*

$$\varphi(\mu,\tau)=\frac{\mu}{1-\tau} \qquad (28)$$

V. Conclusions

In this work, we have considered the fractional differential equations with Caputo-Fabrizio fractional derivative. The (FADM) has been successfully used to obtain the analytical approximate solutions . The obtained solutions were in the form of infinite power series which can be written in a closed form. The example shows that the results of (FADM) are in excellent agreement with the exact solution when $\alpha=1$. Because of the results, we can say that the proposed technique is a powerful mathematical tool for solving fractional differential equations. Also, we can use them to obtain approximate (or even analytical) solutions to other problems.

REFERENCES

[1]. I. Podlubny, Fractional differential equations, San Diego, Academic Press (1999).
[2]. I. Petras, Fractional-order nonlinear systems: modeling, analysis and simulation, Beijing, Higher Education Press, (2011).
[3]. H. K. Jassim, Some Dynamical Properties of Rössler System, Journal of University of Thi-Qar, Vol. 3 No. 1 (2017), 69-76.
[4]. H. Jafari, H. K. Jassim, On the Existence and Uniqueness of Solutions for Local differential equations, Entropy, 18(2016) 1-9.
[5]. S. Xu, X. Ling, Y. Zhao, H. K. Jassim, A Novel Schedule for Solving the Two-Dimensional Diffusion in Fractal Heat Transfer, Thermal Science, 19 (2015) S99-S103.
[6]. S. Q. Wang, Y. J. Yang, and H. K. Jassim, Local Fractional Function Decomposition Method for Solving Inhomogeneous Wave Equations with Local Fractional Derivative, Abstract and Applied Analysis, 2014 (2014) 1-7.
[7]. Z. P. Fan, H. K. Jassim, R. K. Rainna, and X. J. Yang, Adomian Decomposition Method for Three-Dimensional Diffusion Model in Fractal Heat Transfer Involving Local Fractional Derivatives, Thermal Science, 19(2015) S137-S141.
[8]. H. K. Jassim, et al., Fractional variational iteration method to solve one dimensional second order hyperbolic telegraph equations, Journal of Physics: Conference Series, 1032(1) (2018) 1-9.
[9]. S. P. Yan, H. Jafari, and H. K. Jassim, Local Fractional Adomian Decomposition and Function Decomposition Methods for Solving Laplace Equation within Local Fractional Operators, Advances in Mathematical Physics, 2014 (2014) 1-7.
[10]. H. Jafari, H. K. Jassim, F. Tchier, D. Baleanu, On the Approximate Solutions of Local Fractional Differential Equations with Local Fractional Operator, Entropy, 18 (2016) 1-12.
[11]. D. Baleanu, H. K. Jassim, Approximate Analytical Solutions of Goursat Problem within Local Fractional Operators, Journal of Nonlinear Science and Applications, 9(2016) 4829-4837.
[12]. H. K. Jassim, C. Ünlü, S. P. Moshokoa, C. M. Khalique, Local Fractional Laplace Variational Iteration Method for Solving Diffusion and Wave Equations on Cantor Sets within Local Fractional Operators, Mathematical Problems in Engineering, 2015 (2015) 1-9: ID 309870.
[13]. D. Baleanu, H. K. Jassim, M. Al Qurashi, Solving Helmholtz Equation with Local Fractional Derivative Operators, Fractal and Fractional, 3(43) (2019) 1-13.
[14]. H. K. Jassim, Analytical Approximate Solutions for Local Fractional Wave Equations, Mathematical Methods in the Applied Sciences, 43(2) (2020) 939-947.
[15]. H. Jafari, H. K. Jassim, and S. T. Mohyuid-Din, Local Fractional Laplace Decomposition Method for Solving Linear Partial Differential Equations with Local Fractional Derivative. In Fractional Dynamics. C. Cattani, H. M. Srivastava, and X.-J. Yang (Editors), De Gruyter Open, Berlin and Warsaw (2015) 296-316.
[16]. H. K. Jassim, New Approaches for Solving Fokker Planck Equation on Cantor Sets within Local Fractional Operators, Journal of Mathematics, 2015(2015)1-8 :ID 684598.
[17]. H. Jafari, H. K. Jassim, S. P. Moshokoa, V. M. Ariyan and F. Tchier, Reduced differential transform method for partial differential equations within local fractional derivative operators, Advances in Mechanical Engineering, 8(4) (2016) 1-6.
[18]. H. K. Jassim, The Approximate Solutions of Three-Dimensional Diffusion and Wave Equations within Local Fractional Derivative Operator, Abstract and Applied Analysis, 2016 (2016) 1-5: ID 2913539.
[19]. D. Baleanu, H. K. Jassim, H. Khan, A Modification Fractional Variational Iteration Method for solving Nonlinear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operators, Thermal Science, 22(2018) S165-S175.
[20]. H. Jafari, H. K. Jassim, J. Vahidi, Reduced Differential Transform and Variational Iteration Methods for 3D Diffusion Model in Fractal Heat Transfer within Local Fractional Operators, Thermal Science, 22(2018) S301-S307.
[21]. H. K. Jassim, D. Baleanu, A novel approach for Korteweg-de Vries equation of fractional order, Journal of Applied Computational Mechanics, 5(2) (2019) 192-198.
[22]. D. Baleanu, H. K. Jassim, Approximate Solutions of the Damped Wave Equation and Dissipative Wave Equation in Fractal Strings, Fractal and Fractional, 3(26) (2019) 1-12.
[23]. D. Baleanu, H. K. Jassim, A Modification Fractional Homotopy Perturbation Method for Solving Helmholtz and Coupled Helmholtz Equations on Cantor Sets, Fractal and Fractional, 3(30) (2019) 1-8.

