Quest Journals
Journal of Research in Applied Mathematics
Volume 8 ~ Issue 12 (2022) pp: 50-60
ISSN(Online) : 2394-0743 ISSN (Print): 2394-0735
www.questjournals.org

Research Paper

A New Block of Higher Order Hybrid Super Class BDF for Simulating Stiff IVP of ODEs

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ABSTRACT
A new block of higher order hybrid super class of backward differentiation formula for simulating stiff IVP of ODEs was developed. The proposed new scheme can approximate the values of two point and two off-step points at a time per integration step. The scheme is Super class; it comprises a stability control parameter and by varying the value of the free parameter,ρ within the interval $(-1, 1)$in the formula, more zero and A-s stable schemes can be obtained. This research considers $\rho = -\frac{3}{5}$ and arrived at zero and A – stable method, capable of solving stiff IVPs of ODEs. Approximates result from the system of stiff ODE problems considered are found to favourably validate the performance of the new method in terms of accuracy of the scale error and less executional time compared to other schemes considered in the research. Hence, the proposed new scheme (HSBBDF) can be used for integrating stiff IVPs of ODEs.
KEYWORDS: *Block, Implicit, IVPs, Ordinary Differential Equation, Zero stable*

***R**eceived 06 Dec., 2022; **R**evised 18 Dec., 2022; **A**ccepted 20 Dec., 2022 © The author(s) 2022.*
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I. INTRODUCTION

Backward differentiation formula came to existence from the work of Curtiss & Hirschfield [1], then its extended by Cash [2 - 3], its implicit block method by Ibrahim *et al.*,[4], Super class aspect of BBDF formula by Sueiman *et al.*,[5];Musa *et al.*,[6 - 10], diagonally implicit BBDF formula by Zawawi *et al*, [11 - 12], Abdullahi*et al* [12 - 16],Sagir*et al* [17 - 19]. All these methods possess difference degree of accuracy when compared with some existing methods. Still due to the preferences of seeking numerical approximate solutions to most of the modern problems, numerical methods are been developed continuously with various capacities to handle current realities of stiff initial vaue problem of ODEs. A stiff equation is a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small [16]. Some of the recent method with very good stability properties, at one point or the other are found with in [20 - 24]. Most of the methods stated are zero stable, A- stable or both, and displays good accuracy of the scaled error and executional time.

This research aimed at proposing a new higher order hybrid block of super class of backward differentiation formulathat possesses zeroand A- stable propertiesrequired to handle a stiff IVP of ODEs. The proposed scheme is of the form

$$\sum_{j=o}^{2} \alpha_{j,i} y_{n+j-2} + \sum_{j=o}^{1+k} \alpha_{j+3,i} y_{n+(j+1)/2} = h\beta_{k+1,i}[f_{n+k} - \rho f_{n+k-2}] \quad (1)$$

where ρ is a free parameter considered with the same interval of (-1 , 1) as in Musa *et al* [5]. The proposed formula (1) would be used for integrating first order stiff IVPs of the form

$$\left.\begin{aligned} & y' = f(x, \hat{Y}),\ \hat{Y}(a) = \varphi\eta,\ a \le x \le b \\ \text{where}\ & \hat{Y} = (y_1, y_2, y_3, \dots\dots\dots y_n),\ \eta\bar{\varphi} = (\varphi\eta_1, \varphi\eta_2, \varphi\eta_3, \dots, \varphi\eta_n) \end{aligned}\right\} \quad (2)$$

II. MATERIAL AND METHOD

In this section, two approximate solution values y_{n+1} and y_{n+2} with step size$h$, and two off-step points $y_{n+\frac{1}{2}}$ and $y_{n+\frac{3}{2}}$ which are chosen at the point where the step size are halved is formulated in a block simultaneously. The formulae are computed using the back valuesy_n,y_{n-1} andy_{n-2} with step size h.

From (1) consider$k$and$i$ have the same value. The formula (1) is derived using Taylor's series expansion about x_n

The Linear operator $L_i$associated with first, second, third and fourth point of the proposed formulaisdefined as follows

$$\left.\begin{aligned} L_i[y(x_n),h]&: \alpha_{0,i}y_{n-2}+\alpha_{1,i}y_{n-1}+\alpha_{2,i}y_n+\alpha_{3,i}y_{n+\frac{1}{2}}-h\beta_{k+1,i}[f_{n+k}-\rho f_{n+k-2}]=0\\ L_i[y(x_n),h]&: \alpha_{0,i}y_{n-2}+\alpha_{1,i}y_{n-1}+\alpha_{2,i}y_n+\alpha_{3,i}y_{n+\frac{1}{2}}+\alpha_{4,i}y_{n+1}-h\beta_{k+1,i}[f_{n+k}-\rho f_{n+k-2}]=0\\ L_i[y(x_n),h]&: \alpha_{0,i}y_{n-2}+\alpha_{1,i}y_{n-1}+\alpha_{2,i}y_n+\alpha_{3,i}y_{n+\frac{1}{2}}+\alpha_{4,i}y_{n+1}+\alpha_{5,i}y_{n+\frac{3}{2}}-h\beta_{k+1,i}[f_{n+k}-\rho f_{n+k-2}]=0\\ L_i[y(x_n),h]&: \alpha_{0,i}y_{n-2}+\alpha_{1,i}y_{n-1}+\alpha_{2,i}y_n+\alpha_{3,i}y_{n+\frac{1}{2}}+\alpha_{4,i}y_{n+1}+\alpha_{5,i}y_{n+\frac{3}{2}}+\alpha_{6,i}y_{n+2}-h\beta_{k+1,i}[f_{n+k}-\rho f_{n+k-2}]=0 \end{aligned}\right\}$$

