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Review Paper

An analogue of Vizing's Theorem for intersecting hypergraphs

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Abstract
Recall that when considering (proper) edge colourings of a graph G the Theorems of Shannam (Sha 49] and Vizing [Viz64] give the following bounds for the chromatic index of a multigraph G

$$X'(G) \leq 3/2\,\Delta$$

Where A is the maximum degree of the vertices of G is the maximum multiplicity of the edges of G
A natural question to ask would he

"Can these results be generalized to hypertrophies?"

We consider a possible first step towards answering that question, namely

"How many edges can an intersecting hyper graph have?"

Where a hyper graph = (V, E) is intersecting if, for any edges e1,e2 € E

In order to illustrate the connection, we first define the notion of proper coloring that we will use for hyper graph we define a proper k-edge-coloring of a hyper graph.

Be an assignment of a color to each edge such that any two edges sharing at least u vertex receive different colors and at most k colors are used. We can then define the chromatic-index (H) to be the minimum k such that there exist a proper k-edge-coloring of H. Under these definitions, if H is an intersecting. Then
$|e(H)| = X'(H)$.

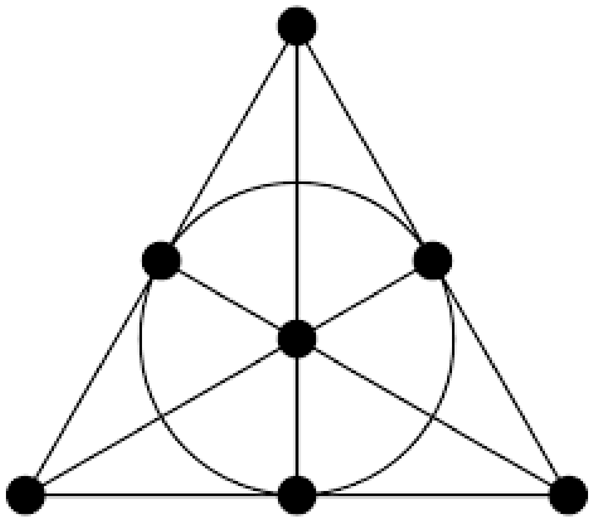
Key Words: *Hyper Grpahs , Vizing's Theorem, Vizing's leemas*

I. Introduction

In Mathematics, hyper graph is a generalization of a graph in which an edge can join any number of vertices.

In contrast, in an ordinary graph, an edge connects exactly two vertices.

Formally, a directed hyper graph is a pair(X, E) where X is a set of elements called *nodes*, *vertices*, *points*, or *elements* and E is a set of pairs of subsets of X. Each of these pairs (D, C) € E is called an *edge* or *hyper edge*; the vertex subset D is known as its *tail* or *domain*, and C as its *head* or *co domain*.



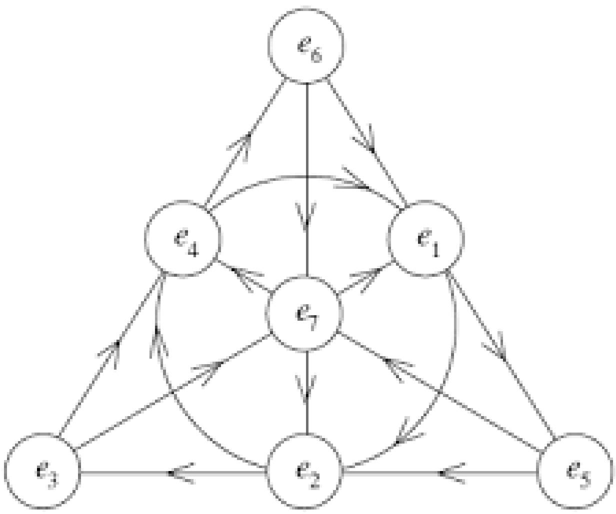
The Fano Plane

We consider intersecting three uniform hypergraphs We prove an upper bound for E of H more similar in forms to Vizing's bound and that this upper bound is obtained only by the Fano plane**.**
Theorem: Light H be an intersecting 3-uniform hyper graph with maximum degree Δ and maximum multiplicity μ.then