[24]. J. Singh, H. K. Jassim, D. Kumar, An efficient computational technique for local fractional Fokker-Planck equation, Physica A: Statistical Mechanics and its Applications, 555(124525) (2020) 1-8.

[25]. H. K. Jassim, J. Vahidi, V. M. Ariyan, Solving Laplace Equation within Local Fractional Operators by Using Local Fractional Differential Transform and Laplace Variational Iteration Methods, Nonlinear Dynamics and Systems Theory, 20(4) (2020) 388-396.

[26]. D. Baleanu, H. K. Jassim, Exact Solution of Two-dimensional Fractional Partial Differential Equations, Fractal Fractional, 4(21) (2020) 1-9 .

[27]. H. K. Jassim, M. G. Mohammed, H. A. Eaued, A Modification Fractional Homotopy Analysis Method for Solving Partial Differential Equations Arising in Mathematical Physics, IOP Conf. Series: Materials Science and Engineering, 928 (042021) (2020) 1-22.

[28]. H. A. Eaued, H. K. Jassim, M. G. Mohammed, A Novel Method for the Analytical Solution of Partial Differential Equations Arising in Mathematical Physics, IOP Conf. Series: Materials Science and Engineering, 928 (042037) (2020) 1-16.

[29]. H. K. Jassim, J. Vahidi, A New Technique of Reduce Differential Transform Method to Solve Local Fractional PDEs in Mathematical Physics, International Journal of Nonlinear Analysis and Applications, 12(1) (2021) 37-44.

[30]. S. M. Kadhim, M. G. Mohammad, H. K. Jassim, How to Obtain Lie Point Symmetries of PDEs, Journal of Mathematics and Computer science, 22 (2021) 306-324.

[31]. H. K. Jassim, M. A. Shareef, On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator, Journal of Mathematics and Computer science, 23 (2021) 58-66.

[32]. H. K. Jassim, S. A. Khafif, SVIM for solving Burger's and coupled Burger's equations of fractional order, Progress in Fractional Differentiation and Applications, 7(1) (2021)1-6.

[33]. H. K. Jassim, H. A. Kadhim, Fractional Sumudu decomposition method for solving PDEs of fractional order, Journal of Applied and Computational Mechanics, 7(1) (2021) 302-311.

[34]. H. Jafari, H. K. Jassim, D. Baleanu, Y. M. Chu, On the approximate solutions for a system of coupled Korteweg-de Vries equations with local fractional derivative, Fractals, 29(5)(2021) 1-7.

[35]. H. K. Jassim, M. G. Mohammed, Natural homotopy perturbation method for solving nonlinear fractional gas dynamics equations, International Journal of Nonlinear Analysis and Applications, 12(1) (2021) 37-44.

[36]. M. G. Mohammed, H. K. Jassim, Numerical simulation of arterial pulse propagation using autonomous models, International Journal of Nonlinear Analysis and Applications, 12(1) (2021) 841-849.