(3)

Consider the following value of k & $i's$ value in (3) for the cases below:

FOR CASE 1, 2, 3 & 4 as in $k=i=\frac{1}{2}$, $=i=1$, $k=i=\frac{3}{2}$&$k=i=2$for the First, Second, Third & Fourth point respectively, withthe associated operator ($L_{\frac{1}{2}}$, L_1 , $L_{\frac{3}{2}}$&L_2) related to (3) written as

$$\left.\begin{aligned} &\alpha_{0,\frac{1}{2}}y(x_n-2h)+\alpha_{1,\frac{1}{2}}y(x_n-h)+\alpha_{2,\frac{1}{2}}y(x_n)+\alpha_{3,\frac{1}{2}}y(x_n+\tfrac{1}{2}h)-h\beta_{\frac{1}{2},\frac{1}{2}}\left[f\left(x_n+\tfrac{1}{2}h\right)+\rho f\left(x_n-\tfrac{3}{2}h\right)\right]=0\\ &\alpha_{0,1}y(x_n-2h)+\alpha_{1,1}y(x_n-h)+\alpha_{2,1}y(x_n)+\alpha_{3,1}y(x_n+\tfrac{1}{2}h)+\alpha_{4,1}y(x_n+h)-h\beta_{1,1}[f(x_n+h)+\rho f(x_n-h)]=0\\ &\alpha_{0,\frac{3}{2}}y(x_n-2h)+\alpha_{1,\frac{3}{2}}y(x_n-h)+\alpha_{2,\frac{3}{2}}y(x_n)+\alpha_{3,\frac{3}{2}}y(x_n+\tfrac{1}{2}h)+\alpha_{4,\frac{3}{2}}y(x_n+h)+\alpha_{5,\frac{3}{2}}y(x_n+\tfrac{3}{2}h)-h\beta_{\frac{3}{2},\frac{3}{2}}\left[f\left(x_n+\tfrac{3}{2}h\right)+\rho f\left(x_n-\tfrac{1}{2}h\right)\right]=0\\ &\alpha_{0,2}y(x_n-2h)+\alpha_{1,2}y(x_n-h)+\alpha_{2,2}y(x_n)+\alpha_{3,2}y(x_n+\tfrac{1}{2}h)+\alpha_{4,2}y(x_n+h)+\alpha_{5,2}y(x_n+\tfrac{3}{2}h)+\alpha_{6,2}y(x_n+2h)-h\beta_{2,2}[f(x_n+2h)+\rho f(x_n)]=0 \end{aligned}\right\}$$

(4)

Expanding$(x_n-2h)(x_n-h)$, $y(x_n)$, $y(x_n+\frac{1}{2}h)$,$y(x_n+h)$, $y(x_n+\frac{3}{2}h)$, $y(x_n+2h)$,$f\left(x_n-\frac{3}{2}h\right)$, $f\left(x_n-\right.$ *12h, fxn+32h,fxn+h , fxn+2h*in (4) with a Taylor's series expansionabout *xn* and collect the like terms gives

$$\left.\begin{aligned} C_{0,\frac{1}{2}}y(x_n)+C_{1,\frac{1}{2}}hy'(x_n)+C_{2,\frac{1}{2}}h^2y''(x_n)+\cdots&=0\\ C_{0,1}y(x_n)+C_{1,1}hy'(x_n)+C_{2,1}h^2y''(x_n)+C_{3,1}h^3y'''(x_n)+\cdots&=0\\ C_{0,\frac{3}{2}}y(x_n)+C_{1\frac{3}{2},}hy'(x_n)+C_{2,\frac{3}{2}}h^2y''(x_n)+C_{3,\frac{3}{2}}h^3y'''(x_n)+C_{4,\frac{3}{2}}h^4y'^{v}(x_n)+\cdots&=0\\ C_{0,\frac{3}{2}}y(x_n)+C_{1\frac{3}{2},}hy'(x_n)+C_{2,\frac{3}{2}}h^2y''(x_n)+C_{3,\frac{3}{2}}h^3y'''(x_n)+C_{4,\frac{3}{2}}h^4y'^{v}(x_n)+\cdots&=0 \end{aligned}\right\} \quad (5)$$

