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Research Paper

New discontinuity results at fixed point for four Self-mappings in G-metric space

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Abstract
The discontinuity problem at a fixed point has recently been researched from various angles. Using appropriate contractive conditions that are strong enough to generate fixed points but do not require the map to be continuous at fixed points, we investigate new discontinuity problem solutions for four self-mappings in this paper. In relation to our issue, an example is also provided.
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I. INTRODUCTION

Fixed points are the points which remain invariant under a transformation. Fixed points tell us which parts of the space are pinned in plane, not moved, by the transformation. The fixed points of a transformation restrict the motion of the space under some restrictions. Moreover, fixed point applications increasesenormously.

Fixed Point Theory is divided into three major areas:

1. Topological Fixed- Point Theory,
2. Metric Fixed Point Theory,
3. Discrete Fixed- Point Theory.

Now we state some important fixed- point theorems:

Brouwer (1912): Every continuous self- map of the closed unit ball in R^n has a fixed point. **Banach (1922)**: Let X be a non -empty set and (X, d) be a complete metric space. If T: $X \rightarrow X$ such that $d\ (Tx, Ty) \leq k\ d\ (x, y)$ for each x, y in X where $0 \leq k < 1$, then T has a unique fixed point in X. i.e., every contraction maps on a complete metric space hasa fixed point.Thistheorem has had many applications, but suffers from one drawback- the definition requires that T is continuous throughout X. After wards, a number of works have appeared which involve contractive definition that does not require the continuity of T.

Schauder (1930) If K is compact, convex subset of a topological vector space V and T is a continuous self-mapping on K, then T has a fixed point.

Kannan (1968): If T is a self -mapping of a contractive metric space X satisfying

$$d\ (Tx, Ty) \leq k[d\ (Tx,x) + d\ (Ty, y)]$$

for all x, y in X and $0 \leq k < 1/2$,then T has unique fixed point in X. We note that map T is not continuous even though T has a fixed point. However, in every case the maps involved were continuous at the fixed point. It may be noted that Kannan's fixed point is not an extension of Banach contraction principle. Therefore, Kannan type and their generalizations have been considered as constituting an important class of mapping in fixed point theory. These theorems are generalized by various authors in various spaces by using different variants of commuting and minimal commutativity maps.

II. PRELIMINARIES

2.1.G-Metric Space:

In 2005, Mustafa and Sims [28] introduced a new class of generalized metric spaces which are called G-metric spaces, as generalization of a metric space (X, d).

Let X be a non-empty set, and $G: X \times X \times X \to R^+$ be a function satisfying the following properties:

(1) $G(x, y, z) = 0$ if and only if $x = y = z$;

(2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G-metric on X and (X, G) is called a G-metric space.

Defination 2.1.1[28] Let (X, G) be a G-metric space. We say that $\{x_n\}$is a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $P \in N$ (the set of all positive integers) such that for all $n, m, l \geq P$, $G(x_n, x_m, x_l) < \varepsilon$.

Defination 2.1.2[28] *Let* (X, G) be a G -metric space. We say that $\{x_n\}$ is a G -convergent sequence to $x \in X$ if, for any $\varepsilon > 0$, there is $P \in N$ such that for all $n, m \geq P$, $G(x, x_n, x_m) < \varepsilon$.

Defination 2.1.3[28] A G-metric space (X, G) is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Defination 2.1.4[28] A G-metric space (X, G) is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

