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Research Paper

Two infinite sums with the Li-Keiper coefficients (at s=2 and at s=1/2)

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ABSTRACT: *In this paper, starting with the infinite set of Equations for the trend, the tiny part and the complete Li-Keiper coefficients, we study two sequences, the first at s=2 (z=1-1/s =1/2), the second at s =1/2 (z=1-1/s= -1).*
In relation with this two series, we present a numerical experiment using a known formula for the coefficients given in the binary system, with the presence of a unique constant, i.e. π (for the first case).
KEYWORDS: *Li-Keiper coefficients binary system, infinite sums outside and inside of the critical strip, numerical experiment, Riemann Hypothesis.*

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I. INTRODUCTION

The starting point of this work is a set of Equations for the trend, for the tiny part and for the complete Li-Keiper coefficients (n=1, 2, 3...) given by [1]:

$$\sum_{k=0}^{n-1} (-1)^{(k)} \cdot \binom{n-1}{k} \cdot \varphi_{n-k} = \left(\frac{\xi^{n}(1)}{\xi(1)} \cdot \frac{1}{\Gamma(n+1)}\right) \tag{1}$$

In the above Eq. (1), ξ (1) is the Xi function of argument s=1 where $\xi(s) = (½)\cdot(s-1)\cdot\Gamma(1+s/2)\cdot\pi^{-s/2}\cdot\zeta(s)$ and where $\zeta(s)$ is the Zeta function .
Moreover:

$$\varphi_n = \sum_{k=1}^{n} \binom{n-1}{k-1} \cdot \left(\frac{\xi^{k}(1)}{\xi(1)} \cdot \frac{1}{\Gamma(k+1)}\right) \tag{1'}$$

For ξ (s=1) on the right hand side of Eq. (1) the Equations concern the full Li-Keiper coefficients $\lambda(n)$. If on the right hand side of Eq. (1) we have instead $((½)\cdot\Gamma(1+s/2)\cdot\pi^{-s/2})|_{s=1}$ with $\varphi_{n-k\ (trend)}$, the set concerns the trend part of the coefficients, i.e. $\lambda_{trend}(n)$.
Moreover, if on the right hand side instead of ξ we have $(s-1)\cdot\zeta(s)|_{s=1}$, with $\varphi_{n-k\ (tiny)}$, the set concerns the tiny part of the coefficients $\lambda_{tiny}(n)$ and $\lambda(n) = \lambda_{trend}(n) + \lambda_{tiny}(n)$.
We notice that φ, $\varphi_{(trend)}$ and $\varphi_{(tiny)}$ are the corresponding cluster functions for the three sets.

II. TWO INFINITE SERIES

We may consider Eq. (1) and sums in an appropriate way the set of Equations. We will treat, the first with a result at s =2, i.e. z=1-1/2=1/2**,** the second with a result at s=1/2 i.e. z= 1-1/(1/2) = -1.

The case s=2.
Summing the right hand sides of the three set of Equations, we obtain a shift of 1, i.e. the argument s=2 at the right hand side and

$$\sum_{n=1}^{\infty} \lambda(n)\cdot\left(\frac{1}{2}\right)^n = log\left(\frac{2\cdot\xi(2)}{\xi(1)}\right) = log\left(2\cdot\left(\frac{2}{2}\right)\cdot(2-1)\cdot\pi^{-1}\cdot(\zeta(2)\right) =$$

$$= log(2\cdot\pi^{-1}\cdot(\zeta(2)) = log\left(2\cdot\pi^{-1}\cdot\frac{\pi^2}{6}\right) = log\left(\frac{\pi}{3}\right)$$

$$\sum_{n=1}^{\infty} \lambda_{trend}(n)\cdot\left(\frac{1}{2}\right)^n = log\left(2\cdot\left(\frac{s}{2}\right)\cdot\Gamma\left(\frac{s}{2}\right)\cdot\pi^{-\frac{s}{2}}\right)\Big|_{s=2} = log\left(\frac{2}{\pi}\right)$$

