Quest Journals
Journal of Research in Applied Mathematics
Volume 8 ~ Issue 11 (2022) pp: 08-15
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

On the Elementary Trigonometric Polynomials and their Generating Functions

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ABSTRACT: *The trigonometric polynomials are simple yet have lots of interest in theories of orthogonal polynomials and in numerical analysis. The elementary trigonometric polynomials are broken series of sin(x) and cos(x). Some of their important properties have been left unsolved. Their integral representations are derived and the polynomials can be generated with them. Their differential equations are solved and the generating functions are created with the help of their recursion relations. A correspondence was observed between the exponential and trigonometric polynomials in the form of a pair of equations. 1*

***R**eceived 28 Oct., 2022; **R**evised 07 Nov., 2022; **A**ccepted 09 Nov., 2022 © The author(s) 2022.*
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I. INTRODUCTION

The trigonometric polynomials are widely studied in relation with orthogonal polynomials and approximation theories and in addition in numerical analysis [3], [5]. Signal analysis and wavelet theories also find use of them [6]. The polynomials resist regular methods [1] for creating simplified closed-form solutions. This subject has gained more interest lately [2]. In the following are first introduced the commonly used trigonometric polynomials and the elementary trigonometric polynomials. Recursion relations are developed for the polynomials. Second order inhomogeneous differential equations are derived for them. The generating functions for the polynomials are based on these and the integral relations obtained, being the main results of this paper. As the last topic, a pair of relations between the exponential and trigonometric polynomials is derived.

1.1 Mathematical Classification

Mathematics Subject Classification: 11L03, 11T23, 33C45, 33C47,33D45

1.2 Keywords

Trigonometric polynomials, orthogonal polynomials, exponential polynomials, approximation theory, applied mathematics

II. THE TRIGONOMETRIC POLYNOMIALS

The Full Trigonometric Polynomial

Often the trigonometric polynomials are defined as [2]

$$p_j(x) = \frac{1}{\sqrt{j}} \sum_{n=0}^{j} [a_n \cdot sin(nx) + b_n \cdot cos(nx)] \qquad (1)$$

with the coefficients aj , bj being standard Gaussian. This common form is applying actual trigonometric functions in the polynomial. Since the polynomials are periodic this gives them interesting features useful in analysis [4]. In general, it is critical to have full knowledge of the coefficients to make progress in analysis.

2.2 The Elementary Trigonometric Polynomials

In this paper, the more elementary trigonometric polynomials are of broken series of sin(x) and cos(x). The sin(x) polynomial with an argument x in R is defined as

$$s_j(x) = \sum_{n=0}^{j} \frac{x^{2n+1}(-1)^n}{(2n+1)!} \tag{2}$$

The cos(x) polynomial is

$$c_j(x) = \sum_{n=0}^{j} \frac{x^{2n}(-1)^n}{(2n)!} \tag{3}$$

They are not periodic with a finite j but can be considered as almost-periodic. The trigonometric polynomials will always be finite when x in **R** is finite. When approaching the infinity in the index j, the polynomials become the actual trigonometric functions. In order to derive generating functions for these two polynomials, both proper recursion formulas and suitable differential equations are required. In the following, similar methods are used for solving the generating functions, as were recently done with the exponential polynomial [7].

2.3 Differential Equations of the Trigonometric Polynomials

The first task is to find differential equations for the s_j and c_j . The following equations are obtained by differentiation of the definitions above in terms of x.

$$s_j' = \sum_{n=0}^{j} \frac{x^{2n}(-1)^n}{(2n)!} = c_j \tag{4}$$

The corresponding cos(x) polynomial's derivative is the following

$$c_j' = \sum_{n=1}^{j} \frac{x^{2n-1}(-1)^n}{(2n-1)!} \tag{5}$$

$$= -\sum_{n=0}^{j-1} \frac{x^{2n+1}(-1)^n}{(2n+1)!} = -s_{j-1} \tag{6}$$

after changing the index a little. The following recursions are direct consequences of the definitions

$$s_{j+1}(x) - s_j(x) = \frac{x^{2j+3}(-1)^{j+1}}{(2j+3)!} \quad (7)$$

$$c_{j+1}(x) - c_j(x) = \frac{x^{2j+2}(-1)^{j+1}}{(2j+2)!} \quad (8)$$

By differentiating (4) one will obtain

$$s_j'' = c_j' = -s_{j-1} \quad (9)$$

One can now substitute the recursion (7) to this and get

$$s_j'' + s_j = \frac{x^{2j+1}(-1)^j}{(2j+1)!} \quad (10)$$

A different path is followed with the cos(x). By differentiating (6) and using equation (4), the following is the outcome

$$c_{j+1}'' = -s_{j-1}' = -c_{j-1} \quad (11)$$

To this the recursion (8) can be applied and it leads to

$$c_j'' + c_j = \frac{x^{2j}(-1)^j}{(2j)!} \quad (12)$$

(10) and (12) are useful differential equations for further progress. They are both of the same form which is second order linear with constant coefficients and an inhomogeneous term. The form is thus

$$y'' + y = Q \quad (13)$$

The general solution is

$$y(x) = e^{ix} \int^x ds \cdot e^{-2is} \int^s du \cdot e^{iu} Q(u) \quad (14)$$

