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Research Paper

Refinement Invariants of Binary Images, II

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ABSTRACT: *A few more uniqueness results are proven for the Euler numbers of binary images.*
KEYWORDS: *Euler number, binary digital image, refinement invariant.*

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I. BASICS

As in the previous paper [8], we define a *binary digital image* to be a function $P : \mathbf{Z}^2 \to \{0,1\}$. Alternatively, it is also defined as an array of 0/1 integer values. A coordinate system in $\mathbf{Z}^2$ is chosen such that the first axis points downward (the *row* axis) and the second axis points to the right (the *column* axis). As usual, an element $(i, j) \in \mathbf{Z}^2$ can be regarded as a point (placed at row i and column j), or as a square placed with its center or with its upper-left corner at coordinates (i, j); such an element is usually called a *pixel*. If $P(i, j) = 0$, the pixel (i, j) is called a *background* point; otherwise, if $P(i, j) = 1$, the pixel (i, j) is called a *foreground* point. We assume the number of 1-pixels to be finite; we can therefore restrict each image to a digital rectangle.

Given a binary digital image P, the set of all pixels defines a graph $gr(P)$. The vertices of this graph are single pixels, represented as $[p]$; its faces are quadruples of pixels, represented as $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$; and its edges are horizontal pairs $[p \;\; q]$, vertical pairs $\begin{bmatrix} p \\ r \end{bmatrix}$, main diagonal pairs $\begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix}$, and secondary diagonal pairs $\begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}$.

II. RESOLUTION REFINEMENTS

Let P be a binary digital image. The *resolution refinement* of P is a binary digital image Q, obtained from P by dividing each pixel into four smaller ones; this division is performed by cutting each square by a horizontal line and a vertical line, both passing through its center.

A convenient way to represent the resolution refinement of a binary image is to place the pixels of the original image P with their corners at integral coordinates, then multiply all coordinates by 2 and perform the subdivision (and finally move back the pixels with their centers at integral coordinates, if desired).

A function ρ on the class of binary digital images will be called a *resolution refinement invariant* (or just a *refinement invariant*), if $\rho(Q) = \rho(P)$ whenever Q is the resolution refinement of P. The aim of this paper is to characterize all linear, real valued, resolution refinement invariants of certain kinds, including the refinement invariants discussed in [8].

In the following we will denote by $\#(P;\pi)$ the number of occurrences of the local pattern π in the graph $gr(P)$.

III. TECHNICAL DETAILS

Consider the function

$$\rho(P) = \alpha_v \cdot \#(P;[1]) + \alpha_{hi} \cdot \#(P;[1\ \ 1]) + \alpha_{vi} \cdot \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$$

$$+ \alpha_{he} \cdot \#(P;[0\ \ 1]) + \alpha_{ve} \cdot \#\left(P;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) + \alpha_{sq} \cdot \#\left(P;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right)$$

where the coefficients $\alpha_v, \alpha_{hi}, \alpha_{vi}, \alpha_{he}, \alpha_{ve}, \alpha_{sq}$ are real numbers.

If Q is a resolution refinement of P, then

- each vertex $[1]$ of P generates in Q: four vertices $[1]$, two horizontal edges $[1\ \ 1]$, two vertical edges $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, and one square $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$;
- each horizontal edge $[1\ \ 1]$ of P generates in Q, in excess: two horizontal edges $[1\ \ 1]$, and one square $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$;
- each vertical edge $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ of P generates in Q, in excess: two vertical edges $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, and one square $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$;
- each horizontal edge $[0\ \ 1]$ of P generates in Q, in excess: two horizontal edges $[0\ \ 1]$;
- each vertical edge $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ of P generates in Q, in excess: two vertical edges $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$;
- each square face $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$ of P generates in Q, in excess: one square $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$.

