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**Research Paper**

# Exact analytic solutions of Two Dimensional Fractional PDEs

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***ABSTRACT:*** *In this work, we provide an approximate solution of a parabolic fractional degenerate problem emerging in a spatial diffusion of biological population model using the fractional Adomian decomposition method (FADM). A new solution is constructed in power series. The time fractional derivatives are described in the Caputo sense. The results prove that the proposed method is very effective and simple for solving fractional partial differential equations.*
***KEYWORDS:*** *Fractional biological population model; Mittag-leffler function; Caputo fractional derivative; Adomian decomposition method.*

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## I. INTRODUCTION

Fractional differential equations (FDEs) have been applied in many fields, such as physics, mechanics, chemistry, engineering etc. There has been a significant progress in ordinary differential equations involving fractional order derivative, see the monographs of Hilfer, Kilbas and Podlubny. Especially, numerous works have been devoted to the study of initial value problems, etc. [1-6].

In recent decades, many of the numerical and analytical techniques have been implemented to solve the differential equations in the Caputo sense such as the fractional differential transform method, fractional variational iteration method, fractional differential transform method, fractional series expansion method, fractional Sumudu variational iteration method, fractional Laplace transform method, fractional homotopy perturbation method, fractional Sumudu decomposition method, fractional Fourier series method, fractional reduced differential transform method, fractional Adomian decomposition method, and another methods [7-71]. Our aim is to present the FADM, and to used it to solve two dimensional FRDE. The remaining sections of this work are organized as follows. In Section 2, some background notations of fractional calculus are presented. In Section 3, the analysis of fractional ADM is discussed. Applications of FADM are shown in Section 4. The conclusion of this paper is given in Section 5.

## II. PRELIMINARIES

Some fractional Calculus definitions and notation needed in the course of this work section are discussed in this section.

**Definition 2.1:** A real function $\varphi(\mu)$ , $\mu > 0$, is said to be in the space $C_\vartheta, \vartheta \in \mathbb{R}$ if there exists a real number $q\ (q > \vartheta)$, such that $\varphi(\mu) = \mu^q\ \varphi_1(\mu)$, where $\varphi_1(\mu) \in [0, \infty)$, and it is said to be in the space $C_\vartheta^m$ if $\varphi^{(m)} \in C_\vartheta, m \in \mathbb{N}$.

**Definition 2.2:** The Riemann Liouville fractional integral operator of order $v \geq 0$ , of a function $\varphi(\mu) \in C_\vartheta, \vartheta \geq -1$ is defined as

$$I^v\varphi(\mu) = \begin{cases} \dfrac{1}{\Gamma(v)}\displaystyle\int_0^\mu (\mu-\tau)^{v-1}\varphi(\tau)\,d\tau, v > 0, \mu > 0. \\ I^0\ \varphi(\mu) = \varphi(\mu) \quad , \quad v = 0 \end{cases} \qquad (1)$$

Where $\Gamma(\cdot)$ is the well-known Gamma function.

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**Properties of operator $I^{v}$, which we will use here, are as follows:**

for $\varphi \in C_{\vartheta}$ ,$\vartheta \geq -1$ , $v,\sigma \geq 0$ ,

1. $I^{v}I^{\sigma}\varphi(\mu) = I^{v+\sigma}\varphi(\mu)$.
2. $I^{v}I^{\sigma}\varphi(\mu) = I^{\sigma}I^{v}\varphi(\mu)$.
3. $I^{v}\mu^{m} = \frac{\Gamma(m+1)}{\Gamma(v+m+1)}\mu^{v+m}$.

**Definition 2.3:** The Caputo derivative of fractional order $v$ of a function $\varphi(\mu)$ is defined as :-

$$D^{v}\varphi(\mu) = I^{m-v}D^{m}\varphi(\mu)$$
$$= \frac{1}{\Gamma(m-v)}\int_{0}^{\mu}(\mu-\tau)^{m-v-1}\varphi^{(m)}(\tau)d\tau \qquad (2)$$

For $m-1 < v < m$ , $m \in N$ , $\mu > 0$ and $\varphi \in C_{-1}^{m}$.