Hence

$$s_j(x) = A_j + e^{ix} \int^x ds \cdot e^{-2is} (\int^s du \cdot e^{iu} \frac{u^{2j+1}(-1)^j}{(2j+1)!} + B_j) \quad (15)$$

and

$$c_j(x) = C_j + e^{ix} \int^x ds \cdot e^{-2is} (\int^s du \cdot e^{iu} \frac{u^{2j}(-1)^j}{(2j)!} + D_j) \qquad (16)$$

The constants of integration A_j ,C_j are observed to be zero since the results must satisfy the original differential equations. The B_j ,D_j are also zero since when integrated, they create extra exponential terms which do not exist in the polynomials. Therefore the following equations remain.

$$s_j(x) = e^{ix} \int^x ds \cdot e^{-2is} \int^s du \cdot e^{iu} \frac{u^{2j+1}(-1)^j}{(2j+1)!} \qquad (17)$$

and

$$c_j(x) = e^{ix} \int^x ds \cdot e^{-2is} \int^s du \cdot e^{iu} \frac{u^{2j}(-1)^j}{(2j)!} \qquad (18)$$

Equations (17) and (18) are integrals which will deliver any of the trigonometric polynomials by integration.

2.4 Generating Functions of the Trigonometric Polynomials

One can multiply the integral (17) above by t^{2j+1} and sum it getting

$$\sum_{j=0}^{\infty} s_j(x) t^{2j+1} = e^{ix} \int^x ds \cdot e^{-2is} \int^s du \cdot e^{iu} \sum_{j=0}^{\infty} \frac{u^{2j+1}(-1)^j t^{2j+1}}{(2j+1)!} \qquad (19)$$

The order of summation and integration can be safely swapped. One can immediately identify the proper series expansion to belong to a sin(x) function leading to

$$\sum_{j=0}^{\infty} s_j(x) t^{2j+1} = e^{ix} \int^x ds \cdot e^{-2is} \int^s du \cdot e^{iu} sin(tu) \qquad (20)$$

The integrals can be sorted out to become

$$\sum_{j=0}^{\infty} s_j(x) t^{2j+1} = \frac{sin(tx)}{1-t^2} \qquad (21)$$

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The steps are repeated below for the cos(x) function below. The integral (18) above is multiplied by t^{2j} and summed getting

$$\sum_{j=0}^{\infty} c_j(x)t^{2j} = e^{ix}\int^{x} ds \cdot e^{-2is}\int^{s} du \cdot e^{iu}\sum_{j=0}^{\infty}\frac{u^{2j}(-1)^j t^{2j}}{(2j)!} \qquad (22)$$

The order of summation and integration have been swapped as well and the series expansion of a cos(x) function is apparent

$$\sum_{j=0}^{\infty} c_j(x)t^{2j} = e^{ix}\int^{x} ds \cdot e^{-2is}\int^{s} du \cdot e^{iu} cos(tu) \qquad (23)$$

The integrals can be solved to produce

$$\sum_{j=0}^{\infty} c_j(x)t^{2j} = \frac{cos(tx)}{1-t^2} \qquad (24)$$

The equations (21) and (24) are generating functions for the elementary trigonometric polynomials. They are finite as long as $|t| < 1$. By division one will obtain an interesting result

$$\sum_{j=0}^{\infty} s_j(x)t^{2j+1} = tan(tx)\sum_{j=0}^{\infty} c_j(x)t^{2j} \qquad (25)$$

2.5 Connection between the Exponential and Trigonometric Polynomials

In would be expected that the trigonometric polynomials would be related in some way to the exponential polynomial

$$y_j(x) = \sum_{n=0}^{j}\frac{x^n}{n!} \qquad (26)$$

The generating function for the exponential polynomial was found to be [7]

$$\sum_{j=0}^{\infty} y_j(x)t^{j} = \frac{e^{tx}}{1-t} \qquad (27)$$

In analogy with regular trigonometric functions one will set out to find what is the result of having an imaginary unit in the argument with x in R.