$$X\ (H) = |e\ (H)| \leq 2\Delta + \mu$$

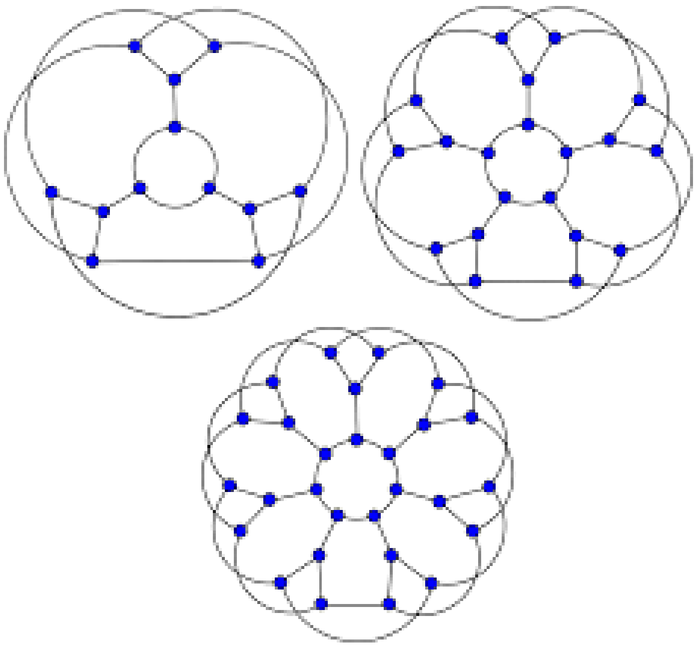
Furthermore the unique structure of achieving this maximum is μ copies of the fanno plane.
Given the form of this bound, It is attempting to conjecture that Vizing's theorems and are special cases of a result that would say that, For R uniform hyper graphs.