The following are the basic properties of the operator $D^{v}$:-

1. $D^{v}k = 0$ , where k is a constant.
2. $D^{v}I^{v}\varphi(\mu) = \varphi(\mu)$,
3. $D^{v}\mu^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-v+1)}\mu^{\sigma-v}$,
4. $D^{v}D^{\sigma}\varphi(\mu) = D^{v+\sigma}\varphi(\mu)$
5. $D^{v}\big(a\,\varphi(\mu) + b\psi(\mu)\big) = aD^{v}\varphi(\mu) + bD^{v}\psi(\mu)$.
6. $I^{v}D^{v}\varphi(\mu) = \varphi(\mu) - \sum_{k=0}^{m-1}\varphi^{(k)}(0)\frac{\mu^{k}}{k!}$.

**Definition 2.4 :** The Mittag-Leffler function $E_{v}(z)$ with $v > 0$ is defined as:-

$$E_{v}(z) = \sum_{m=0}^{\infty}\frac{z^{v}}{\Gamma(mv+1)} \qquad (3)$$

## III. ANALYSIS OF FADM :

Let us consider a generalized non-linear biological population equation of the form:

$$^{c}D_{\tau}^{v}\varphi(\mu,\zeta,\tau) = \frac{\partial^{2}\varphi^{2}}{\partial\mu^{2}} + \frac{\partial^{2}\varphi^{2}}{\partial\zeta^{2}} + h\,\varphi^{a}(1-r\,\varphi^{b}) \qquad (4)$$

with the initial condition:

$$\varphi^{(k)}(\mu,\zeta,0) = C_{k} \qquad , \qquad k = 0,1,\dots,m-1$$

where $\varphi(\mu,\zeta,\tau)$ is an unknown function, $^{c}D_{\tau}^{v}$ is the Caputo operator , $m-1 < v \leq m$ and h,a,b,r are real numbers.

Now, Operating with $I^{v}$ in both sides of Eq.(4), we find

$$\varphi(\mu,\zeta,\tau) = \varphi(\mu,\zeta,0) + I^{v}\left(\varphi_{\mu\mu}^{2} + \varphi_{\zeta\zeta}^{2} + h\,\varphi^{a}(1-r\,\varphi^{b})\right) \qquad (5)$$

Now, we represent solution as an infinite series given below:

$$\varphi(\mu,\zeta,\tau) = \sum_{n=0}^{\infty}\varphi_{n}(\mu,\zeta,\tau) \qquad (6)$$

By substituting Eq. (6) in Eq. (5):

$$\sum_{n=0}^{\infty}\varphi_{n}(\mu,\zeta,\tau) = \varphi(\mu,\zeta,0) + I^{v}\left(\sum_{n=0}^{\infty}A_{n}\right) \qquad (7)$$

where $A_{n}$ are Adomian's polynomials which are derived as:

$$A_{0} = (\varphi_{0}^{2})_{\mu\mu} + (\varphi_{0}^{2})_{\zeta\zeta} + h\varphi_{0}^{a}(1-r\,\varphi_{0}^{b}),$$
$$A_{1} = (2\varphi_{0}\varphi_{1})_{\mu\mu} + (2\varphi_{0}\varphi_{1})_{\zeta\zeta} + ah\varphi_{0}^{a-1}\varphi_{1} + rh(a+b)\varphi_{0}^{a+b-1}\varphi_{1} ,$$
$$A_{2} = (2\varphi_{0}\varphi_{2} + \varphi_{1}^{2})_{\mu\mu} + (2\varphi_{0}\varphi_{2} + \varphi_{1}^{2})_{\zeta\zeta} + ah(\varphi_{0}^{a-1}\varphi_{2}$$
$$+ \tfrac{1}{2}(a-b)\,\varphi_{0}^{a-2}\varphi_{1}^{2}) + rh(a+b)(\varphi_{0}^{a+b-1}\varphi_{2} + \tfrac{1}{2}(a+b-1)\,\varphi_{0}^{a+b-2}\varphi_{1}^{2}.$$
$$\vdots$$

on comparing both sides of Eq.(7), we get

$$\varphi_0(\mu,\zeta,\tau) = \varphi(\mu,\zeta,0)$$
$$\varphi_1(\mu,\zeta,\tau) = I^v(A_0)$$
$$\varphi_2(\mu,\zeta,\tau) = I^v(A_1)$$
$$\vdots$$

and so on,

Thus, the approximate solution of eq.(4) is :

$$\varphi(\mu,\zeta,\tau) = \sum_{n=0}^{\infty} \varphi_n(\mu,\zeta,\tau) \tag{8}$$

