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Research Paper

A Simple Proof of the Formula of Rodrigues

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The Formula of Rodrigues states that in a direction of extremal curvature on a smooth enough surface,

$$\boxed{\mathbf{X}_3{}' + k\mathbf{X}' = \mathbf{O}}$$

where $\mathbf{X}_3$is the surface unit normal vector and $\mathbf{X}$is the position vector of a "principal" curve with extremal normal curvature, k. This formula has implications for analyzing developable surfaces, that is, surfaces with 0 gaussian curvature. [1][2]

We present an easy proof of the formula using the Weingarten Equations. In what follows, indices have the values 1 and 2; g_{ij}denotes the entries of the fundamental metric tensor; $g^{ij}$are the entries of its inverse; $L_{ij}$are the coefficients of the second fundamental form, and $\mathbf{X}_i$and$\mathbf{X}_{ij}$are the first and second partial derivatives with respect to the surface principal parameters, u_i and u_j. Since the parameter curves are in the directions of extremal curvature, we have:

$$L_{12} = L_{21} = g_{12} = g_{21} = g^{12} = g^{21} = 0$$

Since the fundamental metric tensor is diagonal, we have $g^{ii} = \dfrac{1}{g_{ii}}$. We follow the convention that a repeated index in a subscript and superscript implies summation.

The Weingarten Equations are $\boxed{\mathbf{X}_{3i} = -L_{ij}g^{kj}\mathbf{X}_k}$

Since we are employing principal curve parameters, we have

$$\mathbf{X}_{3i} = -L_{ii}g^{ii}\mathbf{X}_i = -\frac{L_{ii}}{g_{ii}}\mathbf{X}_i \Longrightarrow \mathbf{X}_{3i} = -\frac{L_{ii}}{g_{ii}}\mathbf{X}_i \qquad (1)$$

The extremal normal curvatures satisfy $k_i = \dfrac{L_{ii}}{g_{ii}}$, in which (1) becomes

$$\boxed{\mathbf{X}_{3i} + k_i\mathbf{X}_i = \mathbf{O}} \ \blacksquare$$

References

[1]. J.J. Stoker, Differential Geometry, Wiley-Interscience, NY, 1969
[2]. M. Lewinter, Problems in classical differential geometry, *Monographs in Undergraduate Mathematics*, Vol. 14, JUM, Guilford, NC, 1987.