Therefore,

$$\#(Q;[1]) = 4 \cdot \#(P;[1])$$

$$\#(Q;[1\ \ 1]) = 2 \cdot \#(P;[1]) + 2 \cdot (P;[1\ \ 1])$$

$$\#\left(Q;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) = 2 \cdot \#(P;[1]) + 2 \cdot \left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$$

$$\#(Q;[0\ \ 1]) = 2 \cdot \#(P;[0\ \ 1])$$

$$\#\left(Q;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = 2 \cdot \#\left(P;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)$$

$$\#\left(Q;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right) = \#(P;[1]) + \#(P;[1\ \ 1]) + \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) + \#\left(P;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right)$$

and we get

$$\rho(Q) = \alpha_v \cdot \#(Q;[1]) + \alpha_{hi} \cdot \#(Q;[1\ \ 1]) + \alpha_{vi} \cdot \#\left(Q;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) + \alpha_{he} \cdot \#(Q;[0\ \ 1]) + \alpha_{ve} \cdot \#\left(Q;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) + \alpha_{sq} \cdot \#\left(Q;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right)$$

$$= (4\alpha_v + 2\alpha_{hi} + 2\alpha_{vi} + \alpha_{sq}) \cdot \#(P;[1]) + \left(2\alpha_{hi} + \alpha_{sq}\right) \cdot \#(P;[1\ \ 1]) + \left(2\alpha_{vi} + \alpha_{sq}\right) \cdot \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$$

$$+ 2\alpha_{he} \cdot \#(P;[0\ \ 1]) + 2\alpha_{ve} \cdot \#\left(P;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) + \alpha_{sq} \cdot \#\left(P;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right)$$

In the sequel we will need the difference

$$\rho(Q) - \rho(P) = \left(3\alpha_v + 2\alpha_{hi} + 2\alpha_{vi} + \alpha_{sq}\right) \cdot \#(P;[1])$$

$$+ \left(\alpha_{hi} + \alpha_{sq}\right) \cdot \#(P;[1\ \ 1]) + \left(\alpha_{vi} + \alpha_{sq}\right) \cdot \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$$

$$+ \alpha_{he} \cdot \#(P;[0\ \ 1]) + \alpha_{ve} \cdot \#\left(P;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)$$

If $\rho(Q) = \rho(P)$, then for each binary digital image P,

$$(D) \quad \left(3\alpha_v + 2\alpha_{hi} + 2\alpha_{vi} + \alpha_{sq}\right) \cdot \#(P;[1]) + \left(\alpha_{hi} + \alpha_{sq}\right) \cdot \#(P;[1\ \ 1]) + \left(\alpha_{vi} + \alpha_{sq}\right) \cdot \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$$

$$+ \alpha_{he} \cdot \#(P;[0\ \ 1]) + \alpha_{ve} \cdot \#\left(P;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = 0$$

If we extend the function ρ to one of the following three functions

$$\rho(P) = \alpha_{v} \cdot \#(P;[1]) + \alpha_{hi} \cdot \#(P;[0\ \ 1]) + \alpha_{vi} \cdot \#\left(P;\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \alpha_{he} \cdot \#(P;[0\ \ 1]) + \alpha_{ve} \cdot \#\left(P;\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$+ \alpha_{sq} \cdot \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) + \alpha_{md} \cdot \#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$\rho(P) = \alpha_{v} \cdot \#(P;[1]) + \alpha_{hi} \cdot \#(P;[0\ \ 1]) + \alpha_{vi} \cdot \#\left(P;\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \alpha_{he} \cdot \#(P;[0\ \ 1]) + \alpha_{ve} \cdot \#\left(P;\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$+ \alpha_{sq} \cdot \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) + \alpha_{sd} \cdot \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

and

$$\rho(P) = \alpha_{v} \cdot \#(P;[1]) + \alpha_{hi} \cdot \#(P;[0\ \ 1]) + \alpha_{vi} \cdot \#\left(P;\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \alpha_{he} \cdot \#(P;[0\ \ 1]) + \alpha_{ve} \cdot \#\left(P;\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$+ \alpha_{sq} \cdot \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) + \alpha_{md} \cdot \#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + \alpha_{sd} \cdot \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right),$$

where the coefficients α_{md}, α_{sd} are real numbers as well, then we easily see that each of the bit-quads $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in P generates just one bit-quad of the same kind in Q; therefore, there will be no changes in (D).

IV. EULER NUMBERS

We consider the following well known, linear, integer valued functions on the class of binary digital images:

the 4-adjacency Euler number

$$\chi^{(4)}(P) = \#(P;[1]) - \#(P;[1\ \ 1]) - \#\left(P;\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

the main diagonal 6-adjacency Euler number

$$\chi^{(6m)}(P) = \chi^{(4)}(P) - \#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= \#(P;[1]) - \#(P;[1\ \ 1]) - \#\left(P;\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

the secondary diagonal 6-adjacency Euler number

$$\chi^{(6s)}(P) = \chi^{(4)}(P) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$
$$= \#(P;[1]) - \#(P;[1\ \ 1]) - \#\left(P;\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

and the 8-adjacency Euler number

$$\chi^{(8)}(P) = \chi^{(4)}(P) - \#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$
$$= \#(P;[1]) - \#(P;[1\ \ 1]) - \#\left(P;\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) - \#\left(P;\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

For the 4-adjacency and 8-adjacency Euler numbers, see the references [1], [3], [4]; for the two 6-adjacency Euler numbers see the references [1], [2].