$$y_j(ix) = \sum_{n=0}^{j} \frac{i^n x^n}{n!} \quad (28)$$

Since for even n

$$i^n = (-1)^{n/2} \quad (29)$$

and for odd n

$$i^n = i(-1)^{(n-1)/2} \quad (30)$$

the equation above equals to

$$y_j(ix) = \sum_{n=0,2,4..}^{j} \frac{(-1)^{n/2} x^n}{n!} + i \sum_{n=1,3,5..}^{j} \frac{(-1)^{(n-1)/2} x^n}{n!} \quad (31)$$

where the index n will not go over j. One also knows that $c_{-N} = 0$ and $s_{-N} = 0$. There are obviously even and odd cases for the j. A more accurate pair of expressions are, for even j

$$y_j(ix) = \sum_{n=0,2,4..}^{j} \frac{(-1)^{n/2} x^n}{n!} + i \sum_{n=1,3,5..}^{j-1} \frac{(-1)^{(n-1)/2} x^n}{n!} \quad (32)$$

and for odd j

$$y_j(ix) = \sum_{n=0,2,4..}^{j-1} \frac{(-1)^{n/2} x^n}{n!} + i \sum_{n=1,3,5..}^{j} \frac{(-1)^{(n-1)/2} x^n}{n!} \quad (33)$$

After changing the indexes for the first terms as n = 2k for the second terms as n = 2k + 1, one will get

$$y_j(ix) = \sum_{k=0}^{j/2} \frac{(-1)^k x^{2k}}{(2k)!} + i \sum_{k=0}^{(j-2)/2} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad (34)$$

and for odd j

$$y_j(ix) = \sum_{k=0}^{(j-1)/2} \frac{(-1)^k x^{2k}}{(2k)!} + i \sum_{k=0}^{(j-1)/2} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad (35)$$

One can immediately recognize the definitions of the c_j and s_j and go one step further for the even j

$$y_j(ix) = c_{\frac{j}{2}}(x) + is_{\frac{j-2}{2}}(x) \qquad (36)$$

and for the odd j

$$y_j(ix) = c_{\frac{j-1}{2}}(x) + is_{\frac{j-1}{2}}(x) \qquad (37)$$

The generating function will start constructing as follows

$$\sum_{j=0}^{\infty} y_j(ix)t^j = \sum_{j=0,2,4..}^{\infty} (c_{\frac{j}{2}}(x) + is_{\frac{j-2}{2}}(x))t^j + \sum_{j=1,3,5..}^{\infty} (c_{\frac{j-1}{2}}(x) + is_{\frac{j-1}{2}}(x))t^j \qquad (38)$$

By changing the indexes of the $s_{(j-2)/2}$ in the even case and the $c_{(j-1)/2-1}$ odd case as j-1= n, one will obtain by collecting the terms

$$\sum_{j=0}^{\infty} y_j(ix)t^j = \sum_{n=0,2,4..}^{\infty} c_{\frac{n}{2}}(x)(t^n + t^{n+1}) + i \sum_{n=1,3,5..}^{\infty} s_{\frac{n-1}{2}}(x))(t^n + t^{n+1}) \qquad (39)$$

By once again changing the indexes as n/2 = m, n−1/2 = m in the first and second terms correspondingly, one will get

$$\sum_{j=0}^{\infty} y_j(ix)t^j = (1+t)[\sum_{m=0}^{\infty} c_m(x)t^{2m} + i \sum_{m=0}^{\infty} s_m(x)t^{2m+1}] \qquad (40)$$

At this point it is possible to identify the expressions for the generating functions of the trigonometric polynomials and place the known results into the equation.

$$\sum_{j=0}^{\infty} y_j(ix)t^j = \frac{cos(tx) + isin(tx)}{(1-t)} = \frac{e^{itx}}{(1-t)} \qquad (41)$$

which is an identity. This proves that the three functions y_j , c_j and s_j are related as expressed in equations (36) and (37). The trigonometric polynomials seem to behave similar to trigonometric functions.

III. Discussion

The trigonometric polynomials are studied in orthogonal polynomials, numerical analysis and approximation theory. Basic properties of the elementary trigonometric polynomials are commonly ignored, regarding them to be too trivial. In this paper it is shown that even the elementary trigonometric polynomials have specific differential equations (10) and (12). Their solutions are double integrals (17) and (18). Both polynomials have been shown to have generating functions, equations (21) and (22).

The connection between the trigonometric and exponential polynomials was shortly studied. It was established that the equations (36) and (37) give the polynomial level correspondence. That looks a bit complicated but when moving to the generating function level, one will find out a perfect correspondence between the three polynomials, equations (40) and (41). This is also pointed by equation (25).

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ACKNOWLEDGEMENTS

*The author is grateful to Visilab Signal Technologies for supporting this work.
1Visilab Report #2022-11