Where (5) can be evaluated as in (6),(7), (8) & (9) respectively as follows

$$\left.\begin{aligned} C_{0,\frac{1}{2}}&=\alpha_{0,\frac{1}{2}}+\alpha_{1,\frac{1}{2}}+\alpha_{2,\frac{1}{2}}+\alpha_{3,\frac{1}{2}}=0\\ C_{1,\frac{1}{2}}&=-2\alpha_{0,\frac{1}{2}}-\alpha_{1,\frac{1}{2}}+\frac{1}{2}\alpha_{3,\frac{1}{2}}-\beta_{\frac{3}{2},\frac{1}{2}}(1-\rho)=0\\ C_{2,\frac{1}{2}}&=2\alpha_{0,\frac{1}{2}}+\frac{1}{2}\alpha_{1,\frac{1}{2}}+\frac{1}{8}\alpha_{3,\frac{1}{2}}-\beta_{\frac{3}{2},\frac{1}{2}}\left(\frac{1}{2}+\frac{3}{2}\rho\right)=0\\ C_{3,\frac{1}{2}}&=-\frac{4}{3}\alpha_{0,\frac{1}{2}}-\frac{1}{6}\alpha_{1,\frac{1}{2}}+\frac{1}{48}\alpha_{3,\frac{1}{2}}-\beta_{\frac{3}{2},\frac{1}{2}}\left(\frac{1}{8}-\frac{9}{8}\rho\right)=0 \end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned} C_{0,1}&=\alpha_{0,1}+\alpha_{1,1}+\alpha_{2,1}+\alpha_{3,1}+\alpha_{4,1}=0\\ C_{1,1}&=-2\alpha_{0,1}-\alpha_{1,1}+\frac{1}{2}\alpha_{3,1}+\alpha_{4,1}-\beta_{2,1}(1-\rho)=0\\ C_{2,1}&=2\alpha_{0,1}+\frac{1}{2}\alpha_{1,1}+\frac{1}{8}\alpha_{3,1}+\frac{1}{2}\alpha_{4,1}-\beta_{2,1}(1+\rho)=0\\ C_{3,1}&=-\frac{4}{3}\alpha_{0,1}-\frac{1}{6}\alpha_{1,1}+\frac{1}{48}\alpha_{3,1}+\frac{1}{6}\alpha_{4,1}-\beta_{2,1}\left(\frac{1}{2}-\frac{1}{2}\rho\right)=0\\ C_{4,1}&=\frac{2}{3}\alpha_{0,1}+\frac{1}{24}\alpha_{1,1}+\frac{1}{384}\alpha_{3,1}+\frac{1}{24}\alpha_{4,1}-\beta_{2,1}\left(\frac{1}{6}+\frac{1}{6}\rho\right)=0 \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} C_{0,\frac{3}{2}}&=\alpha_{0,\frac{3}{2}}+\alpha_{1,\frac{3}{2}}+\alpha_{2,\frac{3}{2}}+\alpha_{3,\frac{3}{2}}+\alpha_{4,\frac{3}{2}}+\alpha_{5,\frac{3}{2}}=0\\ C_{1,\frac{3}{2}}&=-2\alpha_{0.\frac{3}{2}}-\alpha_{1,\frac{3}{2}}+\frac{1}{2}\alpha_{3,\frac{3}{2}}+\alpha_{4,\frac{3}{2}}+\frac{3}{2}\alpha_{5,\frac{3}{2}}-\beta_{\frac{5}{2},\frac{3}{2}}(1-\rho)=0\\ C_{2,\frac{3}{2}}&=2\alpha_{0\frac{3}{2}}+\frac{1}{2}\alpha_{1,\frac{3}{2}}+\frac{1}{8}\alpha_{3,\frac{3}{2}}+\frac{1}{2}\alpha_{4,\frac{3}{2}}+\frac{9}{8}\alpha_{5,\frac{3}{2}}-\beta_{\frac{5}{2},\frac{3}{2}}\left(\frac{3}{2}+\frac{1}{2}\rho\right)=0\\ C_{3,\frac{3}{2}}&=-\frac{4}{3}\alpha_{0,\frac{3}{2}}-\frac{1}{6}\alpha_{1,\frac{3}{2}}+\frac{1}{48}\alpha_{3,\frac{3}{2}}+\frac{1}{6}\alpha_{4,\frac{3}{2}}+\frac{27}{48}\alpha_{5,\frac{3}{2}}-\beta_{\frac{5}{2},\frac{3}{2}}\left(\frac{9}{8}-\frac{1}{8}\rho\right)=0\\ C_{4,\frac{3}{2}}&=\frac{2}{3}\alpha_{0\frac{3}{2}}+\frac{1}{24}\alpha_{1,\frac{3}{2}}+\frac{1}{384}\alpha_{3,\frac{3}{2}}+\frac{1}{24}\alpha_{4,\frac{3}{2}}+\frac{81}{384}\alpha_{5,\frac{3}{2}}-\beta_{\frac{5}{2},\frac{3}{2}}\left(\frac{27}{48}+\frac{1}{48}\rho\right)=0\\ C_{5,\frac{3}{2}}&=-\frac{4}{15}\alpha_{0,\frac{3}{2}}-\frac{1}{120}\alpha_{1,\frac{3}{2}}+\frac{1}{3840}\alpha_{3,\frac{3}{2}}+\frac{1}{120}\alpha_{4,\frac{3}{2}}+\frac{243}{3840}\alpha_{5,\frac{3}{2}}-\beta_{\frac{5}{2},\frac{3}{2}}\left(\frac{81}{384}-\frac{1}{384}\rho\right)=0 \end{aligned}\right\} \quad (8)$$

&

$$\left.\begin{aligned} C_{0,2} &= \alpha_{0,2}+\alpha_{1,2}+\alpha_{2,2}+\alpha_{3,2}+\alpha_{4,2}+\alpha_{5,2}+\alpha_{6,2}=0\\ C_{1,2} &= -2\alpha_{0,2}-\alpha_{1,2}+\frac{1}{2}\alpha_{3,2}+\alpha_{4,2}+\frac{3}{2}\alpha_{5,2}+2\alpha_{6,2}-\beta_{3,2}(1-\rho)=0\\ C_{2,2} &= 2\alpha_{0,2}+\frac{1}{2}\alpha_{1,2}+\frac{1}{8}\alpha_{3,2}+\frac{1}{2}\alpha_{4,2}+\frac{9}{8}\alpha_{5,2}+2\alpha_{6,2}-2\beta_{3,2}=0\\ C_{3,2} &= -\frac{4}{3}\alpha_{0,2}-\frac{1}{6}\alpha_{1,2}+\frac{1}{48}\alpha_{3,2}+\frac{1}{6}\alpha_{4,2}+\frac{27}{48}\alpha_{5,2}+\frac{4}{3}\alpha_{6,2}-2\beta_{3,2}=0\\ C_{4,2} &= \frac{2}{3}\alpha_{0,2}+\frac{1}{24}\alpha_{1,2}+\frac{1}{384}\alpha_{3,2}+\frac{1}{24}\alpha_{4,2}+\frac{81}{384}\alpha_{5,2}+\frac{2}{3}\alpha_{6,2}-\frac{4}{3}\beta_{3,2}=0\\ C_{5,2} &= -\frac{4}{15}\alpha_{0,2}-\frac{1}{120}\alpha_{1,2}+\frac{1}{3840}\alpha_{3,2}+\frac{1}{120}\alpha_{4,2}+\frac{243}{3840}\alpha_{5,2}+\frac{4}{15}\alpha_{6,2}-\frac{2}{3}\beta_{3,2}=0\\ C_{4,2} &= \frac{4}{45}\alpha_{0,2}+\frac{1}{720}\alpha_{1,2}+\frac{1}{46080}\alpha_{3,2}+\frac{1}{720}\alpha_{4,2}+\frac{729}{46080}\alpha_{5,2}+\frac{4}{45}\alpha_{6,2}-\frac{4}{15}\beta_{3,2}=0 \end{aligned}\right\} \quad (9)$$