Some more linear, integer valued functions on the class of binary digital images were discussed in [9]; this work was inspired by [5], [6], [7]:

the 4-adjacency Euler number

$$\chi^{(4)}(P) = -\#(P;[1]) + \#(P;[0\ \ 1]) + \#\left(P;\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$
$$= -\#(P;[1]) + \#(P;[0\ \ 1]) + \#\left(P;\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$
$$= -\#(P;[1]) + \#(P;[1\ \ 0]) + \#\left(P;\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$
$$= -\#(P;[1]) + \#(P;[1\ \ 0]) + \#\left(P;\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + \#\left(P;\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

the main diagonal 6-adjacency Euler number

$$\chi^{(6m)}(P) = \chi^{(4)}(P) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[0\ \ 1]) + \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[0\ \ 1]) + \#\left(P;\begin{bmatrix}1\\0\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[1\ \ 0]) + \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[1\ \ 0]) + \#\left(P;\begin{bmatrix}1\\0\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

the secondary diagonal 6-adjacency Euler number

$$\chi^{(6s)}(P) = \chi^{(4)}(P) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[0\ \ 1]) + \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[0\ \ 1]) + \#\left(P;\begin{bmatrix}1\\0\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[1\ \ 0]) + \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[1\ \ 0]) + \#\left(P;\begin{bmatrix}1\\0\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

and the 8-adjacency Euler number

$$\chi^{(8)}(P) = \chi^{(4)}(P) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[0\ \ 1]) + \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[0\ \ 1]) + \#\left(P;\begin{bmatrix}1\\0\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[1\ \ 0]) + \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= -\#(P;[1]) + \#(P;[1\ \ 0]) + \#\left(P;\begin{bmatrix}1\\0\end{bmatrix}\right) + \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) - \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

It is enough to use the first formula in each of the four quadruples.

Each of these Euler number functions is a refinement invariant. For a proof, it is enough to replace in (D)

$$\alpha_v = \alpha_{sq} = 1, \quad \alpha_{hi} = \alpha_{vi} = -1, \quad \alpha_{he} = \alpha_{ve} = 0,$$

or

$$\alpha_v = -1, \quad \alpha_{hi} = \alpha_{vi} = 0, \quad \alpha_{he} = \alpha_{ve} = \alpha_{sq} = 1,$$

and we get $\rho(Q) = \rho(P)$.

The four Euler numbers are not independent, since

$$\chi^{(4)}(P) + \chi^{(8)}(P) = \chi^{(6m)}(P) + \chi^{(6s)}(P)$$

On the other hand, as shown in [8], every three of them are independent.

The aim of this paper is to prove some uniqueness results, according to which these four Euler number functions are, essentially, the only extended linear, real valued refinement invariants on the class of binary digital images, generalizing thus the results proven in [8].

But first, we derive an alternative version of identity (D) . By an argument similar to that in [9], it can be easily shown that

$$\#(P;[0\ \ 1]) = \#(P;[1]) - \#(P;[1\ \ 1])$$

$$\#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right) = \#(P;[1]) - \#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right)$$

Replacing in (D) , we get the identity

$$(D_e)\quad \begin{aligned}&\left(3\alpha_v + 2\alpha_{hi} + 2\alpha_{vi} + \alpha_{sq}\right)\cdot \#(P;[1]) + \left(\alpha_{hi} + \alpha_{sq}\right)\cdot \#(P;[1\ 1]) + \left(\alpha_{vi} + \alpha_{sq}\right)\cdot \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)\\ &+\alpha_{he}\cdot\left(\#(P;[1]) - \#(P;[1\ 1])\right) + \alpha_{ve}\cdot\left(\#(P;[1]) - \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)\right) = 0\end{aligned}$$

or

$$(D_e)\quad \begin{aligned}&\left(3\alpha_v + 2\alpha_{hi} + 2\alpha_{vi} + \alpha_{he} + \alpha_{ve} + \alpha_{sq}\right)\cdot \#(P;[1])\\ &+\left(\alpha_{hi} - \alpha_{he} + \alpha_{sq}\right)\cdot \#(P;[1\ 1]) + \left(\alpha_{vi} - \alpha_{ve} + \alpha_{sq}\right)\cdot \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) = 0\end{aligned}$$