[37]. H. K. Jassim, A new approach to find approximate solutions of Burger's and coupled Burger's equations of fractional order, TWMS Journal of Applied and Engineering Mathematics, 11(2) (2021) 415-423.

[38]. L. K. Alzaki, H. K. Jassim, The approximate analytical solutions of nonlinear fractional ordinary differential equations, International Journal of Nonlinear Analysis and Applications, 12(2) (2021) 527-535.

[39]. H. Jafari, and H. K. Jassim, Local Fractional Series Expansion Method for Solving Laplace and Schrodinger Equations on Cantor Sets within Local Fractional Operators, International Journal of Mathematics and Computer Research, vol. 2, no. 11, pp. 736-744, 2014.

[40]. H. Jafari, and H. K. Jassim, Local Fractional Adomian Decomposition Method for Solving Two Dimensional Heat conduction Equations within Local Fractional Operators, Journal of Advance in Mathematics, vol. 9 , No. 4, pp. 2574-2582, 2014.

[41]. H. Jafari, and H. K. Jassim, Local Fractional Laplace Variational Iteration Method for Solving Nonlinear Partial Differential Equations on Cantor Sets within Local Fractional Operators, Journal of Zankoy Sulaimani-Part A, vol. 16, no. 4, pp. 49-57, 2014.

[42]. H. Jafari and H. K. Jassim, Numerical Solutions of Telegraph and Laplace Equations on Cantor Sets Using Local Fractional Laplace Decomposition Method , International Journal of Advances in Applied Mathematics and Mechanics, vol. 2, No. 3, pp. 144-151, 2015.

[43]. H. Jafari and H. K. Jassim, A Coupling Method of Local Fractional Variational Iteration Method and Yang-Laplace Transform for Solving Laplace Equation on Cantor Sets, International Journal of pure and Applied Sciences and Technology, vol. 26, no. 1, pp. 24-33, 2015.

[44]. H. K. Jassim, Local Fractional Laplace Decomposition Method for Nonhomogeneous Heat Equations Arising in Fractal Heat Flow with Local Fractional Derivative, International Journal of Advances in Applied Mathematics and Mechanics, vol.2, no. 4, pp. 1-7, 2015.

[45]. H. K. Jassim, Homotopy Perturbation Algorithm Using Laplace Transform for Newell-Whitehead-Segel Equation, International Journal of Advances in Applied Mathematics and Mechanics, vol.2, no. 4, pp. 8-12, 2015.

[46]. H. Jafari, and H. K. Jassim, Application of the Local fractional Adomian Decomposition and Series Expansion Methods for Solving Telegraph Equation on Cantor Sets Involving Local Fractional Derivative Operators, Journal of Zankoy Sulaimani-Part A, vol. 17, no. 2, pp. 15-22 , 2015.

[47]. H. K. Jassim, Analytical Solutions for System of Fractional Partial Differential Equations by Homotopy Perturbation Transform Method, International Journal of Advances in Applied Mathematics and Mechanics, vol.3, no. 1, pp. 36-40, 2015.

[48]. H. K. Jassim, Analytical Approximate Solution for Inhomogeneous Wave Equation on Cantor Sets by Local Fractional Variational Iteration Method, International Journal of Advances in Applied Mathematics and Mechanics, vol.3, no. 1, pp. 57-61, 2015.

[49]. H. K. Jassim, Local Fractional Variational Iteration Transform Method to Solve partial differential equations arising in mathematical physics, International Journal of Advances in Applied Mathematics and Mechanics, vol.3, no. 1, pp. 71-76, 2015.

[50]. H. Jafari, H. K. Jassim, Local Fractional Variational Iteration Method for Nonlinear Partial Differential Equations within Local Fractional Operators, Applications and Applied Mathematics, vol. 10, no. 2, pp. 1055-1065, 2015

[51]. H. K. Jassim, The Approximate Solutions of Helmholtz and Coupled Helmholtz Equations on Cantor Sets within Local Fractional Operator, Journal of Zankoy Sulaimani-Part A, vol. 17, no. 4, pp. 19-25, 2015.

[52]. H. K. Jassim, The Approximate Solutions of Fredholm Integral Equations on Cantor Sets within Local Fractional Operators, Sahand Communications in Mathematical Analysis, Vol. 16, No. 1, 13-20, 2016.