$$\sum_{n=1}^{\infty} \lambda_{tiny}(n)\cdot\left(\frac{1}{2}\right)^n = log\left(\frac{2\cdot\xi(2)}{\xi(1)}\right) = log\left(\frac{\zeta(2)}{(1-1)\cdot\zeta(1)}\right) = log\left(\frac{\pi^2}{6}\right)$$

$$\sum_{n=1}^{\infty} \lambda(n)\cdot\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \lambda_{trend}(n)\cdot\left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \lambda_{tiny}(n)\cdot\left(\frac{1}{2}\right)^n$$

$$log\left(\frac{\pi}{3}\right) = log\left(\frac{2}{\pi}\right) + log\left(\frac{\pi^2}{6}\right) \tag{2}$$

We may then compute the contributions, i.e. that for the complete coefficient λ's, the trend part λ_{trend}'s and for the tiny part λ_{tiny}'s , using functions on the binary system alone (except for a numerical constant independent of γ and log(π)), obtained by Matiyasevich [2], i.e.

$$N_0 = \sum_{n=1}^{\infty} N_0(n)\cdot\left(\frac{1}{2\cdot n\cdot(2\cdot n+1)}\right)$$

and

$$N_1 = \sum_{n=1}^{\infty} N_1(n)\cdot\left(\frac{1}{2\cdot n\cdot(2\cdot n+1)}\right)$$

where $N_0(n)$.$N_1(n)$ is the number of zeros resp. of the units in the binary representation of the integer n.
Then:

$$\sum_{n=1}^{\infty} \lambda(n)\cdot\left(\frac{1}{2}\right)^n = log\left(\frac{\pi}{3}\right) = N_0 - N_1 + log\left(\frac{4}{3}\right) \tag{3}$$

$$\sum_{n=1}^{\infty} \lambda_{trend}(n)\cdot\left(\frac{1}{2}\right)^n = log\left(\frac{2}{\pi}\right) = -(N_0 - N_1) - log(2) \tag{4}$$

$$\sum_{n=1}^{\infty} \lambda_{tiny}(n)\cdot\left(\frac{1}{2}\right)^n = log\left(\frac{\pi^2}{6}\right) = 2\cdot(N_0 - N_1) + log\left(\frac{8}{3}\right)$$

(5)

In figure 1, the plot of the three functions in the range n= (1..50), log(π/3) ~ 0.046, log(2/ π)~ - 0.451, log(π ^{2}/6) ~ 0.497..

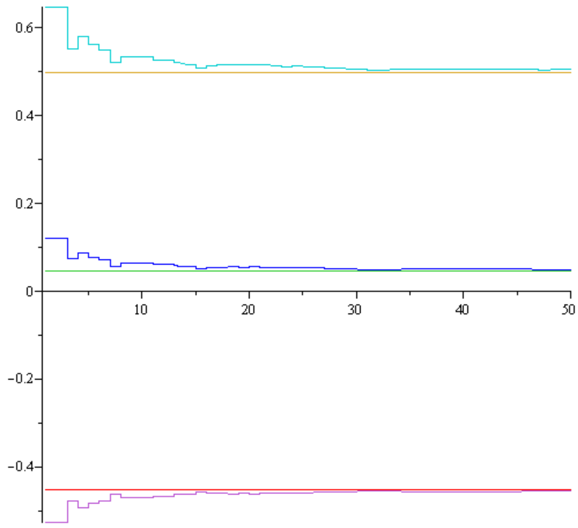

Figure 1. Eq. (3) in blue, Eq. (4) in violet and Eq. (5) in light blue (the tiny part)

Remark
The same treatment may be pursued for values of s = n, n even, a positive integer, with values related to the Bernoulli numbers, i.e. to powers of π.