3.MAIN RESULTS

In this section, our aim is to obtain new solutions to the open problem related to discontinuity problem at the fixed point [8].For this purpose, we consider the number $\mathcal{P}(u, v)$ defined as

$$\mathcal{P}(u,v)=\max\left\{\begin{matrix}\mathcal{G}(u,v,v),\mathcal{G}(\mathcal{A}u,\mathcal{T}u,\mathcal{T}u),\mathcal{G}(\mathcal{B}v,\mathcal{S}v,\mathcal{S}v),\\ \left[\frac{\mathcal{G}(\mathcal{A}u,\mathcal{S}v,\mathcal{S}v)+\mathcal{G}(\mathcal{B}v,\mathcal{T}u,\mathcal{T}u)}{\mathcal{G}(\mathcal{A}u,\mathcal{T}u,\mathcal{T}u)+\mathcal{G}(\mathcal{B}v,\mathcal{S}v,\mathcal{S}v)+1}\right]\mathcal{G}(u,v,v)\end{matrix}\right\}$$

We give the following theorem

Theorem 3.1 Let $(\mathcal{X}, \mathcal{G})$ be a complete $\mathcal{G}$ -metric space. Let $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}$ are four self-mappings on $\mathcal{X}$ such that for any $u, v \in \mathcal{X}$

(1) $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$, $\mathcal{B}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$

(2) There exist a function $\varphi: \mathcal{R}^+ \to \mathcal{R}^+$ such that $\varphi(t) < t$, for each $t > 0$ and

$$\mathcal{G}(\mathcal{T}u, \mathcal{S}v, \mathcal{S}v) \leq \varphi(\mathcal{P}(u, v))$$

(3) For a given $\epsilon > 0$, there exist a $\delta(\epsilon) > 0$ such that $\epsilon < \mathcal{P}(u, v) < \epsilon + \delta$ implies $\mathcal{G}(\mathcal{T}u, \mathcal{S}v, \mathcal{S}v) \leq \epsilon$.

Then $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}$ have aa common fixed point $z \in \mathcal{X}$ and $\mathcal{T}^n u \to z$, $\mathcal{S}^n u \to z$, $\mathcal{A}^n u \to z$ *and* $\mathcal{B}^n u \to z$ for each $u \in \mathcal{X}$. Also, atleast one of $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}$ is discontinuous at z if and only if

$lim_{u \to z} \mathcal{P}(u, z) \neq 0$ or $lim_{v \to z} \mathcal{P}(z, v) \neq 0$

Proof. Let us define a sequence $\{u_n\}$ in $\mathcal{X}$ such that

$u_{2n+1} = \mathcal{T}u_{2n} = \mathcal{B}u_{2n+1}$, $u_{2n+2} = \mathcal{S}u_{2n+1}$ and $u_n = \mathcal{A}u_n$

For n$\in N \cup \{0\}$.Suppose that $\mathcal{G}(u_n, u_{n+1}, u_{n+1})$, by condition (2), we obtain

$s_1 = \mathcal{G}(u_1, u_2, u_2) = \mathcal{G}(\mathcal{T}u_0, \mathcal{S}u_1, \mathcal{S}u_1) \leq \varphi(\mathcal{P}(u_0, u_1))$

$$=\varphi\left(\max\left\{\begin{matrix}\mathcal{G}(u_0,u_1,u_1),\mathcal{G}(\mathcal{A}u_0,\mathcal{T}u_0,\mathcal{T}u_0),\mathcal{G}(\mathcal{B}u_1,\mathcal{S}u_1,\mathcal{S}u_1),\\ \left[\frac{\mathcal{G}(\mathcal{A}u_0,\mathcal{S}u_1,\mathcal{S}u_1)+\mathcal{G}(\mathcal{B}u_1,\mathcal{T}u_0,\mathcal{T}u_0)}{\mathcal{G}(\mathcal{A}u_0,\mathcal{T}u_0,\mathcal{T}u_0)+\mathcal{G}(\mathcal{B}u_1,\mathcal{S}u_1,\mathcal{S}u_1)+1}\right]\mathcal{G}(u_0,u_1,u_1)\end{matrix}\right\}\right)$$