Normalizing the Coefficients$\alpha_{3,\frac{1}{2}}$, $\alpha_{4,1}$, $\alpha_{5,\frac{3}{2}}$&$\alpha_{6,2}$of $y_{n+\frac{1}{2}}$, y_{n+1}, $y_{n+\frac{3}{2}}$&y_{n+2} respectively to 1.

Solving equation (6), (7), (8) & (9) with the aids of Maple Software for the values of $\alpha_{j,i's}$ and $\beta_{j,i's}$and Substituting the values in (4) gives the first, second, third & fourth point as

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\frac{3}{32}y_{n-2}+\frac{25}{16}y_{n-1}-\frac{15}{32}y_n-\frac{75}{64}hf_{n+\frac{1}{2}}+\frac{75}{64}\rho hf_{n-\frac{3}{2}}\\ y_{n+1} &= \frac{2}{155}y_{n-2}+\frac{9}{31}y_{n-1}-\frac{36}{31}y_n+\frac{288}{155}y_{n+\frac{1}{2}}+\frac{15}{62}hf_{n+1}-\frac{15}{62}\rho hf_{n-1}\\ y_{n+\frac{3}{2}} &= \frac{9}{2312}y_{n-2}+\frac{147}{2312}y_{n-1}+\frac{1295}{2312}y_n-\frac{483}{289}y_{n+\frac{1}{2}}+\frac{4725}{2312}y_{n+1}+\frac{525}{2312}hf_{n+\frac{3}{2}}-\frac{525}{2312}\rho hf_{n-\frac{1}{2}}\\ y_{n+2} &= -\frac{6663}{1052555}y_{n-2}+\frac{2452}{30073}y_{n-1}-\frac{10794}{30073}y_n+\frac{209664}{150365}y_{n+\frac{1}{2}}-\frac{72228}{30073}y_{n+1}+\frac{482304}{210511}y_{n+\frac{3}{2}}+\frac{6330}{30073}hf_{n+2}+\frac{6330}{30073}\rho hf_n \end{aligned}\right\}$$

(10)

(10) is calleda new block of higher order hybrid super class of BDF for solving stiff initial value problem of ordinary differential equation (HSBBDF).

III. Analysis of the Method

In this section, order and Stability properties of the proposed method (10) will be analysed.

3.1 Order of the Method

In this section, the order of the proposed methods (10) will be derived.

The proposed method (10) is equivalent to the following form

(11)

$$\begin{aligned} \frac{3}{32}y_{n-2}-\frac{25}{16}y_{n-1}+\frac{15}{32}y_n+y_{n+\frac{1}{2}} &= -\frac{75}{64}hf_{n+\frac{1}{2}}+\frac{75}{64}\rho hf_{n-\frac{3}{2}}\\ -\frac{2}{155}y_{n-2}-\frac{9}{31}y_{n-1}+\frac{36}{31}y_n-\frac{288}{155}y_{n+\frac{1}{2}}+y_{n+1} &= \frac{15}{62}hf_{n+1}-\frac{15}{62}\rho hf_{n-1}\\ -\frac{9}{2312}y_{n-2}-\frac{147}{2312}y_{n-1}-\frac{1295}{2312}y_n+\frac{483}{289}y_{n+\frac{1}{2}}-\frac{4725}{2312}y_{n+1}+y_{n+\frac{3}{2}} &= \frac{525}{2312}hf_{n+\frac{3}{2}}-\frac{525}{2312}\rho hf_{n-\frac{1}{2}}\\ \frac{6663}{1052555}y_{n-2}-\frac{2452}{30073}y_{n-1}+\frac{10794}{30073}y_n-\frac{209664}{150365}y_{n+\frac{1}{2}}+\frac{72228}{30073}y_{n+1}-\frac{482304}{210511}y_{n+\frac{3}{2}}+y_{n+2} &= \frac{6330}{30073}hf_{n+2}+\frac{6330}{30073}\rho hf_n \end{aligned}$$

(11) Can be transforming to a general matrix form as

$\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h\sum_{j=0}^{1} D_j^* Y_{m+j-1}$, (12)