The case s =1/2.
Still using the infinite set of equations above, we may obtain a shift of -1/2 from s =1, thus, an alternating series for $\lambda(n)/n$, $\lambda_{trend}(n)/n$ and $\lambda_{tiny}(n)/n$ are:

$$\sum_{n=1}^{\infty}\left(\frac{\lambda(n)}{n}\right)\cdot(-1)^n = \log\left(2\cdot\xi\left(\frac{1}{2}\right)\right) = -0.0057750873 \qquad (6)$$

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_{trend}(n)}{n}\right)\cdot(-1)^n = \log\left(\left(\frac{1}{2}\right)\cdot\pi^{-\frac{1}{4}}\cdot\Gamma\left(\frac{1}{4}\right)\right) = 0.3086928726 \qquad (7)$$

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_{tiny}(n)}{n}\right)\cdot(-1)^n = \log\left(\left(-\frac{1}{2}\right)\cdot\zeta\left(\frac{1}{2}\right)\right) = -0.3144679599 \qquad (8)$$

and: 0.3086928726 + (-0.3144679599) = - 0.0057750873.
A numerical experiment is given below for the three sums. For the trend we use the Equation given in [3], i.e.

$$\lambda_{trend}(n) = 1 - \frac{1}{2}\cdot(\gamma + \log(4\cdot\pi))\cdot n + \sum_{j=2}^{\infty}(-1)^j\cdot\binom{n}{j}\cdot(1-2^{-j})\cdot\zeta(j)$$

or given in [4]

$$\lambda_{trend}(n) = \frac{1}{2} + \frac{1}{2}\cdot(\gamma - 1 - \log(2\cdot\pi))\cdot n + \frac{1}{2}\cdot n\cdot\log(n)$$

and the second of the above Equations i.e. Eq. (7); in the plot we indicate the "attractor" for the trend given by 0.308...

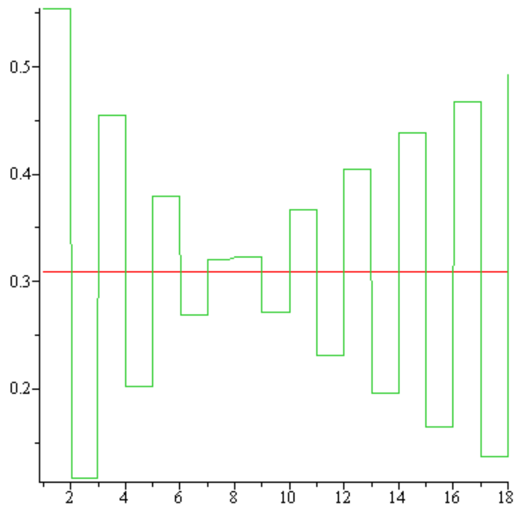

Figure 2. The function for the trend and the constant 0.308**...** (Eq. (7) with 0.308.... in red)

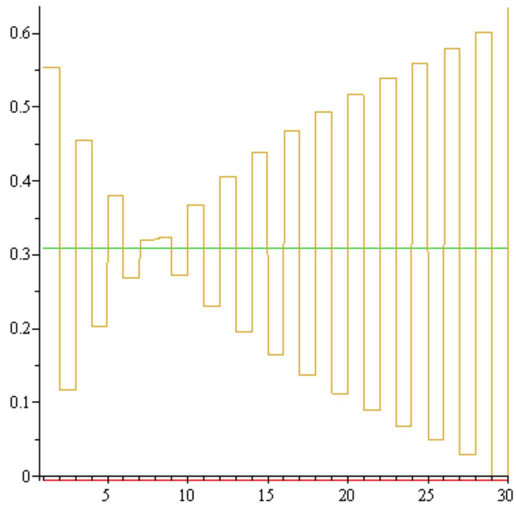

Figure 3. The function, Eq. (7) as above in the range n=1..30 and the constant 0.308.. in green.

We then obtain the plot of the function for the tiny part, given below.

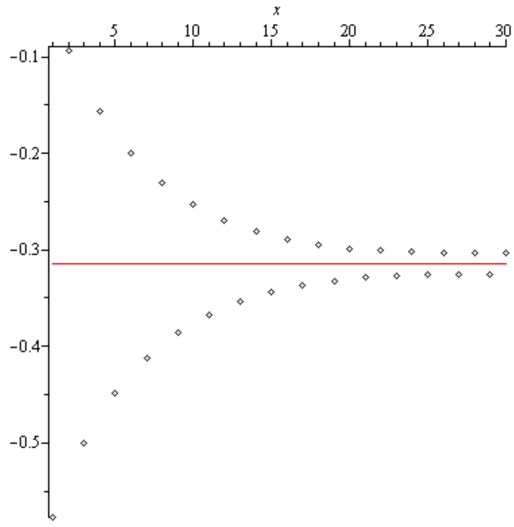

Figure 4. The function for the tiny part (Eq. (8)) in black and the limit -0.3144... (in red), in the range n=[1..30].