## IV. APPLICATIONS OF FADM

**Example 4.1:** Consider the fractional generalized biological population model:

$$^{c}D_{\tau}^{v}\ \varphi(\mu,\zeta,\tau) = \varphi_{\mu\mu}^2 + \varphi_{\zeta\zeta}^2 - \varphi\left(\frac{8}{9}\varphi + 1\right) \tag{9}$$

where $0 < v \le 1$ and subject to the initial condition

$$\varphi(\mu,\zeta,0) = e^{\frac{1}{3}(\mu+\zeta)}.$$

operating with $I^v$ in both sides of Eq. (9),

$$\varphi(\mu,\zeta,\tau) =$$
$$\varphi(\mu,\zeta,0) + I^v\left(\varphi_{\mu\mu}^2 + \varphi_{\zeta\zeta}^2 - \frac{8}{9}\varphi - \varphi\right) \tag{10}$$

$$\text{Let } \varphi(\mu,\zeta,\tau) = \sum_{n=0}^{\infty} \varphi_n(\mu,\zeta,\tau) \tag{11}$$

Substituting the decomposition series (11) into (10), yields:

$$\sum_{n=0}^{\infty} \varphi_n(\mu,\zeta,\tau) = \varphi(\mu,\zeta,0) + I^v\left(\sum_{n=0}^{\infty} A_n\right) \tag{12}$$

where $A_n$ are Adomian's polynomials which are derived as:

$$\begin{aligned} A_0 &= (\varphi_0^2)_{\mu\mu} + (\varphi_0^2)_{\zeta\zeta} - \frac{8}{9}\varphi_0^2 - \varphi_0\,, \\ A_1 &= (2\varphi_0\varphi_1)_{\mu\mu} + (2\varphi_0\varphi_1)_{\zeta\zeta} - \frac{16}{9}\varphi_0\varphi_1 - \varphi_1\,, \\ &\vdots \end{aligned} \tag{13}$$

on comparing both sides of Eq.(12), we get

$$\varphi_0(\mu,\zeta,\tau) = \varphi(\mu,\zeta,0) = e^{\frac{1}{3}(\mu+\zeta)}.$$
$$\begin{aligned} \varphi_1(\mu,\zeta,\tau) &= I^v(A_0) \\ &= I^v\left(0 - e^{\frac{1}{3}(\mu+\zeta)}\right) \\ &= \frac{-\tau^v}{\Gamma(v+1)} e^{\frac{1}{3}(\mu+\zeta)} \end{aligned}$$
$$\begin{aligned} \varphi_2(\mu,\zeta,\tau) &= I^v(A_1) \\ &= I^v\left(0 + \frac{\tau^v}{\Gamma(v+1)} e^{\frac{1}{3}(\mu+\zeta)}\right) \\ &= \frac{\tau^{2v}}{\Gamma(2v+1)} e^{\frac{1}{3}(\mu+\zeta)} \end{aligned}$$
$$\vdots$$

and so on,

then we have the solution $\varphi(\mu,\zeta,\tau)$ in a series form by

$$\varphi(\mu,\zeta,\tau) = e^{\frac{1}{3}(\mu+\zeta)}\left(1 - \frac{\tau^v}{\Gamma_{(v+1)}} + \frac{\tau^{2v}}{\Gamma_{(2v+1)}} - \cdots\right)$$

$$= e^{\frac{1}{3}(\mu+\zeta)}\, E_v(-\tau^v). \tag{14}$$

Where $E_v(-\tau^v)$ is the Mittag-Leffler function.