Where C &D are constant its coefficient matrices obtain as

$$\begin{bmatrix} 0 & 0 & 0 & -\frac{3}{32} \\ 0 & 0 & 0 & -\frac{2}{155} \\ 0 & 0 & 0 & -\frac{9}{2312} \\ 0 & 0 & 0 & \frac{6663}{1052555} \end{bmatrix} \begin{bmatrix} y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{25}{16} & 0 & \frac{15}{32} \\ 0 & -\frac{9}{31} & 0 & \frac{36}{31} \\ 0 & -\frac{147}{2312} & 0 & \frac{1295}{2312} \\ 0 & -\frac{2452}{30073} & 0 & \frac{10794}{30073} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{288}{155} & 1 & 0 & 0 \\ \frac{483}{289} & -\frac{4725}{2312} & 1 & 0 \\ -\frac{209664}{150365} & \frac{72228}{30073} & -\frac{482304}{210511} & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = h \begin{bmatrix} \frac{75}{64}\rho & 0 & -\frac{1}{8} & 0 \\ 0 & -\frac{15}{62}\rho & 0 & -\frac{1}{3} \\ 0 & 0 & -\frac{525}{2312}\rho & 0 \\ 0 & 0 & 0 & \frac{6330}{30073}\rho \end{bmatrix}$$

$$\begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} + \mathrm{h} \begin{bmatrix} -\frac{75}{64} & 0 & 0 & 0 \\ 0 & \frac{15}{62} & 0 & 0 \\ 0 & 0 & \frac{525}{2312} & 0 \\ 0 & 0 & 0 & \frac{6330}{30073} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \tag{13}$$

Whereas the free parameter, ρwill be considers as $\rho = -\frac{3}{5}$.

$$C_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} C_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} C_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} C_3 = \begin{bmatrix} -\frac{3}{32} \\ -\frac{2}{155} \\ -\frac{9}{2312} \\ \frac{6663}{1052555} \end{bmatrix} C_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} C_5 = \begin{bmatrix} -\frac{25}{16} \\ -\frac{9}{31} \\ -\frac{147}{2312} \\ -\frac{2452}{30073} \end{bmatrix} C_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} C_7 = \begin{bmatrix} \frac{15}{32} \\ \frac{36}{31} \\ \frac{1295}{2312} \\ \frac{10794}{30073} \end{bmatrix} C_8 =$$

$$\begin{bmatrix} 1 \\ -\frac{288}{155} \\ \frac{483}{289} \\ -\frac{209664}{150365} \end{bmatrix} C_9 = \begin{bmatrix} 0 \\ 1 \\ -\frac{4725}{2312} \\ \frac{72228}{30073} \end{bmatrix} C_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{482304}{210511} \end{bmatrix} C_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} D_4 = \begin{bmatrix} \frac{75}{64}\rho \\ 0 \\ 0 \\ 0 \end{bmatrix} D_5 = \begin{bmatrix} 0 \\ \frac{25}{62}\rho \\ 0 \\ 0 \end{bmatrix} D_6 =$$

$$\begin{bmatrix} -\frac{1}{8} \\ 0 \\ -\frac{252}{2312}\rho \\ 0 \end{bmatrix}$$

$$D_7 = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 0 \\ \frac{6330}{30073}\rho \end{bmatrix} D_8 = \begin{bmatrix} \frac{75}{64} \\ 0 \\ 0 \\ 0 \end{bmatrix} D_9 = \begin{bmatrix} 0 \\ \frac{15}{62} \\ 0 \\ 0 \end{bmatrix} D_{10} = \begin{bmatrix} 0 \\ 0 \\ \frac{525}{2312} \\ 0 \end{bmatrix} D_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{6330}{30073} \end{bmatrix}$$

Definition 3.1: The numerical method is said to be of order p if,
$E_0 = E_1 = E_2 = \ldots \quad \ldots \ldots E_p = 0$ But $E_{q+1} \neq 0$, where E_{q+1} is the error constant of the method and q is unique integer such that

$$E_0 = \sum_{J=0}^{11} C_J = 0$$

$$E_1 = \sum_{J=0}^{11} \left[jC_j - 2D_j \right] = 0$$

$$E_2 = \sum_{J=0}^{11} \left[\frac{1}{2!} j^2 C_j - 2jD_j \right] = 0$$

$$E_3 = \sum_{J=0}^{11} \left[\frac{1}{3!} j^3 C_j - 2\frac{1}{2!} j^2 D_j \right] = 0$$

$$E_4 = \sum_{J=0}^{11} \left[\frac{1}{4!} j^4 C_j - 2\frac{1}{3!} j^3 D_j \right] = 0$$

$$E_5 = \sum_{J=0}^{11} \left[\frac{1}{5!} j^5 C_j - 2\frac{1}{4!} j^4 D_j \right] = 0$$

$$E_6 = \sum_{J=0}^{11} \left[\frac{1}{6!} j^6 C_j - 2\frac{1}{5!} j^5 D_j \right] = 0$$

$$E_7 = \sum_{J=0}^{11} \left[\frac{1}{7!} j^7 C_j - 2\frac{1}{6!} j^6 D_j \right] = 0$$

$$E_8 = \sum_{J=0}^{11} \left[\frac{1}{8!} j^8 C_j - 2\frac{1}{7!} j^7 D_j \right] = \begin{bmatrix} -\frac{7}{32} \\ -\frac{2}{155} \\ \frac{525}{2312} \\ -\frac{4096}{150365} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the method is of order 7, with error constant $E_8 = \begin{bmatrix} -\frac{7}{32} \\ -\frac{2}{155} \\ \frac{525}{2312} \\ -\frac{4096}{150365} \end{bmatrix}$ (15)

3.2 Stability Analysis of the Method

In this section, we investigate the Zero and A- Stability property of the proposed method (10).