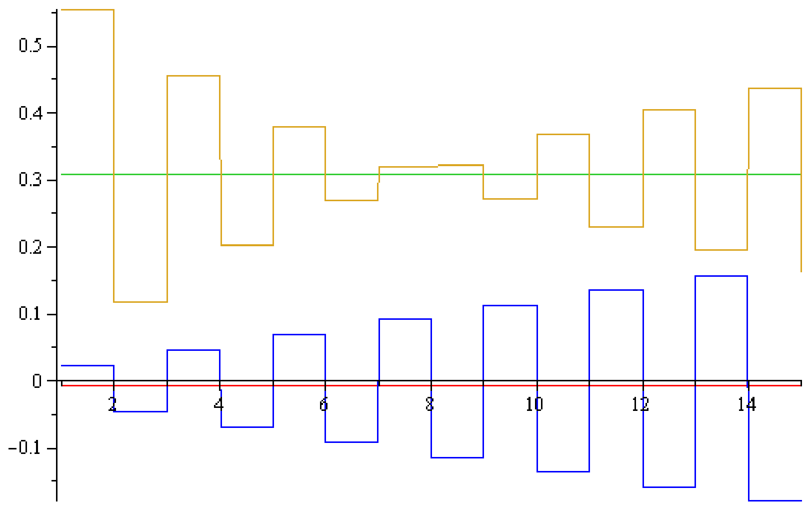

Figure 5. The two functions Eq. (6) in blue with the constant - 0.0057 (in red) and Eq. (7) (in maroon), with the value 0.308.. (in green), in the range n= (1..15). (Notice that -0.0057- 0.308 is equal to the limit -0.314.. on Figure 4).

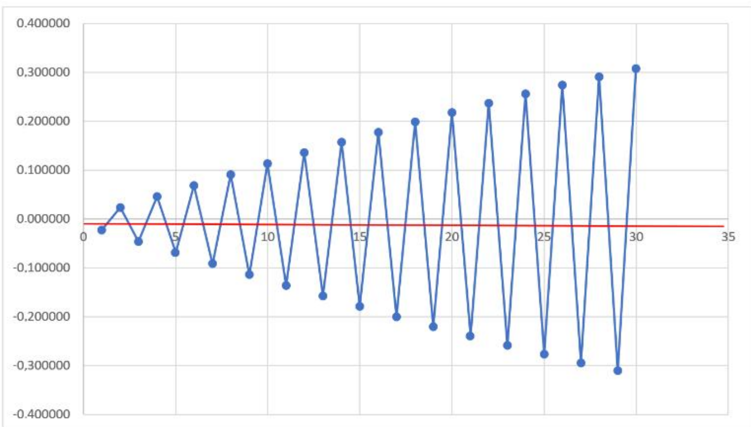

Figure 6. The function for the complete Li-Keiper coefficients computed with Fungrim at some decimals up to n=30. Eq. (6) and the value -0.0057...

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Remark

Since $\lambda(n)/n$ and $\lambda_{trend}(n)/n$ in the above alternating sums have marked fluctuations it is important to consider the difference between the partial sums of $(-1)^n\cdot\lambda(n)/n$ and that of $(-1)^n\cdot\lambda_{trend}(n)/n$ to obtain convergent values for the resulting alternating series of $\lambda_{tiny}(n)/n$, i.e. the limit given by -0.314... The plot of the above difference is given below confirming the limit -0.3144..