When $v = 1$

$$\varphi(\mu,\zeta,\tau) = e^{\frac{1}{3}(\mu+\zeta)}\left(1-\tau+\frac{\tau^2}{2!}-\cdots\right)$$

$$= e^{\frac{1}{3}(\mu+\zeta)-\tau}. \tag{15}$$

## V. CONCLUSION

The approximate solutions of nonlinear fractional order biological population model with time fractional derivatives have been obtained by successful application of FADM. The obtained solutions were in the form of infinite power series which can be written in a closed form. In view of the results, we can say that this technique is powerful mathematical tool for solving fractional PDEs. Also, we can use it to obtain approximate solution of other problems.

## REFERENCES

[1] R. Hilfer, Application of fractional Calculus in Physics, World Scientific, Singapore, 1999.
[2] R. Hilfer, Y. Luchko, Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Lioville fractional derivative, Fract. Calc. Appl. Anal., 12 (2009), 229-318.
[3] I. Petras, Fractional-order nonlinear systems: modeling, analysis and simulation, Beijing, Higher Education Press, (2011).
[4] H. K. Jassim, Some Dynamical Properties of Rössler System, Journal of University of Thi-Qar, Vol. 3 No. 1 (2017), 69-76.
[5] I. Podlubny, Fractional differential equations, San Diego, Academic Press (1999).
[6] H. Jafari, H. K. Jassim, On the Existence and Uniqueness of Solutions for Local differential equations, Entropy, 18(2016) 1-9.
[7] S. Xu, X. Ling, Y. Zhao, H. K. Jassim, A Novel Schedule for Solving the Two-Dimensional Diffusion in Fractal Heat Transfer, Thermal Science, 19 (2015) S99-S103.
[8] S. Q. Wang, Y. J. Yang, and H. K. Jassim, Local Fractional Function Decomposition Method for Solving Inhomogeneous Wave Equations with Local Fractional Derivative, Abstract and Applied Analysis, 2014 (2014) 1-7.
[9] Z. P. Fan, H. K. Jassim, R. K. Rainna, and X. J. Yang, Adomian Decomposition Method for Three-Dimensional Diffusion Model in Fractal Heat Transfer Involving Local Fractional Derivatives, Thermal Science, 19(2015) S137-S141.
[10] H. K. Jassim, et al., Fractional variational iteration method to solve one dimensional second order hyperbolic telegraph equations, Journal of Physics: Conference Series, 1032(1) (2018) 1-9.
[11] S. P. Yan, H. Jafari, and H. K. Jassim, Local Fractional Adomian Decomposition and Function Decomposition Methods for Solving Laplace Equation within Local Fractional Operators, Advances in Mathematical Physics, 2014 (2014) 1-7.
[12] H. Jafari, H. K. Jassim, F. Tchier, D. Baleanu, On the Approximate Solutions of Local Fractional Differential Equations with Local Fractional Operator, Entropy, 18 (2016) 1-12.
[13] D. Baleanu, H. K. Jassim, Approximate Analytical Solutions of Goursat Problem within Local Fractional Operators, Journal of Nonlinear Science and Applications, 9(2016) 4829-4837.
[14] H. K. Jassim, C. Ünlü, S. P. Moshokoa, C. M. Khalique, Local Fractional Laplace Variational Iteration Method for Solving Diffusion and Wave Equations on Cantor Sets within Local Fractional Operators, Mathematical Problems in Engineering, 2015 (2015) 1-9: ID 309870.
[15] D. Baleanu, H. K. Jassim, M. Al Qurashi, Solving Helmholtz Equation with Local Fractional Derivative Operators, Fractal and Fractional, 3(43) (2019) 1-13.
[16] H. K. Jassim, Analytical Approximate Solutions for Local Fractional Wave Equations, Mathematical Methods in the Applied Sciences, 43(2) (2020) 939-947.
[17] H. Jafari, H. K. Jassim, and S. T. Mohyuid-Din, Local Fractional Laplace Decomposition Method for Solving Linear Partial Differential Equations with Local Fractional Derivative. In Fractional Dynamics. C. Cattani, H. M. Srivastava, and X.-J. Yang (Editors), De Gruyter Open, Berlin and Warsaw (2015) 296-316.
[18] H. K. Jassim, New Approaches for Solving Fokker Planck Equation on Cantor Sets within Local Fractional Operators, Journal of Mathematics, 2015(2015)1-8 :ID 684598.
[19] H. Jafari, H. K. Jassim, S. P. Moshokoa, V. M. Ariyan and F. Tchier, Reduced differential transform method for partial differential equations within local fractional derivative operators, Advances in Mechanical Engineering, 8(4) (2016) 1-6.
[20] H. K. Jassim, The Approximate Solutions of Three-Dimensional Diffusion and Wave Equations within Local Fractional Derivative Operator, Abstract and Applied Analysis, 2016 (2016) 1-5: ID 2913539.
[21] D. Baleanu, H. K. Jassim, H. Khan, A Modification Fractional Variational Iteration Method for solving Nonlinear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operators, Thermal Science, 22( 2018) S165-S175.
[22] H. Jafari, H. K. Jassim, J. Vahidi, Reduced Differential Transform and Variational Iteration Methods for 3D Diffusion Model in Fractal Heat Transfer within Local Fractional Operators, Thermal Science, 22( 2018) S301-S307.
[23] H. K. Jassim, D. Baleanu, A novel approach for Korteweg-de Vries equation of fractional order, Journal of Applied Computational Mechanics, 5(2) (2019) 192-198.
[24] D. Baleanu, H. K. Jassim, Approximate Solutions of the Damped Wave Equation and Dissipative Wave Equation in Fractal Strings, Fractal and Fractional, 3(26) (2019) 1-12.
[25] D. Baleanu, H. K. Jassim, A Modification Fractional Homotopy Perturbation Method for Solving Helmholtz and Coupled Helmholtz Equations on Cantor Sets, Fractal and Fractional, 3(30) (2019) 1-8.
[26] J. Singh, H. K. Jassim, D. Kumar, An efficient computational technique for local fractional Fokker-Planck equation, Physica A: Statistical Mechanics and its Applications, 555(124525) (2020) 1-8.