Definition 3.1 A linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple [5].
Definition 3.2 A linear multistep Method is said to be an A-stable method if its stability region covers the entire negativeleft half-plane [5].
The stability of the method (10) can be obtains by applying the standard test equation of the form
$y' = \lambda y \qquad \lambda \text{ is a complex number}, \quad Re(\lambda) < 0$ (16)
To get the following solutions

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= -\frac{3}{32}y_{n-2} + \frac{25}{16}y_{n-1} - \frac{15}{32}y_n - \frac{75}{64}h\lambda y_{n+\frac{1}{2}} + \frac{75}{64}\rho h\lambda y_{n-\frac{3}{2}} \\
y_{n+1} &= \frac{2}{155}y_{n-2} + \frac{9}{31}y_{n-1} - \frac{36}{31}y_n + \frac{288}{155}y_{n+\frac{1}{2}} + \frac{15}{62}h\lambda y_{n+1} - \frac{15}{62}\rho h\lambda y_{n-\frac{3}{2}} \\
y_{n+\frac{3}{2}} &= \frac{9}{2312}y_{n-2} + \frac{147}{2312}y_{n-1} + \frac{1295}{2312}y_n - \frac{483}{289}y_{n+\frac{1}{2}} + \frac{4725}{2312}y_{n+1} + \frac{525}{2312}h\lambda y_{n+\frac{3}{2}} - \frac{525}{2312}\rho h\lambda y_{n-\frac{1}{2}} \\
y_{n+2} &= -\frac{6663}{1052555}y_{n-2} + \frac{2452}{30073}y_{n-1} - \frac{10794}{30073}y_n + \frac{209664}{150365}y_{n+\frac{1}{2}} - \frac{72228}{30073}y_{n+1} + \frac{482304}{210511}y_{n+\frac{3}{2}} + \frac{6330}{30073}h\lambda y_{n+2} + \frac{6330}{30073}\rho h\lambda y_n
\end{aligned}\right\}$$
(17)
(17) Can be also be written as

$$\begin{bmatrix} 1-\frac{75}{64}h\lambda & 0 & 0 & 0 \\ -\frac{288}{155} & 1-\frac{15}{62}h\lambda & 0 & 0 \\ \frac{483}{289} & -\frac{4725}{2312} & 1-\frac{525}{2312}h\lambda & 0 \\ -\frac{209664}{150365} & \frac{72228}{30073} & -\frac{482304}{210511} & 1-\frac{6330}{30073}h\lambda \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = h$$

$$\begin{bmatrix} \frac{75}{64}\rho h\lambda & \frac{25}{16} & 0 & -\frac{15}{32} \\ 0 & \frac{9}{31}-\frac{15}{62}\rho h\lambda & 0 & -\frac{36}{31} \\ 0 & \frac{147}{2312} & \frac{525}{2312}\rho h\lambda & \frac{1295}{2312} \\ 0 & \frac{2452}{30073} & 0 & \frac{10794}{30073}-\frac{6330}{30073}\rho h\lambda \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & -\frac{3}{32} \\ 0 & 0 & 0 & \frac{12}{155} \\ 0 & 0 & 0 & \frac{9}{2312} \\ 0 & 0 & 0 & -\frac{6663}{1052555} \end{bmatrix}$$

$$\begin{bmatrix} y_{n-\frac{7}{2}} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \end{bmatrix} (18)$$

From (18) it follows that the coefficient matrices are given as

$$A = \begin{bmatrix} 1-\frac{75}{64}h\lambda & 0 & 0 & 0 \\ -\frac{288}{155} & 1-\frac{15}{62}h\lambda & 0 & 0 \\ \frac{483}{289} & -\frac{4725}{2312} & 1-\frac{525}{2312}h\lambda & 0 \\ -\frac{209664}{150365} & \frac{72228}{30073} & -\frac{482304}{210511} & 1-\frac{6330}{30073}h\lambda \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{75}{64}\rho h\lambda & \frac{25}{16} & 0 & -\frac{15}{32} \\ 0 & \frac{9}{31}-\frac{15}{62}\rho h\lambda & 0 & -\frac{36}{31} \\ 0 & \frac{147}{2312} & \frac{525}{2312}\rho h\lambda & \frac{1295}{2312} \\ 0 & \frac{2452}{30073} & 0 & \frac{10794}{30073}-\frac{6330}{30073}\rho h\lambda \end{bmatrix} \&$$

$$C = \begin{bmatrix} 0 & 0 & 0 & -\frac{3}{32} \\ 0 & 0 & 0 & \frac{12}{155} \\ 0 & 0 & 0 & \frac{9}{2312} \\ 0 & 0 & 0 & -\frac{6663}{1052555} \end{bmatrix}$$

The stability polynomial of the proposed method will be computed with the aid of the Maple Software using the relation

$\det(A * t^2 - B * t - C) =$

$$-\frac{1869328125}{137945091584} t^8 h^4 + \frac{299176875}{137945091584} t^5 h^2 + \frac{132512932125}{137945091584} t^7 h^2 + \frac{499417605}{1014302144} t^8 h$$
$$- \frac{88647569325}{137945091584} t^8 h^2 + \frac{76829764095}{17243136448} t^7 h + \frac{46488988125}{275890183168} t^8 h^3$$
$$+ \frac{944409375}{137945091584} t^7 h^3 + \frac{3146634315}{17243136448} t^6 h - \frac{192021975}{4756727296} t^6 h^2$$
$$+ \frac{517045185}{17243136448} t^5 h + \frac{112438125}{275890183168} t^6 h^3 + t^8 - \frac{168261849}{269424007} t^7 - \frac{183982971}{269424007} t^6$$
$$+ \frac{105356685}{269424007} t^5 - \frac{21527179695}{17243136448} t^6 h p - \frac{43575306075}{137945091584} t^6 h^2 p^2 + \frac{57189825}{59664832} t^7 h^2 p$$
$$+ \frac{19368331875}{275890183168} t^6 h^3 p^2 - \frac{40451410125}{137945091584} t^5 h^2 p^2 - \frac{79417648125}{275890183168} t^7 h^3 p$$
$$- \frac{1869328125}{68972545792} t^5 h^4 p^3 - \frac{5753845125}{8621568224} t^6 h^2 p - \frac{944409375}{137945091584} t^5 h^3 p^2$$
$$+ \frac{1869328125}{68972545792} t^7 h^4 p - \frac{944409375}{137945091584} t^6 h^3 p + \frac{1869328125}{137945091584} t^4 h^4 p^4$$
$$+ \frac{944409375}{137945091584} t^4 h^3 p^3 + \frac{112438125}{275890183168} t^3 h^3 p^3 + \frac{13560328125}{275890183168} t^5 h^3 p^3$$
$$- \frac{960065205}{1014302144} t^7 h p - \frac{13048308315}{17243136448} t^5 h p + \frac{1197559215}{17243136448} t^4 h p$$
$$+ \frac{5771099475}{137945091584} t^4 h^2 p^2 - \frac{25307775}{17243136448} t^5 h^2 p - \frac{112438125}{275890183168} t^4 h^3 p^2$$
$$- \frac{299176875}{137945091584} t^3 h^2 p^2 - \frac{112438125}{275890183168} t^5 h^3 p$$