$$X\ (H) \leq (R\text{-}1)\ \Delta + \mu$$

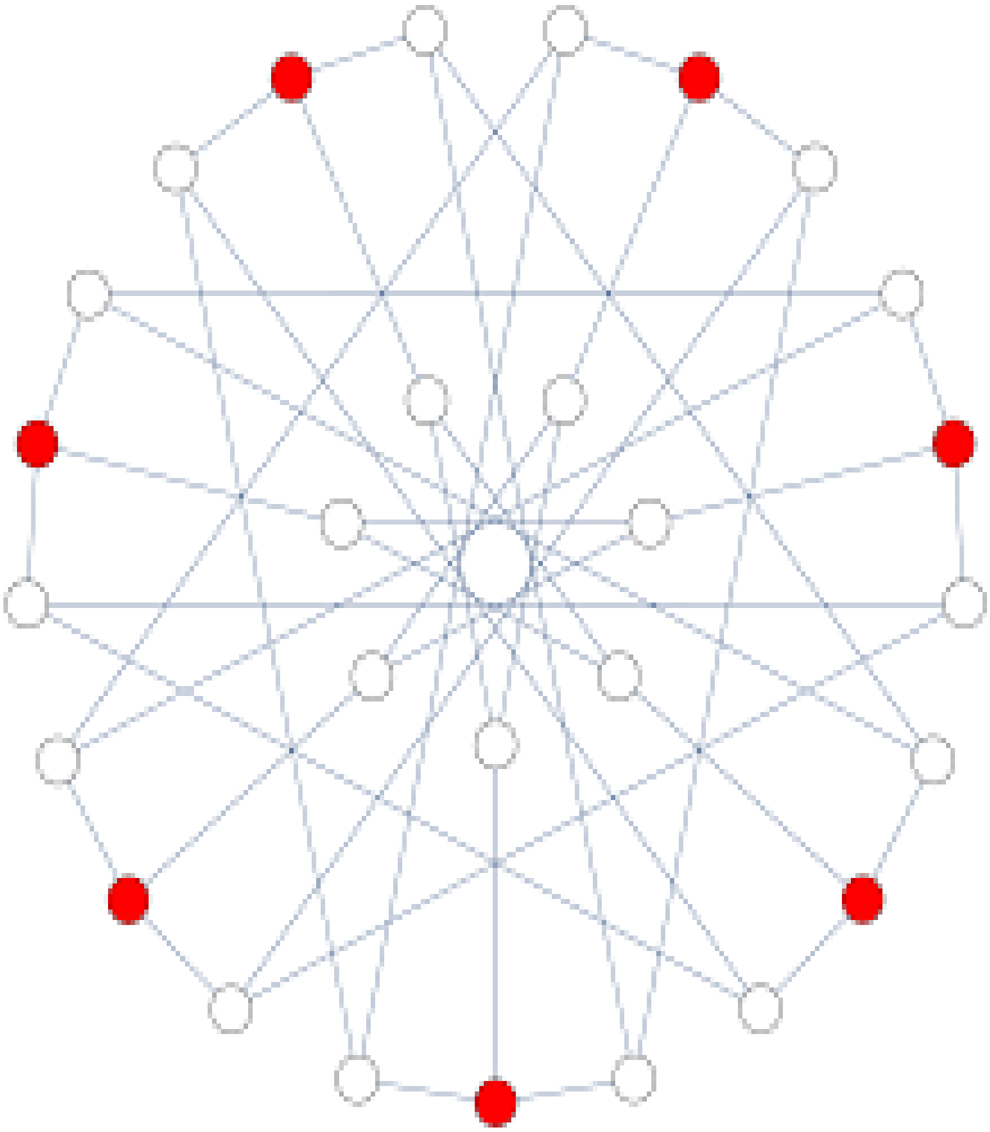


Definition .
The degree of a vertex v, denoted by d(v), is the number of edges of G which have v as a vertex. The maximum degree of a graph is denoted by Δ(G) and the minimum degree of a graph is denoted by δ(G). Vizing's Theorem is the central theorem of edge-chromatic graph theory, since it provides an upper and lower bound for the chromatic index χ 0 (G) of any graph G. Moreover, the upper and lower bound have a difference of 1. That is, for all finite, simple graphs G, $\Delta(G) \leq \chi\ 0\ (G) \leq \Delta(G) + 1$. This theorem motivates the study of the properties of graphs where Vizing's lower bound holds (class one graphs) and graphs where the upper bound holds (class two graphs), and characterizations of each.

Theorem : (Vizing) For any finite, simple graph G, $\Delta(G) \leq \chi 0\ (G) \leq \Delta(G)+1$
Proof: The lower bound, $\Delta(G)$, is trivial, since if G has a vertex v of degree d, then at least d edges share v as a vertex and cannot be colored with less than d colors.
Now, suppose for contradiction that there exist counterexamples to Vizing's upper bound. Of these counterexamples, let G be a counterexample of minimal size – that is, if one edge of G is removed, G becomes $(\Delta(G) + 1)$-edge-colorable.
Let e = {v, w0} be the edge that, if removed, reduces the chromatic index of G to $\Delta(G) + 1$. We construct a sequence of edges {v, w0}, {v, w1}, {v, w2}, ...a sequence of colors c0, c1, c2, ... called a Kempe Chain as follows
Let ci be a color absent at wi . Let {v, wi+1} be an edge colored ci . The Kempe Chain stops at $k \in N$ when either ck is a color absent at v, or ck is already used on {v, wj} for $j < k$. If ck is absent at v, then we can reassign colors ci to {v, wi} for $i \in [k]$ and we are done. So now assume ck is not absent at v.
Let cq be a color absent at v (We know that this color exists because we are allowing ourselves $\Delta + 1$ colors where maximum degree is Δ). Then recolor {v, wi} for $i \in [j - 1]$, and remove the color from {v, wj}. We now must find a way to color {v, wj}. Note that ck is absent at both wj and wk.
Case 1: If ck is absent at v, then colour {v, wj} with ck. Case 2: If cq is absent at wj , then color {v, wj} blue. Case 3: If cq is absent at wk, then color {v, wi} with ci for $j \leq i < k$ and color {v, wk} with cq (since none of the {v, wi}, $j \leq i < k$ are colored with either ck or cq).
If none of these conditions hold, then consider the sub graph G0 of G consisting only of edges colored with ck or cq, and their corresponding vertices. Note that the components of G0 are either paths or cycles. Since none of the above conditions hold, v, wj , and wk must all be endpoints of paths, and so they cannot all be part of the same component. In the component containing exactly one of these vertices, switch ck with cq.