$$=\varphi\left(\max\left\{\begin{matrix}\mathcal{G}(u_0,u_1,u_1),\mathcal{G}(u_0,u_1,u_1),\mathcal{G}(u_1,u_2,u_2),\\ \left[\frac{\mathcal{G}(u_0,u_2,u_2)+\mathcal{G}(u_1,u_1,u_1)}{\mathcal{G}(u_0,u_1,u_1)+\mathcal{G}(u_1,u_2,u_2)+1}\right]\mathcal{G}(u_0,u_1,u_1)\end{matrix}\right\}\right)$$

$$\leq \varphi\left(\max\left\{\begin{matrix}\mathcal{G}(u_0,u_1,u_1),\mathcal{G}(u_1,u_2,u_2),\\ \left[\frac{\mathcal{G}(u_0,u_1,u_1)+\mathcal{G}(u_1,u_2,u_2)}{\mathcal{G}(u_0,u_1,u_1)+\mathcal{G}(u_1,u_2,u_2)+1}\right]\mathcal{G}(u_0,u_1,u_1)\end{matrix}\right\}\right) \text{ (by triangle inequality)}$$

$$=\varphi(\max\{\mathcal{G}(u_0,u_1,u_1),\mathcal{G}(u_1,u_2,u_2)\}) \quad (3.1)$$

Suppose that $\mathcal{G}(u_0, u_1, u_1) \leq \mathcal{G}(u_1, u_2, u_2)$.Then using the inequality (3.1) and the property of φ, we get

$$\mathcal{G}(u_1, u_2, u_2) \leq \varphi(\mathcal{G}(u_1, u_2, u_2)) < \mathcal{G}(u_1, u_2, u_2)$$

A contradiction. It shouldbe

$\mathcal{G}(u_1, u_2, u_2) < \mathcal{G}(u_0, u_1, u_1)$ *i.e.* $\mathcal{G}(u_n, u_{n+1}, u_{n+1}) < \mathcal{G}(u_{n-1}, u_n, u_n)$ (3.2)

If we put $s_n = \mathcal{G}(u_n, u_{n+1}, u_{n+1})$, then from inequality (3.2), we obtain

$s_n < s_{n-1}$ (3.3)

$s_1 = \mathcal{G}(u_1, u_2, u_2) \leq \varphi(\mathcal{G}(u_0, u_1, u_1)) = \varphi(s_0)$ (using 3.2)

Using the same arguments, we find

$s_2=\mathcal{G}(u_2,u_3,u_3) \leq \varphi(\mathcal{G}(u_1,u_2,u_2)) \leq \varphi^2(s_0)$

By the mathematical induction, we have

$$s_n \leq \varphi^n(s_0)$$

Hences_n is a strictly decreasing sequence which tends to a limit s≥0. Suppose s> 0, there exist a positive integer k∈ N such that n≥ k implies

$$s< s_n < s+\delta(s) \quad (3.4)$$

Using the condition (3) and the inequality (3.3), we get

$\mathcal{G}(\mathcal{T}u_{n-1},\mathcal{S}u_{n+2},\mathcal{S}u_{n+2})=\mathcal{G}(u_n,u_{n+1},u_{n+1}) = s_n < s(3.5)$

For n≥ k, the inequality (3.5) contradicts the inequality (3.4). Then it should be s=0.Now we show that $\{u_n\}$ is a Cauchy sequence. Let $m > n$, we can write

$\mathcal{G}(u_{2n},u_{2m},u_{2m}) \leq \mathcal{G}(u_{2n},u_{2n+1},u_{2n+1}) + \mathcal{G}(u_{2n+1},u_{2n+2},u_{2n+1})$ ………… +… $\mathcal{G}(u_{2m-1},u_{2m},u_{2m})$

$$\leq \varphi^{2n}(\mathcal{G}(u_0,u_1,u_1)) + \varphi^{2n+1}(\mathcal{G}(u_0,u_1,u_1)) + \cdots \varphi^{2m-1}(\mathcal{G}(u_0,u_1,u_1)) \quad (3.6)$$