[27] H. K. Jassim, J. Vahidi, V. M. Ariyan, Solving Laplace Equation within Local Fractional Operators by Using Local Fractional Differential Transform and Laplace Variational Iteration Methods, Nonlinear Dynamics and Systems Theory, 20(4) (2020) 388-396.
[28] D. Baleanu, H. K. Jassim, Exact Solution of Two-dimensional Fractional Partial Differential Equations, Fractal Fractional, 4(21) (2020) 1-9 .
[29] H. K. Jassim, M. G. Mohammed, H. A. Eaued, A Modification Fractional Homotopy Analysis Method for Solving Partial Differential Equations Arising in Mathematical Physics, IOP Conf. Series: Materials Science and Engineering, 928 (042021) (2020) 1-22.
[30] H. A. Eaued, H. K. Jassim, M. G. Mohammed, A Novel Method for the Analytical Solution of Partial Differential Equations Arising in Mathematical Physics, IOP Conf. Series: Materials Science and Engineering, 928 (042037 ) (2020) 1-16.
[31] H. K. Jassim, J. Vahidi, A New Technique of Reduce Differential Transform Method to Solve Local Fractional PDEs in Mathematical Physics, International Journal of Nonlinear Analysis and Applications, 12(1) (2021) 37-44.
[32] S. M. Kadhim, M. G. Mohammad, H. K. Jassim, How to Obtain Lie Point Symmetries of PDEs, Journal of Mathematics and Computer science, 22 (2021) 306-324.
[33] H. K. Jassim, M. A. Shareef, On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator, Journal of Mathematics and Computer science, 23 (2021) 58-66.
[34] H. K. Jassim, S. A. Khafif, SVIM for solving Burger's and coupled Burger's equations of fractional order, Progress in Fractional Differentiation and Applications, 7(1) (2021)1-6.
[35] H. K. Jassim, H. A. Kadhim, Fractional Sumudu decomposition method for solving PDEs of fractional order, Journal of Applied and Computational Mechanics, 7(1) (2021) 302-311.
[36] H. Jafari, H. K. Jassim, D. Baleanu, Y. M. Chu, On the approximate solutions for a system of coupled Korteweg-de Vries equations with local fractional derivative, Fractals, 29(5)(2021) 1-7.
[37] H. K. Jassim, M. G. Mohammed, Natural homotopy perturbation method for solving nonlinear fractional gas dynamics equations, International Journal of Nonlinear Analysis and Applications, 12(1) (2021) 37-44.
[38] M. G. Mohammed, H. K. Jassim, Numerical simulation of arterial pulse propagation using autonomous models, International Journal of Nonlinear Analysis and Applications, 12(1) (2021) 841-849.
[39] H. K. Jassim, A new approach to find approximate solutions of Burger's and coupled Burger's equations of fractional order, TWMS Journal of Applied and Engineering Mathematics, 11(2) (2021) 415-423.
[40] L. K. Alzaki, H. K. Jassim, The approximate analytical solutions of nonlinear fractional ordinary differential equations, International Journal of Nonlinear Analysis and Applications, 12(2) (2021) 527-535.
[41] H. Jafari, and H. K. Jassim, Local Fractional Series Expansion Method for Solving Laplace and Schrodinger Equations on Cantor Sets within Local Fractional Operators, International Journal of Mathematics and Computer Research, vol. 2, no. 11, pp. 736-744, 2014.
[42] H. Jafari, and H. K. Jassim, Local Fractional Adomian Decomposition Method for Solving Two Dimensional Heat conduction Equations within Local Fractional Operators, Journal of Advance in Mathematics, vol. 9 , No. 4, pp. 2574-2582, 2014.
[43] H. Jafari, and H. K. Jassim, Local Fractional Laplace Variational Iteration Method for Solving Nonlinear Partial Differential Equations on Cantor Sets within Local Fractional Operators, Journal of Zankoy Sulaimani-Part A, vol. 16, no. 4, pp. 49-57, 2014.
[44] H. Jafari and H. K. Jassim, Numerical Solutions of Telegraph and Laplace Equations on Cantor Sets Using Local Fractional Laplace Decomposition Method , International Journal of Advances in Applied Mathematics and Mechanics, vol. 2, No. 3, pp. 144-151, 2015.