$$f(N) = \sum_{n=1}^{N}\left(\frac{\lambda(n)}{n}\right)\cdot(-1)^n - \sum_{n=1}^{N}\left(\frac{\lambda_{trend}(n)}{n}\right)\cdot(-1)^n$$

and $\lim_{N\to\infty} f(N) = -0.314..$

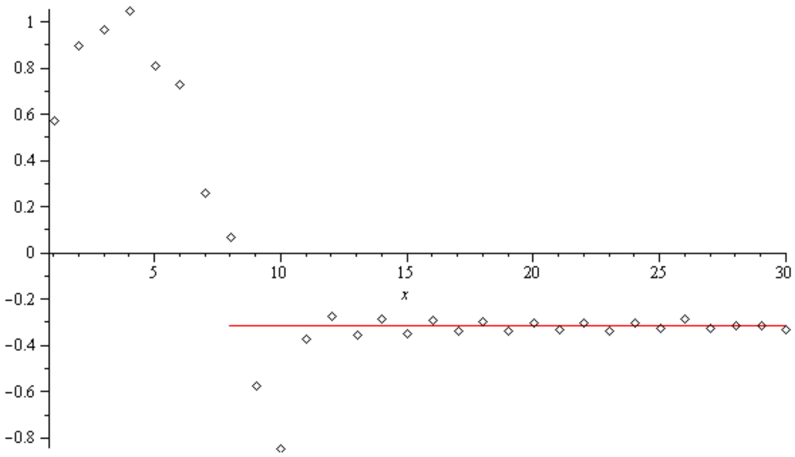

Figure 7. Plot of f(N) up to N=30 and the limit -0.3144.. of Eq. (8).

Convergence with Eq. (1)

We also compute the sums for the tiny part given in terms of the associated cluster functions $\varphi_{tiny,n}$ or with the associated function $\xi_{tiny}(s) = (s-1)\cdot\zeta(s)$.The corresponding set of Equations is given by:

$$\sum_{k=0}^{n-1}(-1)^{(k)}\cdot\binom{n-1}{k}\cdot\varphi_{tiny,n-k} = \left(\frac{\xi^{n}_{tiny}(1)}{\xi_{tiny}(1)}\cdot\frac{1}{\Gamma(n+1)}\right) \tag{1''}$$

and the cluster functions

$$\varphi_{tiny,n} = \sum_{k=1}^{n}\binom{n-1}{k-1}\cdot\left(\frac{\xi^{k}_{tiny}(1)}{\xi_{tiny}(1)}\cdot\frac{1}{\Gamma(k+1)}\right) \tag{2'}$$

We may compute the above sums of interest with the quantities, i.e. the right hand side of Eq.(1) but it is also of interest to give the Table of the $\varphi_{tiny,\, n}$ calculated with Eq.(2').

Below, for the first sum, we give the numerical results up to N=7 (Summation of the right hand side of Eq.(1),i.e. the Taylor series around s=1 i.e. z=0); i.e. with $\xi^0_{tiny}(1)/\xi_{tiny}(1) = 1$,

$1+ [(\xi_{tiny}{}^1(1)/\xi_{tiny}(1))1!]\cdot(2\text{-}1) +[(\xi_{tiny}{}^2(1)/\xi_{tiny}(1))/2!]\cdot(2\text{-}1)^2+ [(\xi_{tiny}{}^3(1)/\xi_{tiny}(1))/3!].(2\text{-}1)^3$ +...

The Table 1 reports the above first few values for small n.

N	$\sum_{n=0}^{N}\left(\xi^{n}_{tiny}(1)/\xi_{tiny}(1)\right)/\Gamma(n+1)\cdot(2-1)^n$
0	1

1	1.5772156644901532
2	1.6500315103852087
3	1.6451863287887725
4	1.6448440230520553
5	1.6449409134714498
6	1.6449343024396390
7	1.6449343024396390

Table 1

This short sequence gives with a=log($\varphi_{tiny,7}$) the result: a = 0.4977004156931398 to be compared with a precision of 10^{-7} (See Eq.(2)). In the same way we may treat the second sum still using Eq.(1) or with the table of the corresponding cluster functions (given below as an example) for the tiny part (Eq.(1")).

n	$\varphi_{tiny,\,n}$
0	1
1	0.5772156649015329
2	0.6500315103852096
3	0.7180021742724499
4	0.7807853508265358
5	0.8381356247301436
6	0.9360002575986541

Table 2.