V. UNIQUENESS THEOREMS

Theorem 1. Every refinement invariant of the form

$$\rho(P) = \alpha_v\cdot \#(P;[1]) + \alpha_{hi}\cdot\#(P;[1\ 1]) + \alpha_{vi}\cdot\#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) + \alpha_{he}\cdot\#(P;[0\ 1]) + \alpha_{ve}\cdot\#\left(P;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) + \alpha_{sq}\cdot\#\left(P;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right)$$

$$\alpha_v, \alpha_{hi}, \alpha_{vi}, \alpha_{he}, \alpha_{ve}, \alpha_{sq} \in R$$

can be expressed in a unique way as a linear combination

$$\rho(P) = \alpha\cdot\chi^{(4)}(P),\quad \alpha\in R\ .$$

Proof: We use identity (D_e), since ρ is a refinement invariant.

By choosing the image $P=[1]$ (only one foreground pixel in the image), we get $\#(P;[1]) = 1$, $\#(P;[1\ 1]) = 0$, $\#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) = 0$, $\#(P;[0\ 1]) = 1$, $\#\left(P;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) = 1$; therefore,

$3\alpha_v + 2\alpha_{hi} + 2\alpha_{vi} + \alpha_{he} + \alpha_{ve} + \alpha_{sq} = 0$.

By choosing the image $P=[1\ 1]$ (only two foreground pixels in the image, forming a horizontal edge), we get $\#(P;[1]) = 2$, $\#(P;[1\ 1]) = 1$, $\#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) = 0$; therefore, $6\alpha_v + 5\alpha_{hi} + 4\alpha_{vi} + \alpha_{he} + 2\alpha_{ve} + 3\alpha_{sq} = 0$.

By choosing the image $P=\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ (only two foreground pixels in the image, forming a vertical edge), we get $\#(P;[1]) = 2$, $\#(P;[1\ 1]) = 0$, $\#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) = 1$; therefore, $6\alpha_v + 4\alpha_{hi} + 5\alpha_{vi} + 2\alpha_{he} + \alpha_{ve} + 3\alpha_{sq} = 0$.

We derived the following system of equations

$$\begin{aligned}3\alpha_v + 2\alpha_{hi} + 2\alpha_{vi} + \alpha_{he} + \alpha_{ve} + \alpha_{sq} &= 0\\ \alpha_{hi} - \alpha_{he} + \alpha_{sq} &= 0\\ \alpha_{vi} - \alpha_{ve} + \alpha_{sq} &= 0\end{aligned}$$

and from these it turns out that

$\alpha_v + \alpha_{he} + \alpha_{ve} = \alpha_{sq} = \alpha$, $\alpha_{hi} - \alpha_{he} = \alpha_{vi} - \alpha_{ve} = -\alpha$, for some $\alpha\in R$. We can now rewrite the invariant formula as

$$\begin{aligned}\rho(P) &= \alpha_v\cdot \#(P;[1]) + \alpha_{hi}\cdot\#(P;[1\ 1]) + \alpha_{vi}\cdot\#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) + \alpha_{he}\cdot\#(P;[0\ 1]) + \alpha_{ve}\cdot\#\left(P;\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right) + \alpha_{sq}\cdot\#\left(P;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right)\\ &= \alpha_v\cdot \#(P;[1]) + \alpha_{hi}\cdot\#(P;[1\ 1]) + \alpha_{vi}\cdot\#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)\\ &\quad + \alpha_{he}\cdot\left(\#(P;[1]) - \#(P;[1\ 1])\right) + \alpha_{ve}\cdot\left(\#(P;[1]) - \#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)\right) + \alpha_{sq}\cdot\#\left(P;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right)\\ &= \left(\alpha_v + \alpha_{he} + \alpha_{ve}\right)\cdot\#(P;[1]) + \left(\alpha_{hi} - \alpha_{he}\right)\cdot\#(P;[1\ 1]) + \left(\alpha_{vi} - \alpha_{ve}\right)\cdot\#\left(P;\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) + \alpha_{sq}\cdot\#\left(P;\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\right)\end{aligned}$$

Therefore, $\rho(P) = \alpha\cdot\chi^{(4)}(P)$.