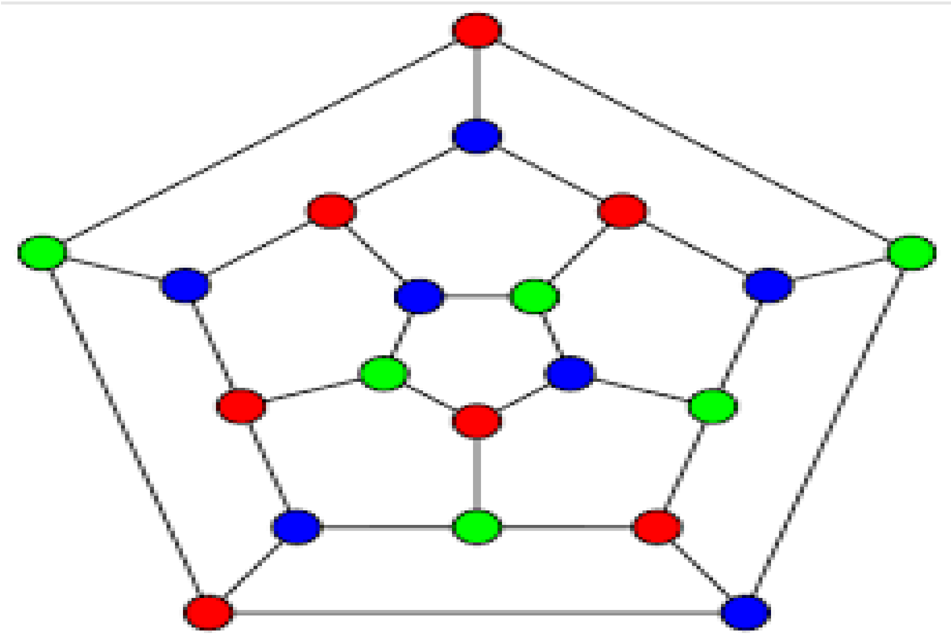
Conjecture:
Every Snark is contractible to the Petersen Graph. That is, every snark can be reduced to the Petersen graph by deleting certain edges and contracting others. This was conjectured by Tuttle, and recently four mathematicians named Robertson, Sanders, Seymour, and Thomas announced that they had discovered a proof. As of August 2015, this proof remains unpublished. This theorem is exceptionally significant, as it would characterize snarks much more strongly than the"bridgeless 3-regular class two" definition. In addition to this, it provides yet another proof of the Four Color Theorem (every planar graph can be vertex-colored by at most four colors)

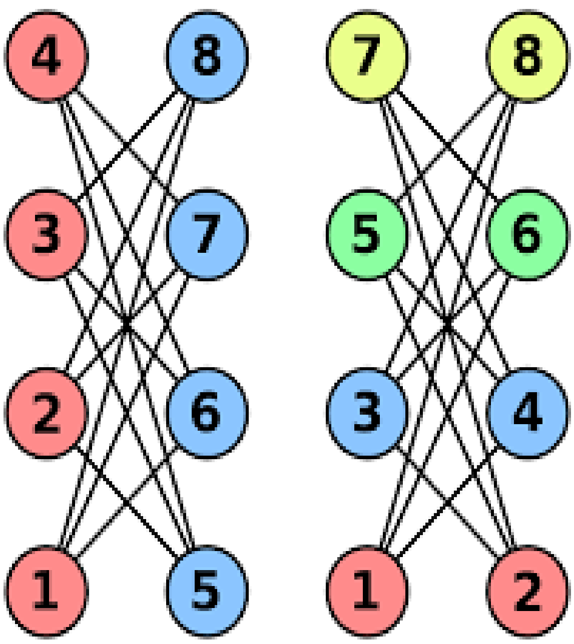


Corollary 1: Let G be a multigraph of maximum degree Δ and of maximum multiplicity μ.If the set of vertices of maximum degree is independent, then $\Delta + \mu - 1$ colors suffice to color the edge-set of G.
Proof: This follows immediately from the main theorem when we let $D = \Delta$ and $t = \mu$.
The next corollary shows us that we can safely pre-color the edges of any maximal matching.

Corollary 2: Let G be a multigraph of maximum degree Δ and of multiplicity μ, and let M be a maximal matching of G. The edges of G can be colored with $\Delta + \mu$ colors so that all the edges in M get the same color.
Proof: Let $G - M$ be the graph obtained by removing the edges of the maximal matching M.
Since M is a matching, when we remove them, we remove exactly one edge from each vertex incident to those edges.
Any vertices which were not incident the edges of M are not adjacent to each other (else their shared edge would be part of M).
Thus, any remaining vertices of degree Δ form an independent set (or an empty set). By the first corollary, we know that $\chi'(G - M) \leq \Delta + \mu - 1$. Color all the edges of M with one new color and we have that $\chi'(G) \leq \Delta + \mu$.
Corollary 2 provides a nice algorithm for properly edge coloring a multigraph with $\Delta + \mu$ colors.
Begin by finding a maximal matching M and assign all of those edges one color. Color the remaining edges with $\Delta + \mu - 1$ colors.
If this proves difficult, we can find an edge $e_0 \in G - M$ such that $(G - M) - e_0$ can be properly edge colored with $\Delta + \mu - 1$ colors and then perform the sequential f-recoloring outlined in the theorem.