[45] H. Jafari and H. K. Jassim, A Coupling Method of Local Fractional Variational Iteration Method and Yang-Laplace Transform for Solving Laplace Equation on Cantor Sets, International Journal of pure and Applied Sciences and Technology, vol. 26, no. 1, pp. 24-33, 2015.
[46] H. K. Jassim, Local Fractional Laplace Decomposition Method for Nonhomogeneous Heat Equations Arising in Fractal Heat Flow with Local Fractional Derivative, International Journal of Advances in Applied Mathematics and Mechanics, vol.2, no. 4, pp. 1-7, 2015.
[47] H. K. Jassim, Homotopy Perturbation Algorithm Using Laplace Transform for Newell-Whitehead-Segel Equation, International Journal of Advances in Applied Mathematics and Mechanics, vol.2, no. 4, pp. 8-12, 2015.
[48] H. Jafari, and H. K. Jassim, Application of the Local fractional Adomian Decomposition and Series Expansion Methods for Solving Telegraph Equation on Cantor Sets Involving Local Fractional Derivative Operators, Journal of Zankoy Sulaimani-Part A, vol. 17, no. 2, pp. 15-22 , 2015.
[49] H. K. Jassim, Analytical Solutions for System of Fractional Partial Differential Equations by Homotopy Perturbation Transform Method, International Journal of Advances in Applied Mathematics and Mechanics, vol.3, no. 1, pp. 36-40, 2015.
[50] H. K. Jassim, Analytical Approximate Solution for Inhomogeneous Wave Equation on Cantor Sets by Local Fractional Variational Iteration Method, International Journal of Advances in Applied Mathematics and Mechanics, vol.3, no. 1, pp. 57-61, 2015.
[51] H. K. Jassim, Local Fractional Variational Iteration Transform Method to Solve partial differential equations arising in mathematical physics, International Journal of Advances in Applied Mathematics and Mechanics, vol.3, no. 1, pp. 71-76, 2015.
[52] H. Jafari, H. K. Jassim, Local Fractional Variational Iteration Method for Nonlinear Partial Differential Equations within Local Fractional Operators, Applications and Applied Mathematics, vol. 10, no. 2, pp. 1055-1065, 2015
[53] H. K. Jassim, The Approximate Solutions of Helmholtz and Coupled Helmholtz Equations on Cantor Sets within Local Fractional Operator, Journal of Zankoy Sulaimani-Part A, vol. 17, no. 4, pp. 19-25, 2015.
[54] H. K. Jassim, The Approximate Solutions of Fredholm Integral Equations on Cantor Sets within Local Fractional Operators, Sahand Communications in Mathematical Analysis, Vol. 16, No. 1, 13-20, 2016.
[55] H. Jafari, H. K. Jassim, Approximate Solution for Nonlinear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operator, Journal of Zankoy Sulaimani-Part A, vol. 18, no.1, pp.127-132, 2016
[56] H. K. Jassim, Application of Laplace Decomposition Method and Variational Iteration Transform Method to Solve Laplace Equation, Universal Journal of Mathematics, Vol. 1, No. 1, pp.16-23, 2016.
[57] H. Jafari, H. K. Jassim, A new approach for solving a system of local fractional partial differential equations, Applications and Applied Mathematics, Vol. 11, No. 1, pp.162-173, 2016.
[58] H. K. Jassim, Local Fractional Variational Iteration Transform Method for Solving Couple Helmholtz Equations within Local Fractional Operator, Journal of Zankoy Sulaimani-Part A, Vol. 18, No. 2, pp.249-258, 2016.
[59] H. K. Jassim, On Analytical Methods for Solving Poisson Equation, Scholars Journal of Research in Mathematics and Computer Science, Vol. 1, No. 1, pp. 26- 35, 2016.
[60] H. K. Jassim, Hussein Khashan Kadhim, Application of Local Fractional Variational Iteration Method for Solving Fredholm Integral Equations Involving Local Fractional Operators, Journal of University of Thi-Qar, Vol. 11, No. 1, pp. 12-18, 2016.