Theorem 2. Every refinement invariant of the form

$$\rho(P) = \alpha_v \cdot \#(P;[1]) + \alpha_{hi} \cdot \#(P;[1\ 1]) + \alpha_{vi} \cdot \#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) + \alpha_{he} \cdot \#(P;[0\ 1]) + \alpha_{ve} \cdot \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$+ \alpha_{sq} \cdot \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) + \alpha_{md} \cdot \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$
$$\alpha_v, \alpha_{hi}, \alpha_{vi}, \alpha_{he}, \alpha_{ve}, \alpha_{sq}, \alpha_{md} \in R$$

can be expressed in a unique way as a linear combination

$$\rho(P) = \alpha \cdot \chi^{(4)}(P) + \delta \cdot \chi^{(6m)}, \quad \alpha, \delta \in R .$$

Proof: The parameter α_{md} does not appear in (D_e), so it is independent. As in the proof of Theorem 1, we get $\alpha_v + \alpha_{he} + \alpha_{ve} = \alpha_{sq} = \beta$, $\alpha_{hi} - \alpha_{he} = \alpha_{vi} - \alpha_{ve} = -\beta$, for some $\beta \in R$. Therefore,

$$\rho(P) = \beta \cdot \chi^{(4)}(P) + \alpha_{md} \cdot \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$$

or $\rho(P) = \alpha \cdot \chi^{(4)}(P) + \delta \cdot \chi^{(6m)}$,

where $\alpha = \beta + \alpha_{md}$, $\delta = -\alpha_{md}$.

Theorem 3. Every refinement invariant of the form

$$\rho(P) = \alpha_v \cdot \#(P;[1]) + \alpha_{hi} \cdot \#(P;[1\ 1]) + \alpha_{vi} \cdot \#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) + \alpha_{he} \cdot \#(P;[0\ 1]) + \alpha_{ve} \cdot \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$+ \alpha_{sq} \cdot \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) + \alpha_{sd} \cdot \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$
$$\alpha_v, \alpha_{hi}, \alpha_{vi}, \alpha_{he}, \alpha_{ve}, \alpha_{sq}, \alpha_{sd} \in R$$

can be expressed in a unique way as a linear combination

$$\rho(P) = \alpha \cdot \chi^{(4)}(P) + \delta \cdot \chi^{(6s)}, \quad \alpha, \delta \in R .$$

Proof: Similar to that of Theorem 2.

Theorem 4. Every refinement invariant of the form

$$\rho(P) = \alpha_v \cdot \#(P;[1]) + \alpha_{hi} \cdot \#(P;[1\ 1]) + \alpha_{vi} \cdot \#\left(P;\begin{bmatrix}1\\1\end{bmatrix}\right) + \alpha_{he} \cdot \#(P;[0\ 1]) + \alpha_{ve} \cdot \#\left(P;\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$+ \alpha_{sq} \cdot \#\left(P;\begin{bmatrix}1&1\\1&1\end{bmatrix}\right) + \alpha_{md} \cdot \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) + \alpha_{sd} \cdot \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$
$$\alpha_v, \alpha_{hi}, \alpha_{vi}, \alpha_{he}, \alpha_{ve}, \alpha_{sq}, \alpha_{md}, \alpha_{sd} \in R$$

can be expressed in a unique way as a linear combination

$$\rho(P) = \alpha \cdot \chi^{(4)}(P) + \delta_m \cdot \chi^{(6m)} + \delta_s \cdot \chi^{(6s)}, \quad \alpha, \delta_m, \delta_s \in R .$$

Proof: As in the proofs of Theorems 2 and 3, we get $\alpha_v + \alpha_{he} + \alpha_{ve} = \alpha_{sq} = \beta$, $\alpha_{hi} - \alpha_{he} = \alpha_{vi} - \alpha_{ve} = -\beta$, for some $\beta \in R$. Therefore,

$$\rho(P) = \beta \cdot \chi^{(4)}(P) + \alpha_{md} \cdot \#\left(P;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right) + \alpha_{sd} \cdot \#\left(P;\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

or $\rho(P) = \alpha \cdot \chi^{(4)}(P) + \delta_m \cdot \chi^{(6m)} + \delta_s \cdot \chi^{(6s)}$,

where $\alpha = \beta + \alpha_{md} + \alpha_{sd}$, $\delta_m = -\alpha_{md}$, $\delta_s = -\alpha_{sd}$.

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