Let $r_{2n}=\sum_{k=0}^{k=2n} \varphi^k(\mathcal{G}(u_0,u_1,u_1))$

By (3.6), we have

$$\mathcal{G}(u_{2n},u_{2m},u_{2m}) \leq r_{2m-1} - r_{2n-1} \quad (3.7)$$

By the defination of φ , there exist$r$∈ [0,∞), such that

$\lim_{n\to\infty} r_{2n}=$ r

By (3.7), we get

$\lim_{n,m\to\infty} \mathcal{G}(u_{2n},u_{2m},u_{2m})=0$

Therefore, $\{u_n\}$is a Cauchy. Since (($\mathcal{X},\mathcal{G}$) is a complete$\mathcal{G}-$ metric space, there exists a point $z \in \mathcal{X}$ such that $u_n \to z$ as $n \to \infty$..Also we have $\mathcal{T}^n u_n \to z$,$\mathcal{S}^n u_n \to z$, $\mathcal{A}^n u_n \to z$,$\mathcal{B}^n u_n \to z.$

We prove that z is a common fixed point of $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$, that is $\mathcal{A}z = \mathcal{B}z = \mathcal{S}z = \mathcal{T}z = z$. At first, we suppose that $\mathcal{S}z \neq z$. Using condition (2)and the triangle inequality, we obtain

$$\mathcal{G}(z,\mathcal{S}z,\mathcal{S}z) \leq \mathcal{G}(z,u_{2n+1},u_{2n+1})+\mathcal{G}(u_{2n+1},\mathcal{S}z,\mathcal{S}z) = \mathcal{G}(z,u_{2n+1},u_{2n+1})+\mathcal{G}(\mathcal{T}u_{2n},\mathcal{S}z,\mathcal{S}z)$$

$$\leq \mathcal{G}(z,u_{2n+1},u_{2n+1})+\varphi(\mathcal{P}(u_{2n},z))$$

$$=\mathcal{G}(z,u_{2n+1},u_{2n+1})+ \varphi\left(\max\left\{\begin{matrix}\mathcal{G}(u_{2n},z,z),\mathcal{G}(\mathcal{A}u_{2n},\mathcal{T}u_{2n},\mathcal{T}u_{2n}),\mathcal{G}(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z),\\ \left[\dfrac{\mathcal{G}(\mathcal{A}u_{2n},\mathcal{S}z,\mathcal{S}z)+\mathcal{G}(\mathcal{B}z,\mathcal{T}u_{2n},\mathcal{T}u_{2n})}{\mathcal{G}(\mathcal{A}u_{2n},\mathcal{T}u_{2n},\mathcal{T}u_{2n})+\mathcal{G}(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z)+1}\right]\mathcal{G}(u_{2n},z,z)\end{matrix}\right\}\right)$$

Taking $n \to \infty$, we have

$$\mathcal{G}(z,\mathcal{S}z,\mathcal{S}z) \leq \varphi(\mathcal{G}(z,\mathcal{S}z,\mathcal{S}z)) < \mathcal{G}(z,\mathcal{S}z,\mathcal{S}z)$$

A contradiction. Hence $\mathcal{S}z = z$.Similarly , we have$\mathcal{A}z = z$, $\mathcal{B}z$=z,,$\mathcal{T}z = z$

Therefore $\mathcal{A}z = \mathcal{B}z = \mathcal{S}z = \mathcal{T}z = z.$

Finally, we show that at least one of the self- mappings $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$is discontinuous at the point z if and only if