Now, we use Eq.(1) with the Taylor expansion given with $\xi_{tiny}{}^{n}(1)/\xi_{tiny}(1)$: after multiplication of the right hand side of Eq.(1) for the tiny part with $(-1/2)^{n-1}$ and after summation of the first n=6 terms, we obtain:

1+(-1/2)·5772156644901532+

+$(-1/2)^2$·0.07281584548367672+

+$(-1/2)^3$·(-0.0048451181596436159 +

+$(-1/2)^4$·(-0.0003423057367172244 +

+$(-1/2)^5$·(0.00009689041939447084+

+ $(-1/2)^6$·(-000006611031810842189 =

= 0.7301773546855564.

log(0.7301773546855564)~ - 0.31446796 ~ - 0.31446795.. = log((-1/2)·ζ(1/2)), exact to 7 digits, (See Eq.(8)); moreover (since s=1/2 → z=(1-1/(1/2))= -1), we have :

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_{tiny}(n)}{n}\right)\cdot(z)^n \to \sum_{n=1}^{\infty}\left(\frac{\lambda_{tiny}(n)}{n}\right)\cdot(-1)^n = log\left(\left(-\frac{1}{2}\right)\cdot\zeta\left(\frac{1}{2}\right)\right) = -0.3144679599..$$

In the same way, for the complete Li-Keiper coefficients, multiplication of the right hand side of Eq. (1) with $(-1/2)^{(n-1)}$ after summation of the first 7 terms we obtain: -0.005775100686, exact to 7 digits as may be verified with Eq. (6).

Finally, we still consider formally the case s=1/2 i.e. z = -1 but for the derivative and given by:

$$\sum_{n=1}^{\infty}(\lambda(n))\cdot(-1)^{n-1} =$$

$$= \frac{1}{4}\cdot\left(-\left(\frac{1}{2}\right)\cdot\log(\pi) + \left(\frac{1}{2}\right)\cdot\Gamma'\left(\frac{1}{4}\right) + \frac{\zeta'\left(\frac{1}{2}\right)}{\zeta\left(\frac{1}{2}\right)}\right) =$$

=0.555282..

(9)

$$\sum_{n=1}^{\infty} (\lambda_{trend}(n)) \cdot (-1)^{n-1} =$$

$$= \frac{1}{4} \cdot \left(2 - \left(\frac{1}{2}\right) \cdot \log(\pi) + \left(\frac{1}{2}\right) \cdot \frac{\Gamma'\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right) =$$

= 0.399683..
(10)

$$\sum_{n=1}^{\infty} (\lambda_{tiny}(n)) \cdot (-1)^{n-1} = -2 + \frac{\zeta'\left(\frac{1}{2}\right)}{\zeta\left(\frac{1}{2}\right)} =$$

=0.171522..
(11)

i.e. 0.399683... + 0.171522 = 0.555282.

We notice that the numbers above are only mean values since in all the three cases (s=1/2, z=-1) for the derivatives), the limits do not exist.
Below, in the numerical experiment, we present the plot of the left hand side of Eq.(11), calculated to 16 digits using the above Equation for the trend and the first 26 results from [5] for the complete Li-Keiper coefficients.

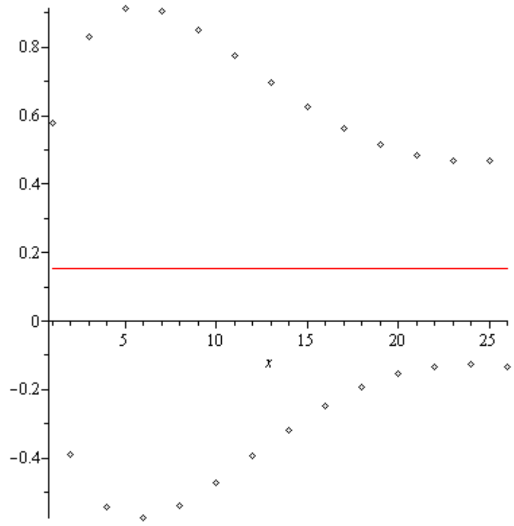

Figure 8. Left hand side of Eq.(11) in the range n=(1..26) and the "limit" 0.171522 (i.e. a mean value in red).