(19)

Where $h = h\lambda$.

Substituting $\rho = -\frac{3}{5}$ and $h = 0$ in (19) solving for t, we have(20)

$$R(t,0) = t^8 - \frac{168261849}{269424007} t^7 - \frac{183982971}{269424007} t^6 + \frac{105356685}{269424007} t^5 \quad (21)$$

$$t = 1, 0, 0, 0, 0, 0 \; -0.8112957430 \;\; \& \;\; 0.5351662297$$

3.3 A - Stability of the proposed Method

In this section, the region for the absolute stability of the proposed methodsis plotted, by considering the stability polynomials (19)with $\rho = -\frac{3}{5}$. The set of point defined by$t = e^{i\theta}$, $0 \leq \theta \leq 2\pi$ describes the boundary of the stability region. The following stability region was the complex plot of the proposed method with the aid of Maple Software.

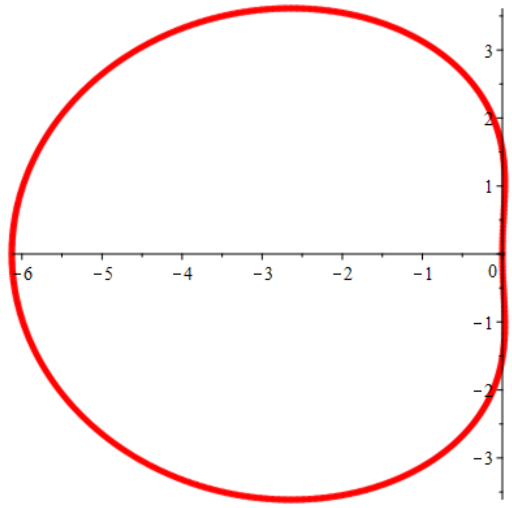

Figure 2: A-Stability region of the Proposed Method (HSBBDF)

IV. Test Problems

To validate the method developed, the following stiff IVPs are solved.

Problem 1: $y' = 5e^{5x}(y-x)^2 + 1$ $\quad y(0) = 0 \quad 0 \leq x \leq 1$

Exact solution:
$y(x) = x - e^{-5x}$
Source: (Lee *et al*, 2002)

Problem 2 : $y_1' = -20y_1 - 19y_2 \quad y_1(0) = 2$
$\qquad 0 \leq x \leq 20$
$y_2' = -19y_1 - 20y_2 y_2(0) = 0$
Exact Solution:
$y_1(x) = e^{-39x} + e^{-x}$
$y_2(x) = e^{-39x} - e^{-x}$
Source: (Cheney and Kincaid 2012)

Problem 3: $y_1' = 198y_1 + 199y_2 \quad y_1(0) = 1 \quad 0 \leq x \leq 10$
$y_2' = -398y_1 - 399y_2 \quad y_1(0) = -1$
Exact solution
$y_1(x) = e^{-x}$
$y_2(x) = -e^{-x}$
Eigen values -1 and -200
Source: (Ibrahim *et al*,2007);

V. Result and Discussions

To validate the method developed some IVPs would be solve and the results are tabulated and compared with some existing methods. The graphs highlighting the performance of these methods are also plotted .The acronyms below are used in the tables.

H= step-size;
MHTD =Method
MAX-ERR = Maximum Error;
EXE-TIME=Executional Time in second;
RDIBM = A Robust Diagonally Implicit Block Method for Solving First Order Stiff IVP of ODEs
3NBBDF = A New Fifth Order implicit block method for Solving First Order Stiff Ordinary Differential Equations

3ESBBDF =Extended 3-Point Super Class Of Block Backward Differentiation Formula For Solving Stiff Initial Value Problems
3BDF= Implicit r-point block backward differentiation formula for solving first-order stiff ODEs
HSBBDF = A Block of Higher Order Hybrid Super Class BDF for Integrating Stiff IVP of ODEs