[61] H. Jafari, H. K. Jassim, Application of Local Fractional Variational Iteration Method to Solve System of Coupled Partial Differential Equations Involving Local Fractional Operator, Applied Mathematics & Information Sciences Letters, Vol. 5, No. 2, pp. 1-6, 2017.
[62] H. K. Jassim, The Analytical Solutions for Volterra Integro-Differential Equations Involving Local fractional Operators by Yang-Laplace Transform, Sahand Communications in Mathematical Analysis, Vol. 6 No. 1 (2017), 69-76.
[63] H. K. Jassim, A Coupling Method of Regularization and Adomian Decomposition for Solving a Class of the Fredholm Integral Equations within Local Fractional Operators, Vol. 2, No. 3, 2017, pp. 95-99.
[64] H. K. Jassim, On Approximate Methods for Fractal Vehicular Traffic Flow, TWMS Journal of Applied and Engineering Mathematics, Vol. 7, No. 1, pp. 58-65, 2017
[65] H. K. Jassim, A Novel Approach for Solving Volterra Integral Equations Involving Local Fractional Operator, Applications and Applied Mathematics, Vol. 12, Issue 1 (2017), pp. 496 – 505.
[66] H. K. Jassim, A New Adomian Decomposition Method for Solving a Class of Volterra Integro-Differential Equations within Local Fractional Integral Operators, Journal of college of Education for Pure Science, Vol. 7, No. 1, pp. 19-29, 2017.
[67] H. K. Jassim, A. A. Neamah, Analytical Solution of The One Dimensional Volterra Integro Differential Equations within Local Fractional Derivative, Journal of Kufa for Mathematics and Computer, V ol.4 ,No.1, 2017, pp. 46-50
[68] H. K. Jassim, Solving Poisson Equation within Local Fractional Derivative Operators, Research in Applied Mathematics, vol. 1 (2017, pp. 1-12.
[69] H. K. Jassim, An Efficient Technique for Solving Linear and Nonlinear Wave Equation within Local Fractional Operators, The Journal of Hyperstructures, vol.6, no. 2, 2017, 136-146.
[70] H. K. Jassim, M. G. Mohammed, S. A. Khafif, The Approximate solutions of time-fractional Burger's and coupled time-fractional Burger's equations, International Journal of Advances in Applied Mathematics and Mechanics,6(4)(2019)64-70.
[71] Mayada Gassab Mohammad, H. K. Jassim, S. M. Kadhim, Symmetry Classification of First Integrals for Scalar Linearizable, International Journal of Advances in Applied Mathematics and Mechanics, 7(1) (2019) 20-40.