$lim_{u\to z}\mathcal{P}(u,z) \neq 0$ or $lim_{v\to z}\mathcal{P}(z,v) \neq 0$

To do this we prove that if $lim_{u\to z}\mathcal{P}(u,z)$=0 and $lim_{v\to z}\mathcal{P}(z,v)$ =0 then all $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$ are continuous at the common fixed point z. Let $lim_{u\to z}\mathcal{P}(u,z)$=0 and $lim_{v\to z}\mathcal{P}(z,v)$ =0.Using the definition of$\mathcal{P}(u,z)$and $\mathcal{P}(z,v)$ we have,

$$lim_{u\to z}\left\{\max\left\{\begin{matrix}\mathcal{G}(u,z,z),\mathcal{G}(\mathcal{A}u,\mathcal{T}u,\mathcal{T}u),\mathcal{G}(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z),\\ \left[\frac{\mathcal{G}(\mathcal{A}u,\mathcal{S}z,\mathcal{S}z)+\mathcal{G}(\mathcal{B}z,\mathcal{T}u,\mathcal{T}u)}{\mathcal{G}(\mathcal{A}u,\mathcal{T}u,\mathcal{T}u)+\mathcal{G}(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z)+1}\right]\mathcal{G}(u,z,z)\end{matrix}\right\}\right\}=0$$

And

$$lim_{v\to z}\left\{\max\left\{\begin{matrix}\mathcal{G}(z,v,v),\mathcal{G}(\mathcal{A}z,\mathcal{T}z,\mathcal{T}z),\mathcal{G}(\mathcal{B}v,\mathcal{S}v,\mathcal{S}v),\\ \left[\frac{\mathcal{G}(\mathcal{A}z,\mathcal{S}v,\mathcal{S}v)+\mathcal{G}(\mathcal{B}v,\mathcal{T}z,\mathcal{T}z)}{\mathcal{G}(\mathcal{A}z,\mathcal{T}z,\mathcal{T}z)+\mathcal{G}(\mathcal{B}v,\mathcal{S}v,\mathcal{S}v)+1}\right]\mathcal{G}(z,v,v)\end{matrix}\right\}\right\}=0$$

Which implies that the self- mappings $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$ are continuous at the point z. Using the similar arguments, the converse statement can be easily checked.

Now we give an example.

Example 3.1. Let$\mathcal{X}$=[0,2] and $(\mathcal{X},\mathcal{G})$ be a $\mathcal{G}$-metric space defined by $\mathcal{G}(x,y,z) = |x-y| + |y-z| + |z-x|$ .Let us define the self- mappings $\mathcal{A},\mathcal{B},\mathcal{S}$,$\mathcal{T}$:$\mathcal{X} \to \mathcal{X}$by

$\mathcal{T}u=\begin{Bmatrix}1, & u \in [0,1]\\ 0.87, & u \in (1,2]\end{Bmatrix}$,$\mathcal{A}u=\begin{Bmatrix}1, & u \in [0,1]\\ 0.86, & u \in (1,2]\end{Bmatrix}$

$\mathcal{S}\,u=\begin{Bmatrix}1, & u\in[0,1]\\ 0.89, & u\in(1,2]\end{Bmatrix}$, $\mathcal{B}u=\begin{Bmatrix}1, & u\in[0,1]\\ 0.88, & u\in(1,2]\end{Bmatrix}$

Then the point$u$=1 is a common fixed point of the self- mappings $\mathcal{A},\mathcal{B},\mathcal{S}$ *and* $\mathcal{T}$ and all these self-mappings are discontinuous at this fixed point u=1.Indeed $\mathcal{A},\mathcal{B},\mathcal{S}$,$\mathcal{T}$ satisfy the condition (2) with

$$\varphi(t)=\begin{Bmatrix}0.13, & t>0.13\\ 0.11, & 0.11<t\leq 0.13\\ \frac{t}{3}, & 0<t\leq 0.11\end{Bmatrix}$$

And satisfy the condition (3) with

$$\boldsymbol{\delta}(\boldsymbol{\epsilon})\begin{Bmatrix}2, & \epsilon\geq 0.13\\ 3-\epsilon, & \epsilon<0.13\end{Bmatrix}$$

Also it can be easily seen that

$lim_{u\to z}\mathcal{P}(u,z)\neq 0$ or $lim_{v\to z}\mathcal{P}(z,v)\neq 0$

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