Table 5.1: Comparison of Errors for Problem 1

h	Mtd	NS	Max-err	Exec-Time
10^{-2}	3NBBDF	333	3.51456(−3)	5.52416(−4)
	RDIBM	555	2.61015(−3)	3.11121(−5)
	HSBBDF	100	2.22716(−5)	1.32053(−5)
	3ESBBDF	333	4.83217(−3)	6.23441(−5)
10^{-3}	3NBBDF	3,333	4.90191(−5)	4.50367(−3)
	RDIBM	5,555	3.73116(−5)	2.96482(−4)
	HSBBDF	1,000	2.23872(−7)	2.10521(−3)
	3ESBBDF	3,333	5.95338(−5)	6.65467(−4)
10^{-4}	3NBBDF	33,333	5.20417(−7)	4.36918(−2)
	RDIBM	55,555	3.73371(−7)	2.94261(−3)
	HSBBDF	10,000	2.24942(−9)	2.20813(−3)
	3ESBBDF	33,333	5.95692(−7)	6.48433(−3)
10^{-5}	3NBBDF	333,333	5.25030(−9)	4.34808(−1)
	RDIBM	555,555	3.73652(−9)	2.92149(−2)
	HSBBDF	100,000	2.46821(−11)	2.27160(−2)
	3ESBBDF	333,333	5.959740(−9)	6.58687(−2)
10^{-6}	3NBBDF	3,333,333	5.25648(−11)	4.35791(+0)
	RDIBM	5,555,555	4.05313(−11)	2.90945(−1)
	HSBBDF	1,000,000	2.48213(−13)	2.32751(−1)
	3ESBBDF	3,333,333	6.186362(−11)	6.23434(−1)

Table 5.2: Comparison of Errors for Problem 2

h	Mtd	NS	Max-err	Exec-Time
10^{-2}	3NBBDF	333	6.98707(−2)	2.63337(−2)
	RDIBM	555	4.45713(−3)	2.41226(−2)
	HSBBDF	100	3.23210(−3)	9.54381(−3)
	3ESBBDF	100	8.83217(−4)	7.68676(−2)
10^{-3}	3NBBDF	3,333	5.40956(−3)	2.60816(−1)
	RDIBM	5,555	3.74938(−5)	2.42705(−1)
	HSBBDF	1,000	3.20945(−5)	4.49315(−2)
	3ESBBDF	1,000	6.05338(−5)	7.64515(−1)
10^{-4}	3NBBDF	33,333	3.08942(−5)	2.60725(+0)
	RDIBM	55,555	3.52727(−7)	2.40503(+0)
	HSBBDF	10,000	3.20032(−7)	2.92755(−1)
	3ESBBDF	10,000	6.26692(−6)	7.68143(−1)
10^{-5}	3NBBDF	333,333	3.18534(−7)	2.60597(+1)
	RDIBM	555,555	3.31505(−9)	2.40064(+1)
	HSBBDF	100,000	3.20019(−9)	3.63812(−1)
	3ESBBDF	100,000	6.32740(−8)	7.59821(+0)
10^{-6}	3NBBDF	3,333,333	3.19872(−9)	2.60700(+2)
	RDIBM	5,555,555	3.11313(−11)	2.40003(+2)
	HSBBDF	1,000,000	3.17623(−11)	2.39636(+0)
	3ESBBDF	1,000,000	6.33362(−10)	7.53567(+1)

Table 5.3: Comparison of Errors for Problem 3

h	Mtd	NS	Max-err	Exec-Time
10^{-2}	3NBBDF	333	1.94447(−4)	1.20394(−2)
	RDIBM	555	1.52564(−4)	3.93719(−3)
	HSBBDF	100	4.86193(−6)	6.51294(−4)
	3BBDF	333	1.07308(−2)	31,867μs
10^{-3}	3NBBDF	3,333	2.07993(−6)	1.19193(−1)
	RDIBM	5,555	1.76763(−6)	1.87573(−2)
	HSBBDF	1,000	4.82038(−8)	3.72066(−3)
	3BBDF	3,333	1.10060(−3)	258,361μs
10^{-4}	3NBBDF	33,333	2.09995(−8)	1.19296e(0)
	RDIBM	55,555	1.79766(−8)	1.66571(−1)
	HSBBDF	10,000	4.79902(−10)	3.88525 (−2)
	3BBDF	33,333	1.10333(−4)	2,582,756μs
10^{-5}	3NBBDF	333,333	2.10257(−8)	1.19173(1)
	RDIBM	555,555	1.82566(−8)	1.43458(0)
	HSBBDF	100,000	4.75952(−12)	2.39452(−1)
	3BBDF	333,333	1.10361(−5)	26,011,417μs
10^{-6}	3NBBDF	3,333,333	1.41029(−11)	1.19110(2)
	RDIBM	5,555,555	1.85567(−12)	1.28786(1)
	HSBBDF	1,000,000	4.72882(−14)	2.29562(+0)

3BBDF	3,333,333	1.10363(−6)	260,435,329μs

From table 5.1, 5.2 & 5.3 consisting the approximates result of problem 1, 2 and 3, it has shown that the newly derived scheme, HSBBDF performed better than 3BBDF, ABISBDF and 3NBBDF in terms of computational time and accuracy in problems 1, 2 and 3. However, in problem 1 and 2 the proposed scheme HSBBDF competes closely with RDIBM in the scaled error with new method having a little advantage over RDIBM. While, RDIBM has good scaled error and executional time than 3BBDF, ABISBDF and 3ESBBDF in problems 1, 2 and 3. However, the 3NBBDF and 3ESBBDF competes closely in terms of accuracy of the scale errors in problems 1. Similarly, the accuracy of the scale errors and executional time of the proposed methods, HSBBDF found to be better than all the methods compared in this research.The proposed method can be an alternative solver for first order stiff IVP of ODEs.

VI. Conclusion

A block of higher order hybrid super class of backward differentiation formula for simulating stiff IVP of ODEs is derived. The new scheme is block and hybrid, can generate four solution values, two point and two off – step point at a time per integration step. The properties of the proposed method have been checked, the method is found to be of order 7, Zero and A Stable, capable of solving stiff IVP of ODEs. Computedand approximated resultvalidated the performance of the method in terms of accuracy of the scale error and executional time with respect to other schemes considered in this research. Hence, the proposed method can be an alternative solver of first order system of stiff IVP of ordinary differential equations.

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