Remark

As for the trend and for the complete Li-Keiper coefficients (which are diverging as $n \rightarrow \infty$) the above alternating sum retains its oscillatory character (growing oscillations as n increases). On the Fig. 7. it is visible the first maximum and the first minimum of the oscillations of $\lambda_{tiny}(n)$ studied extensively in [6]. For the sum in Eq. (8) the denominator n kills the oscillations , a "symptom" that $\lambda_{tiny}(n)$ may be bounded by $c \cdot n$ for all n, with c a constant.

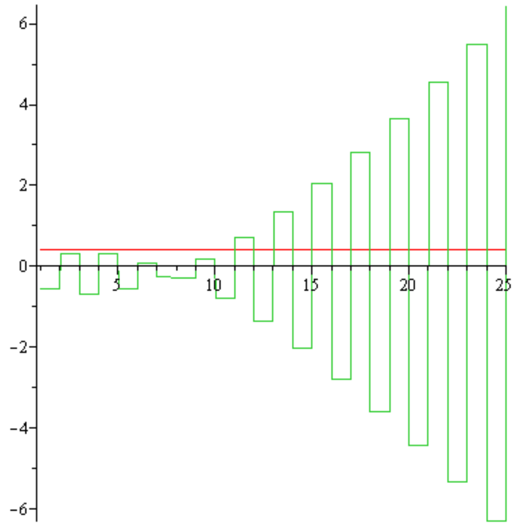

Figure 9. The alternating sum of the trend Eq. (10) (the value 0.399683...in red), in the range N=1..25. (The numerical values are symmetric around this constant).

For the complete Li-Keiper coefficients the Figure is similar. To conclude, we consider the left hand side of Eq. (11) and we take for $\lambda_{tiny}(n)$ our asymptotic formula of Equation:

$$\left(A \cdot \sqrt{n} \cdot \log(n) + B \cdot \log(n)\right)$$

with appropriate values of A and B given in [6]; below we plot the function

$$f(N) = \sum_{n=1}^{N} (-1)^{n-1} \cdot \left(A \cdot \sqrt{n} \cdot \log(n) + B \cdot \log(n)\right)$$

as well as the mean values c = (f(N)+f(N+1))/2. For the values N of a few thousand, we find a value c~0.171.. which supports the above asymptotic envelop of the extreme tiny fluctuation and which is in agreement with the value given by Eq. (11).

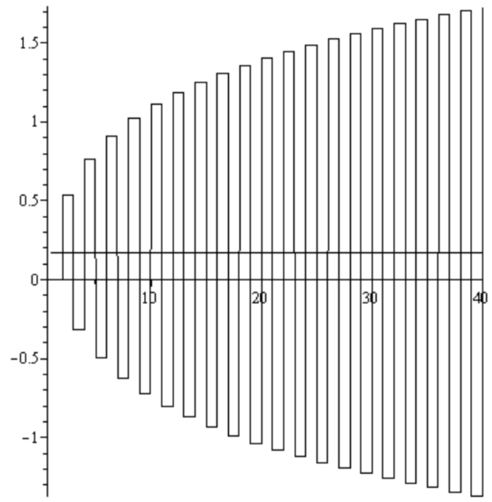

Figure 10. Plot of f(N) and the constant c ~ 0.171...

III. CONCLUSION

The formal series for s >1/2, i.e. |z|<1, given by $\sum\lambda(n)\cdot z^n/n$, (z=1-1/s), is exponential decaying as a function of n and thus converging, while at (z)= -1 (s=1/2) the values of the three series are obtained with few terms of Eq. (1).

For the formal Series $\sum(-1)^{n-1}\cdot\lambda(n)\cdot z^{n-1}$ at z= -1 (s=1/2) i.e. the derivative (with respect to z), we may define mean values in the three cases (complete Li-Keiper, trend and tiny part) and given by Eq. (9), Eq. (10), Eq. (11)); moreover, the asymptotic mean value of the asymptotic function considered above [6] is satisfactory.
For a "concrete" form of the Li-Keiper coefficients (i.e. involving only elementary functions), in connection with the binomial transform, we refer to [6].

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