[53]. H. Jafari, H. K. Jassim, Approximate Solution for Nonlinear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operator, Journal of Zankoy Sulaimani-Part A, vol. 18, no.1, pp.127-132, 2016

[54]. H. K. Jassim, Application of Laplace Decomposition Method and Variational Iteration Transform Method to Solve Laplace Equation, Universal Journal of Mathematics, Vol. 1, No. 1, pp.16-23, 2016.

[55]. H. Jafari, H. K. Jassim, A new approach for solving a system of local fractional partial differential equations, Applications and Applied Mathematics, Vol. 11, No. 1, pp.162-173, 2016.

[56]. H. K. Jassim, Local Fractional Variational Iteration Transform Method for Solving Couple Helmholtz Equations within Local Fractional Operator, Journal of Zankoy Sulaimani-Part A, Vol. 18, No. 2, pp.249-258, 2016.

[57]. H. K. Jassim, On Analytical Methods for Solving Poisson Equation, Scholars Journal of Research in Mathematics and Computer Science, Vol. 1, No. 1, pp. 26- 35, 2016.

[58]. H. K. Jassim, Hussein Khashan Kadhim, Application of Local Fractional Variational Iteration Method for Solving Fredholm Integral Equations Involving Local Fractional Operators, Journal of University of Thi-Qar, Vol. 11, No. 1, pp. 12-18, 2016.

[59]. H. Jafari, H. K. Jassim, Application of Local Fractional Variational Iteration Method to Solve System of Coupled Partial Differential Equations Involving Local Fractional Operator, Applied Mathematics & Information Sciences Letters, Vol. 5, No. 2, pp. 1-6, 2017.

[60]. H. K. Jassim, The Analytical Solutions for Volterra Integro-Differential Equations Involving Local fractional Operators by Yang-Laplace Transform, Sahand Communications in Mathematical Analysis, Vol. 6 No. 1 (2017), 69-76.

[61]. H. K. Jassim, A Coupling Method of Regularization and Adomian Decomposition for Solving a Class of the Fredholm Integral Equations within Local Fractional Operators, Vol. 2, No. 3, 2017, pp. 95-99.

[62]. H. K. Jassim, On Approximate Methods for Fractal Vehicular Traffic Flow, TWMS Journal of Applied and Engineering Mathematics, Vol. 7, No. 1, pp. 58-65, 2017

[63]. H. K. Jassim, A Novel Approach for Solving Volterra Integral Equations Involving Local Fractional Operator, Applications and Applied Mathematics, Vol. 12, Issue 1 (2017), pp. 496 – 505.

[64]. H. K. Jassim, A New Adomian Decomposition Method for Solving a Class of Volterra Integro-Differential Equations within Local Fractional Integral Operators, Journal of college of Education for Pure Science, Vol. 7, No. 1, pp. 19-29, 2017.

[65]. H. K. Jassim, A. A. Neamah, Analytical Solution of The One Dimensional Volterra Integro Differential Equations within Local Fractional Derivative, Journal of Kufa for Mathematics and Computer, V ol.4 ,No.1, 2017, pp. 46-50

[66]. H. K. Jassim, Solving Poisson Equation within Local Fractional Derivative Operators, Research in Applied Mathematics, vol. 1 (2017, pp. 1-12.

[67]. H. K. Jassim, An Efficient Technique for Solving Linear and Nonlinear Wave Equation within Local Fractional Operators, The Journal of Hyperstructures, vol.6, no. 2, 2017, 136-146.

[68]. H. K. Jassim, M. G. Mohammed, S. A. Khafif, The Approximate solutions of time-fractional Burger's and coupled time-fractional Burger's equations, International Journal of Advances in Applied Mathematics and Mechanics,6(4)(2019)64-70.

[69]. Mayada Gassab Mohammad, H. K. Jassim, S. M. Kadhim, Symmetry Classification of First Integrals for Scalar Linearizable, International Journal of Advances in Applied Mathematics and Mechanics, 7(1) (2019) 20-40.

[70]. M. Caputo, M. Fabrizio, A new Definition of Fractional Derivative without Singular Kernel, Progress in Fractional Differentiation and Applications, **1(2)** 73-85(2015).

[71]. M. Al-Refai, K. Pal, New Aspects of Caputo-Fabrizio Fractional Derivative, Progress in Fractional Differentiation and Applications, **5(2)** 157-166 (2019).