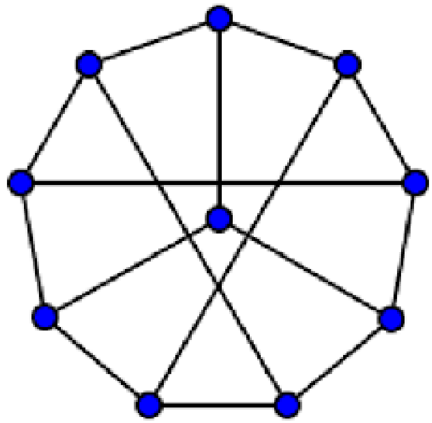
A practical application for proper edge-colorings of a graph results from Corollary 2. Suppose that the maximal matching M represents a pairing of pre-assigned matches on a given day or time slot between teams. The graph represents all the pairings that need to happen for the season or event. The chromatic index of this graph can show us how many more days or time slots would be required to achieve all pairings.





Theorem :(Vizing's theorem for simple graphs). $\Delta(G) \leq \chi 0 (G) \leq \Delta(G) + 1$ for any simple graph G.
Proof. The inequality $\Delta(G) \leq \chi 0 (G)$ being trivial, we show $\chi 0 (G) \leq \Delta(G)+ 1$.
To prove this inductively, it suffices to show for any simple graph G:
Let v be a vertex such that v and all its neighbors have degree at most k, while at most one neighbor has degree precisely k. Then if $G - v$ is k-edge-colorable, also G is k-edge-colorable.
We prove (1) by induction on k. We can assume that each neighbor u of v has degree $k - 1$, except for one of degree k, since otherwise we can add a new vertex w and an edge uw without violating the conditions in (1). We can do this till all neighbors of v have degree $k - 1$, except for one having degree k.

Consider any k-edge-coloring of G − v. For i = 1, k, let Xi be the set of neighbours of v that are missed by color i. So all but one neighbor of v is in precisely two of the Xi, and one neighbor is in precisely one Xi.
Hence X k i=1 |Xi | = 2 deg (v) − 1 < 2k. We can assume that we have chosen the coloring such that i=1 |Xi | 2 is minimized. Then for all i, j = 1, k: ||Xi | − |Xj || ≤ 2. For if, say, |X1| > |X2| + 2, consider the sub graph H made by all edges of colors 1 and 2. Each component of H is a path or circuit.
At least one component of H contains more vertices in X1 than in X2. This component is a path P starting in X1 and not ending in X2. Exchanging colors 1 and 2 on P reduces |X1| 2 + |X2| 2 , contradicting our minimalist assumption. This proves (3). This implies that there exists an i with |Xi | = 1, since otherwise by (2) and (3) each |Xi | is 0 or 2, while their sum is odd, a contradiction. So we can assume |Xk| = 1, say Xk := {u}. Let G0 be the graph obtained from G by deleting edge vu and deleting all edges of color k. So G0 − v is (k − 1)-edge-coloured. Moreover, in G0 , vertex v and all its neighbors have degree at most k − 1, and at most one neighbor has degree k − 1. So by the induction hypothesis, G0 is (k − 1)-edge-colorable. Restoring color k, and giving edge vu color k, gives a k-edge-coloring of G.



II. Conclusion:

In this paper we introduced new constructive heuristic for edge coloring problem on simple graphs based on Vizing's theorem. Being a simple modification of Vizing's algorithm, new heuristic guarantees that, when it is not able to find proven optimal solution to the problem (matching the Δ lower bound), it finds a solution using at most one more color than the optimals.

Experimental results showed that the new heuristic was capable of finding a Δ coloring for all benchmark instance considered. In terms of computational times, the new heuristic is significantly faster than previous approaches in the literatures.

As future work, we intend to extend the proposed heuristic to consider multigraphs. In fact, Vizing's theorem states that the chromatic index of a multigraph is between Δ and Δ + μ being μ the multiplicity of the graph. Vizing's algorithm finds Δ + μ colorings of multigraphs. A heuristic similar to the one developed in this work may be able to find better solutions in almost the same computational time

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